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INTERPRETATION OF THE MOTION OF A HEAVY BODY AROUND A STATIONARY AXIS AND DYNAMIC (KINETIC) PRESSURES ON BEARING BY MEANS OF THE MASS MOMENT VECTOR FOR THE POLE AND THE AXIS

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Abstract

This paper presents the derivation of the dynamic (kinetic) rotation equations and/or oscillations of a rigid body around a stationary axis in the Earth gravitational field. In the general case the rotation axis is not horizontal. The kinetic equations of the rigid body motion around the stationary axis are interpreted by means of the introduced vectors: $\vec{\mathfrak{S}}_n^{(A)}$ of the body mass linear moment for the point in the stationary (fixed) bearing A and for the rotation axis oriented by unit vector $\vec{n}$: $\vec{\mathfrak{J}}_n^{(A)}$ of the body mass inertia moment and its deviation part of the vector $\vec{\mathfrak{D}}_n^{(A)}$ of the deviation load by the body mass inertia moment of the rotation axis for the pole in the stationary (fixed) bearing.

The formed kinetic equations are used for determining scalar equations of the rigid body rotation/oscillation around the stationary rotation axis as well as for determining the bearings reaction components: that is, axial and deviational reaction components – resistance (pressure) of the stationary bearings and deviational (normal to the axis) reaction – the moveable bearing resistance (pressure).

The first integral, that is, the integral of energy is determined from the scalar equation of the rigid body rotation/oscillation and by means of it the motion character analysis is performed by means of the phase trajectories and constant energy curves in the phase plane. The analysis of the singularity and phase trajectories in the phase plane leads to the conclusion about a possible appearence of the rigid body asymptotic stochastic behaviour in its rotation around the stationary axis in the Earth gravitational field in the case when at the initial moment the body is communicated a certain bifurcational value of the overall energy by mean of the kinetic and/or potential energy, that is, when at the initial moment the body is communicated a certain angular velocity and/or

initial elongation. The time interval in which the motion is done is also taken into consideration in this case as well as the time elongation at inginitum.

The appearence of the singularity triggers coupled set is pointed out on the phase portrait and at this bifurcational value of the initial kinetic parameters the sensitivity of the moment character is also pointed out within the scope of that value of the initial motion parameters.

From the expressions for the bearing resistances (pressures) parts are selected corresponding to the kinetic pressures – dynamic bearing resistance from the parts that would correspond to the bearing resistances in the case of the determined system static equilibrium. These parts – the kinetic pressures are expressed by means of the vector $\vec{\mathfrak{S}}_n^{(A)}$ of the body mass linear moment and by the vector $\vec{\mathfrak{D}}_n^{(A)}$ of the deviation load by the body mass inertia moment of the rotation axis for the rotation axis and for the pole in the stationary (fixed) bearing.

On the basis of the expression for the dynamic pressures it can be seen that a part of the stationary bearing reaction coming from the body dynamic properties with respect to the rotation axis its rotation around it depends on the rotator vector $\vec{\mathfrak{R}}$ and the absolute value of the vector $\vec{\mathfrak{S}}_n^{(A)}$ of the body mass linear moment for the rotation axis and for the pole in the stationary (fixed) bearing while the reaction part of both the moveable and the stationary (fixed) bearing which also comes from the rigid body moment properties which rotates for the rotation axis and for the pole in the stationary (fixed) bearing depends on the rotator vector and on the absolute value of the vector $\vec{\mathfrak{D}}_n^{(A)}$ of the deviation load by the body mass inertia moment for the rotation axis.

Further on in the paper the rotator vector $\vec{\mathfrak{R}}$ behaviour is discussed as well as the change of its intensity in the body rotation or oscillation around the stationary axis in the Earth gravitational field. The numerical experiment is performed and the graphical a chemes are formed of the phase trajectories in the phase plane, of the rotator change as the elongation function and the rotator change as the versor in the plane perpendicular to the rotation axis. The diagram shows that the vector has the zero value only at the bifurcational value and that it has extreme values at the singular points; that is, it has maximum in the stable equilibrium positions, and minimums in the unstable equilibrium positions, namely, at the reciprocating points. The reciprocating points appear in oscillation and they correspond to the maximal elongations and the angular velocities are equal to zero at them.

I am sure that in this paper I have given a modest contribution to this much explored topic. In my opinion this contribution is in the interpretation of the kinetic equations by means of two newly – introduced vectors $\vec{\mathfrak{S}}_n^{(A)}$ and $\vec{\mathfrak{J}}_n^{(A)}$ and in their use in interpreting the kinetic pressures as well as in the introduction of the rotator vectors.

Introduction

A body rotation around a stationary axis in the Earth gravitational field is the topic very much explored and the question may be ascked if there is still something new to find out and say about it. However, the presented interpretation of the kinetic pressure obviously gives an affirmative answer.

The Figure N° 1 shows a rigid body that can rotate around a stationary axis which is like a rigid shaft without mass supported on the stationary bearing A and on the moveable sliding one along the rotation axis. The rotation axis forms an angle α with the horizon and it exist in the Earth gravitational field together with the body.

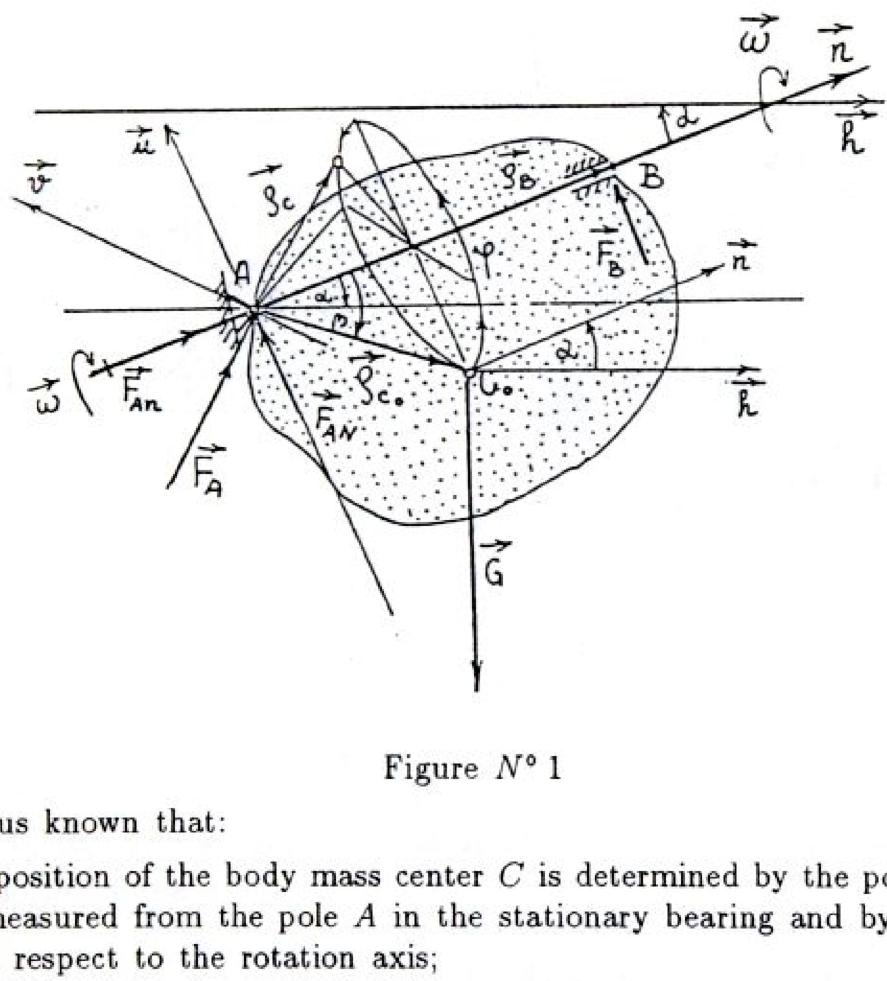


Figure N° 1

Let us known that:

a° the position of the body mass center C is determined by the position vector $\vec{\varrho}_c$ measured from the pole A in the stationary bearing and by the angle β with respect to the rotation axis;

b° the vector $\vec{\mathfrak{J}}_n^{(A)}$ of the rigid body mass inertia moment vector for the pole in the stationary bearing A and for the rotation axis $\vec{n}$ and its deviational part – the vector $\vec{\mathfrak{D}}_n^{(A)}$ of the deviation load by the mass inertia moment of the rotation axis for the pole A and for the rotation axis; acording to the Ref. [10] the vector $\vec{\mathfrak{J}}_n^{(A)}$ is:

$$\vec{\mathfrak{J}}_n^{(A)} = \iiint_V [\vec{\varrho}, [\vec{n}, \vec{\varrho}]]\, dm \tag{1}$$

while

$$\vec{\mathfrak{D}}_n^{(A)} = \left[\vec{n}, \left[\vec{\mathfrak{J}}_n^{(A)}, \vec{n}\right]\right] \tag{2}$$

c° the vector $\vec{\mathfrak{S}}_n^{(A)}$ of the body mass linear moment for the rotation axis $\vec{n}$ and for the pole A in the moveable bearing:

$$\vec{\mathfrak{J}}_n^{(A)} = \iiint\limits_V [\vec{n}, \vec{\varrho}]\, dm \tag{3}$$

The Figure N° 2 graphically shows following vectors: $\vec{\mathfrak{J}}_n^{(A)}$, $\vec{\mathfrak{S}}_n^{(A)}$, $\vec{\varrho}_c$, $\vec{G}$ and the bearings reaction components: $\vec{F}_{An}$, $\vec{F}_A^d$ i $\vec{F}_B^d$.

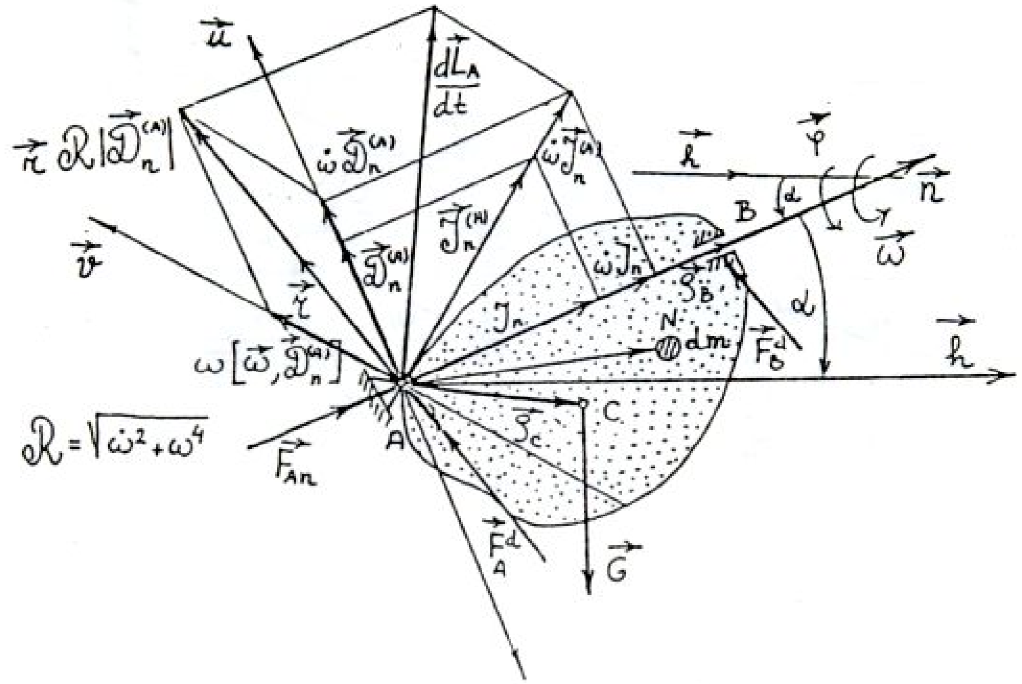


Figure N° 2

In the general case let a rigid body be subjected to a system of forces $\vec{F}_k$ whose points of application N_{ko} are determined by the position vectors $\vec{\varrho}_k$ with respect to the pole in the stationary bearing.

Let's denote the rotation angle of the body around the stationary axis oriented by the unit vector $\vec{n}$ with $\vec{\varphi} = \varphi\,\vec{n}$; let's denote the angular velocity and the body angular acceleration in this rotation with $\vec{\omega}$ and $\mathring{\vec{\omega}}$.

Kinetic equations of dynamic equilibrium

Following the idea in the [1]–[4], [11]–[14] and [16] for the linear momentum $\vec{K}$ and the angular momentum $\vec{L}_A$ for the pole A in the stationary bearing and for the body rotation around the stationary axis oriented by the unit vector $\vec{n}$

we can write the following expressions:

$$\vec{K} = M\,\vec{v}_A + \omega\,\vec{\mathfrak{S}}_n^{(A)} \tag{4}$$

$$\vec{L}_A = \left[\vec{\mathfrak{S}}_n^{(A)},\, \vec{v}_A\right] + \omega\,\vec{\mathfrak{J}}_n^{(A)} + \left[\vec{r}_A,\, M\,\vec{v}_A + \omega\,\vec{\mathfrak{S}}_n^{(A)}\right] \tag{5}$$

that is, for $\vec{v}_A = 0$ and $\vec{r}_A = 0$ follows:

$$\vec{K} = \omega\,\vec{\mathfrak{S}}_n^{(A)} \quad \vec{L}_A = \omega\,\vec{\mathfrak{J}}_n^{(A)} \tag{6}$$

Using the basic laws of the dynamics starting that the linear momentum derivative in time is equal to the sum of all the active and reactive forces, and that the angular momentum derivative in time is equal to the sum of the active and reactive forces moments we write the following two vector equations:

$$\frac{\mathrm{d}\vec{K}}{\mathrm{d}t} = \mathring{\omega}\,\vec{\mathfrak{S}}_n^{(A)} + \omega\left[\vec{\omega},\, \vec{\mathfrak{S}}_n^{(A)}\right] = \sum_{k=1}^{k=N} \vec{F}_k \tag{7}$$

$$\frac{\mathrm{d}\vec{L}_A}{\mathrm{d}t} = \mathring{\omega}\,\vec{\mathfrak{J}}_n^{(A)} + \omega\left[\vec{\omega},\, \vec{\mathfrak{J}}_n^{(A)}\right] = \sum_{k=1}^{k=N} \left[\vec{\varrho}_k,\, \vec{F}_k\right] \tag{8}$$

namely,

$$\mathring{\omega}\,\vec{\mathfrak{S}}_n^{(A)} + \omega\left[\vec{\omega},\, \vec{\mathfrak{S}}_n^{(A)}\right] = \vec{F}_A + \vec{F}_B + \vec{G} + \sum_{k=1}^{k=N} \vec{F}_k \tag{7*}$$

$$\mathring{\vec{\omega}}\, J_n^{(A)} + \mathring{\omega}\vec{\mathfrak{D}}_n^{(A)} + \omega\left[\vec{\omega},\, \vec{\mathfrak{D}}_n^{(A)}\right] = \left[\vec{\varrho}_B,\, \vec{F}_B\right] + \left[\vec{\varrho}_C,\, \vec{G}\right] + \sum_{k=1}^{k=N} \left[\vec{\varrho}_k,\, \vec{F}_k\right] \tag{8*}$$

These two vector equations are kinetic equations of dynamic equilibrium in motion–rotation of the body around the stationary axis in the Earth gravitational field.

If we now multiply scalarly and vectorly the equations (7*) and (8*) by the unit vectors $\vec{n}$ and having in mind that $\vec{\varrho}_B = \varrho_B\,\vec{n}$, we obtain:

1° The rotation equation around the axis oriented by the unit vector $\vec{n}$ in the form:

$$\left(\vec{\mathfrak{J}}_n^{(C)},\, \mathring{\vec{\omega}}\right) = \left(\left[\vec{\varrho}_C,\, \vec{G}\right],\, \vec{n}\right) + \sum_{k=1}^{k=N} \left(\left[\vec{\varrho}_k,\, \vec{F}_k\right],\, \vec{n}\right) \tag{9}$$

and

2° The equations for determining the bearing kinetic resistances (pressures), that is pressures upon the bearings $\vec{F}_A$ and $\vec{F}_B$, that is, their components in the axis direction $\vec{n}$ and normal to the rotation axis:

$$\left(\vec{F}_A,\, \vec{n}\right) + \left(\vec{G},\, \vec{n}\right) + \sum_{k=1}^{k=N} \left(\vec{F}_k,\, \vec{n}\right) = 0 \tag{10}$$

$$\overset{\circ}{\omega}\,\vec{\mathfrak{S}}_n^{(A)} + \omega\left[\vec{\omega},\, \vec{\mathfrak{S}}_n^{(A)}\right] = \left[\vec{n},\, \left[\vec{F}_A,\, \vec{n}\right]\right] + \left[\vec{n},\, \left[\vec{G},\, \vec{n}\right]\right] + \\ + \left[\vec{n},\, \left[\vec{F}_B,\, \vec{n}\right]\right] + \sum_{k=1}^{k=N}\left[\vec{n},\, \left[\vec{F}_k,\, \vec{n}\right]\right] \qquad (11)$$

$$\overset{\circ}{\omega}\vec{\mathfrak{D}}_n^{(A)} + \omega\left[\vec{\omega},\, \vec{\mathfrak{D}}_n^{(A)}\right] = \left[\vec{n},\, \left[\left[\vec{\varrho}_C,\, \vec{G}\right],\, \vec{n}\right]\right] + \left[\vec{n},\, \left[\left[\vec{\varrho}_B,\, \vec{F}_B\right],\, \vec{n}\right]\right] + \\ + \sum_{k=1}^{k=N}\left[\vec{n},\, \left[\left[\vec{\varrho}_C,\, \vec{G}\right],\, \vec{n}\right]\right] \qquad (12)$$

$$\vec{\mathfrak{R}}_1\left|\vec{\mathfrak{S}}_n^{(A)}\right| = \left[\vec{n},\, \left[\vec{F}_A,\, \vec{n}\right]\right] + \left[\vec{n},\, \left[\vec{G},\, \vec{n}\right]\right] + \sum_{k=1}^{k=N}\left[\vec{n},\, \left[\vec{F}_k,\, \vec{n}\right]\right] + \\ + \left[\vec{n},\, \left[\vec{F}_B,\, \vec{n}\right]\right] \qquad (11^*)$$

$$\vec{\mathfrak{R}}\left|\vec{\mathfrak{D}}_n^{(A)}\right| = \left[\vec{n},\, \left[\left[\vec{\varrho}_C,\, \vec{G}\right],\, \vec{n}\right]\right] + \left[\vec{n},\, \left[\left[\vec{\varrho}_B,\, \vec{F}_B\right],\, \vec{n}\right]\right] + \\ + \sum_{k=1}^{k=N}\left[\vec{n},\, \left[\left[\vec{\varrho}_k,\, \vec{F}_k\right],\, \vec{n}\right]\right] \qquad (12^*)$$

where from we determine the bearings resistance (pressure) components themselves in the form:

$$\vec{F}_{AN} = \left[\vec{n},\, \left[\vec{F}_A,\, \vec{n}\right]\right] = \vec{F}_{As}^{(\mathrm{dev})} + \vec{F}_{Ad}^{(\mathrm{dev})}$$

$$\vec{F}_{AN}^{s} = \vec{F}_{As}^{(\mathrm{dev})} = \vec{\mathfrak{R}}_1\left|\vec{\mathfrak{S}}_n^{(A)}\right| - \left[\vec{n},\, \left[\vec{G},\, \vec{n}\right]\right] - \sum_{k=1}^{k=N}\left[\vec{n},\, \left[\vec{F}_A,\, \vec{n}\right]\right] \qquad (10^{**})$$

$$\vec{F}_{AN}^{d} = \vec{F}_{Ad}^{(\mathrm{dev})} = -\vec{F}_B = -\frac{1}{\varrho_B}\vec{\mathfrak{R}}\left|\vec{\mathfrak{D}}_n^{(A)}\right| + \frac{1}{\varrho_B}\left[\vec{n},\, \left[\vec{\varrho_C},\, \vec{G}\right],\, \vec{n}\right] + \\ + \frac{1}{\varrho_B}\sum_{k=1}^{k=N}\left[\vec{n},\, \left[\vec{\varrho}_k,\, \vec{F}_k\right],\, \vec{n}\right] \qquad (10^{***})$$

$$\vec{F}_{An} = \left(\vec{F}_A,\, \vec{n}\right)\vec{n} = -\left(\vec{G},\, \vec{n}\right)\vec{n} - \vec{n}\sum_{k=1}^{k=N}\left(\vec{F}_k,\, \vec{n}\right) \qquad (11^{**})$$

$$\left(\vec{F}_B,\, \vec{n}\right) = 0 \quad \Rightarrow \quad \vec{F}_B = \vec{F}_B^{(dev)}$$

$$\vec{F}_B = \frac{1}{\varrho_B}\vec{\mathfrak{R}}\left|\vec{\mathfrak{D}}_n^{(A)}\right| - \frac{1}{\varrho_B}\left[\vec{n},\, \left[\vec{\varrho}_C,\, \vec{G}\right],\, \vec{n}\right] - \frac{1}{\varrho_B}\sum_{k=1}^{k=N}\left[\vec{n},\, \left[\vec{\varrho}_k,\, \vec{F}_k\right],\, \vec{n}\right] \qquad (12^{**})$$

where is:

$$\mathfrak{R} = \sqrt{\overset{\circ}{\omega}^2 + \omega^4} \quad \vec{\mathfrak{R}} = \mathfrak{R}\vec{n} = \frac{\vec{a}}{r} = \overset{\circ}{\omega}\vec{u} + \omega^2\vec{v} \tag{13}$$

From the expression for the bearing resistances we select a part which is the result of the action of the external active forces of constant intensity or variable in time and whose influence upon the bearings resistances in possibly variable in time only due to the change of their line of application with respect to the configuration of the body which is rotating such as the case when the force of the body's own weight which retains the application line direction and thus its position with respect to the body configuration although in doing this it retains the application point constantly in the body mass center which rotates around the rotation axis together with the body. The body mass center describes a circle or an arc in the plane through the masses center normal to the rotation axis.

The other part of the bearings kinetic resistances (pressures) in the body rotation around the stationary axis is the result exclusively of the kinetic-internal body properties with respect to the rotation axis and the rotation kinematics and the rigid body rotation kinematics around the stationary axis. These parts appear as parameters depending on the rotator vector $\vec{\mathfrak{R}}$ which in itself contains the angular velocity and the angular acceleration of the body rotation around the rotation axis and the rigid body mass moment properties with respect to the pole A in the stationary bearing and the rotation axis expressed by the vectors $\vec{\mathfrak{S}}_n^{(A)}$ and $\vec{\mathfrak{J}}_n^{(A)}$.

In order to discuss the rotor effect on the kinetic pressures upon the bearings in which the rigid body shaft axis is rotating it is necessary to know the angular acceleration $\overset{\circ}{\vec{\omega}}$ and the angular velocity $\vec{\omega}$ and in order to do this it is necessary to solve the body rotation/oscillation equation around the axis (9), namely, to determine $\vec{\varphi}(t)$ and $\vec{\omega}(t)$ and $\varphi(\omega)$.

A special case of the body rotation around the stationary axis in the earth gravitational field

From now on we concetrate on the consideration og the rigid body motion around the stationary axis in the Earth gravitational field assuming that other forces do not exist. For this case the rotation/oscillation equation is reduced to the form:

$$\left(\vec{\mathfrak{J}}_n^{(C)}, \overset{\circ}{\vec{\omega}}\right) = \left(\left[\vec{\varrho}_C, \vec{G}\right], \vec{n}\right) \tag{9*}$$

Since it is:

$$\vec{n} = \cos\alpha\,\vec{j} + \sin\alpha\,\vec{k}, \quad \vec{u} = \sin\alpha\,\vec{j} - \cos\alpha\,\vec{k}, \quad \vec{v} = -\vec{i} \tag{14}$$

$$\vec{\varrho}_C = \varrho_c \cos\beta\,\vec{n} + \varrho_C \sin\beta\cos\varphi\,\vec{u} + \varrho_C \sin\beta\sin\varphi\,\vec{u} \tag{15}$$

$$\vec{\varrho}_C = \varrho_c \Big[-\vec{i} \sin\beta \sin\varphi + \vec{j} (\cos\beta \cos\alpha + \sin\beta \sin\alpha \cos\varphi) + \\ + \vec{k} (\cos\beta \sin\alpha - \sin\beta \cos\alpha \cos\varphi) \Big] \tag{15*}$$

if it is $\overset{\circ}{\omega} = \ddot{\varphi}$ the motion equation (9*) can be written in the form:

$$\ddot{\varphi} + \frac{M\, g\, \varrho_C \, \sin\beta \cos\alpha}{J_n^{(A)}} \sin\varphi = 0 \tag{9**}$$

If we denote the following expression with Ω^2:

$$\Omega^2 = \frac{M\, g\, \varrho_C \, \sin\beta \cos\alpha}{J_n^{(A)}} = \frac{g}{l_r} \tag{16}$$

on which l_r denotes the expression of the form:

$$l_r = \frac{\left(\vec{n}, \vec{\mathfrak{J}}_n^{(C)}\right) + \left(\vec{n}, [\vec{\varrho}_C^{*}, [\vec{n}, \vec{\varrho}_C^{*}]]\right)}{M\left(\left[\vec{\varrho}_C^{*}, \vec{k}\right], \left[\vec{n}, \vec{k}\right]\right)} \tag{17}$$

namely,

$$l_r = \left(\frac{i_n^{(C)2}}{\varrho_C \sin\beta} + \varrho_C \sin\beta \right) \frac{1}{\cos\alpha} \tag{17*}$$

then the motion equation (9*) is reduced to the form:

$$\ddot{\varphi} + \frac{g}{l_r} \sin\varphi = 0 \Big/ \, 2\overset{\circ}{\varphi}\, dt = 2\, d\varphi \tag{9**}$$

which is known in the References as a mathematical model for the heavy material point movement along the circle of the radius in the vertical plane.

This equation of the motion (9**) is non-linear and by multiplying with $2\overset{\circ}{\varphi}\, dt = 2\, d\varphi$ it is reduced to the form that enables integratization after which we obtain the following relation between the generalized coordinate φ and the angular velocity $\omega = \overset{\circ}{\varphi}$:

$$\overset{\circ}{\varphi} = \sqrt{h + 2\left(\frac{g}{l_r}\right)(\cos\varphi_o - 1)} \tag{18}$$

In the previous equation h denotes integracional constant which has the following value for the known initial conditions: the angle φ_o and the angular velocity $\overset{\circ}{\varphi}_o$:

$$h = \overset{\circ}{\varphi}{}_o^2 + 2\left(\frac{g}{l_r}\right)(1 - \cos\varphi_o)$$

The equation (18) represens an energy integral which can be written in the form:

$$\mathbb{E}_k + \mathbb{E}_p = \mathbb{E}_o, \quad \frac{1}{2}\omega\left(\vec{\omega}, \vec{\mathfrak{J}}_n^{(A)}\right) + \left(\vec{G}, \vec{\varrho}_C - \vec{\varrho}_C^{*}\right) = \mathbb{E}_o \tag{20}$$

in which $\mathbb{E}_o$ is the overall system energy which is equal to the energy which is communicated to the body at the initial moment of time measurement by means of the initial elongation of the angle φ_o and the initial angular velocity $\omega_o = \overset{\circ}{\varphi}_o$ and it has the value:

$$\mathbb{E}_o = \frac{1}{2} J_n^{(A)} \overset{\circ}{\varphi}{}_o^2 + M g \varrho_c \sin\beta \cos\alpha \,(1 - \cos\varphi_o) = \frac{1}{2} J_n^{(A)} h \qquad (20^*)$$

Phase portrait analysis and energy constant curves analysis

On the basic of the numerical analysis of the equation (18) by changing the value of the parameters of the initial energy h the family of the integral curves is obtained – of the phase trajectories whose scheme is shown in the Figure $N^\circ 3$ and which corresponds to the known one given in the literature for the motion of a heavy material point along the circle in the vertical plane (gravitational pendulum). See the Ref. [7], [17], for instance.

The numerical analysis is carried out for the following values of the integrational constant – the system energy parameters: $h = n\,g/l_r$ when it is $= 1, 3/2, 2, 5/2, 3, 7/2, 4, 9/2, 5, 11/2, 6, \ldots$ By analysing the phase trajectories family in the phase plane for the heavy rigid body motion by its rotation around the stationary axis we can conclude the following:

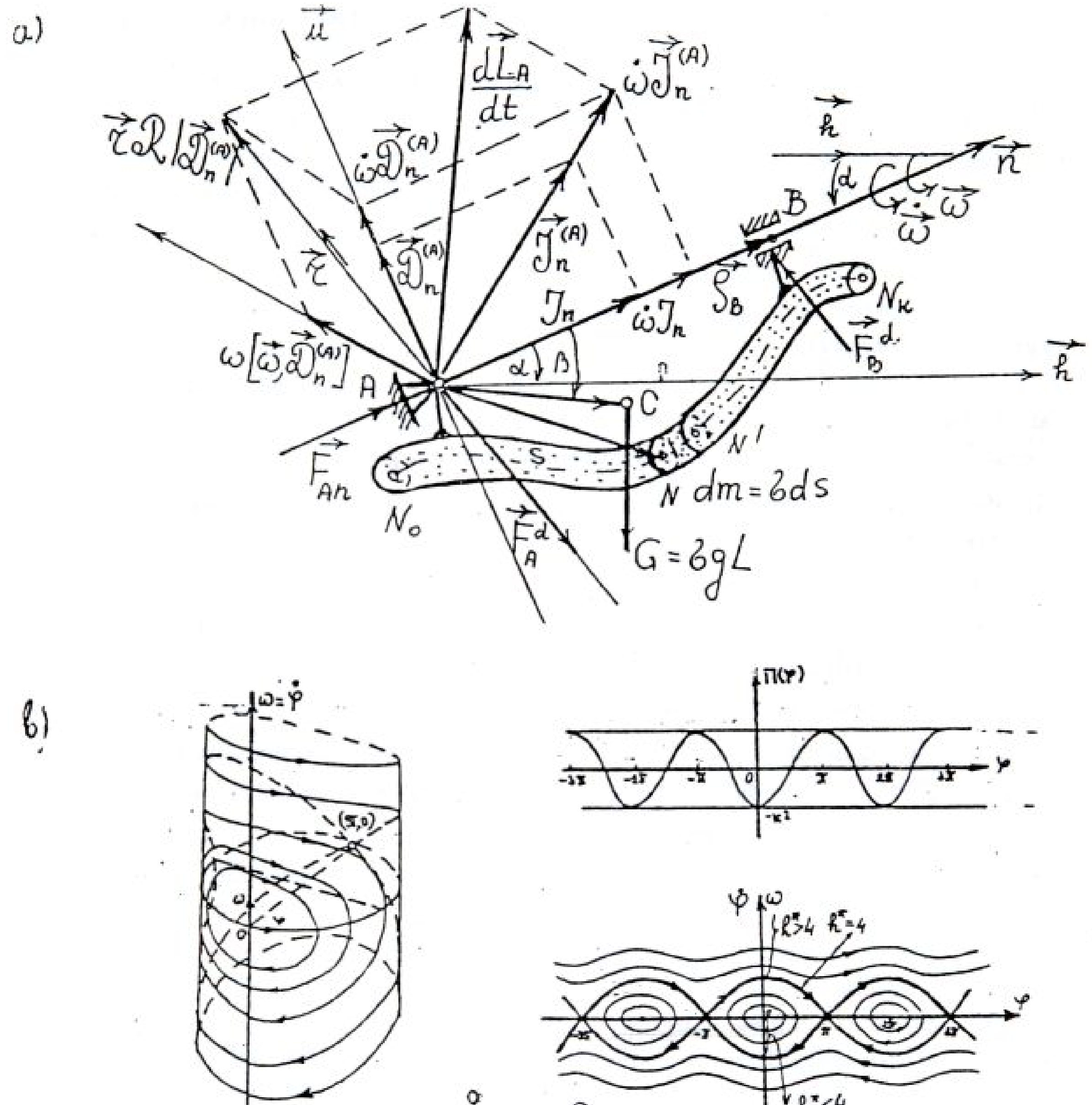
1° depending on the value of the integrating constant – system energy parameter, there are three types of phase trajectories:

- a° closed phase trajectories of ellipsoidal shape appear for $h < 4\,g/l_r$;
- b° open – progressive trajectories for $h > g/l_r$ at which the velocity oscillates within certain limits but which are always of the same sign while the elongation is constantly increasing; and
- c° closed phase trajectories for $h = 4\,g/l_r$ of the separatrice – separating phase trajectories as the boundary curves between the two previous types of the integral curves;

2° There are two types of the singular points; they are: the stable center for $\omega = 0$, $\varphi = 2n\pi$ and unstable saddles for $\omega = 0$ and for $\varphi = (2n+1)\,\pi$:

- a° closed phase trajectories for $h < 4g/l_r$ comprise the concentrically singular points $\omega = 0$, $\varphi = 2n\pi$, $n = \pm1, \pm2, \pm3, \ldots$ which are the centers. These singular points of the saddle type correspond to the stable equilibrium positions of the rigid body on the rotation axis;
- b° separatrices – separating curves for $h = 4\,g/l_r$ comprise the centers and all the closed phase trajectories and pass through the singular points of the sadlle type $\omega = 0$, $\varphi = (2n+1)\pi$, $n = \pm1, \pm2, \pm3, \ldots$. These singular points of the sadlle type correspond to the unstable equilibrium positions of the rigid body on the rotation axis;
- c° phase trajectories for $h = 4g/l_r$, while passing through the stable equilibrium positions $\varphi = 2n\pi$ have the maximal values of the angular

velocities ω, that is, the body has the maximal kinetic energy and the minimal potential energy. In this type of the motion, while passing through the unstable equilibrium positions $\varphi = (2n+1)\pi$ which the singularities of the saddle type correspond to, the angular velocities are minimal, that is, the body kinetic energy is minimal whereas the system potential energy is maximal;



$$h^* = \frac{J_n^{(A)}}{M g \rho_c \sin\beta \cos\alpha} h$$

Figure N° 3

d° closed phase trajectories correspond to the rigid body oscillations around the stable equilibrium position and for small values of the total system energy the oscillations are small whereas for greater values of the initial angles and for the angular velocities at which it still is that $h < 4g/l_r$, the oscillations are of great amplitudes $|\varphi_{\max}| < \pi$.

For small values $h \leq g/3l_r$ in the equation (9*) we can carry out linearization and write the oscillation law in the form:

$$\varphi(t) = \varphi_o \cos t\sqrt{\frac{g}{l_r}} + \mathring{\varphi}_o\sqrt{\frac{l_r}{g}} \sin t\sqrt{\frac{g}{l_r}} \tag{21}$$

then we speak of a physical pendulum which oscillates around the axis which is at an angle α with respect to the horizon. In this case we speak of isochronous oscillations with a constant oscillation period which does not depend upon the initial conditions.

When the constant is $g/3l_r < h < 4g/l_r$ we cannot perform the linearization and then we speak about non linear oscillations with which the oscillation period depends on the initial conditions and the oscillation is not isochronous. The oscillation period will be determined in the next section.

The rigid body motion in the phase plane is shown by the representational point movement along the phase trajectory. Since in the given case the phase trajectories are at the same time the constant energy curves in the phase plane this means that the body motion in the Earth gravitational field is performed at the system constant energy and the identical energy the body has received or it had at the initial moment of the motion observation.

By considering the phase trajectories according to the type the following qeustions are asced: Are the separating curves (separatrices) continuous transitions from the oscillatory to the progressive period motions? How is it possible that, for the local zone of the initial conditions about $h = 4g/l_r$ there is qualitatively and in time a jump in time for which one oscillation is performed as well as one revolution in the progressive motion while an infinitely long time is needed on the separatrice - on the separating curve?

Which properties of the system sensitivity to the initial conditions are needed to obtain qualitatively different motions for the initial conditions selection from small surroundings of the denoted values of the parameter h.

The rigid body periodic motion for the initial values $h > 4g/l_r$ is progressive but it is not oscillatory.

If we consider the representational point movement along the separating curve which will get to the point of the saddle type after an infinitely long period of time a question can be asced about the way it is going to behave after getting to the point of the saddle type which is unstable. The conclusion is obviously leading to a possible stochasticity due to the movement continuation since there are two types of further motion, which could not be controled by the given initial value of the initial conditions. The separating curve contains in itself two types of the phase trajectories, a series of the closed phase trajectories as well as two progressive trajectories of the periodic trajectory. These trajectories comprise singular points of the center type and they pass through the singular point of the saddle type. An infinite secvence of equilibrium state triggers can be noticed on them – that is, singular points: two unstable ones and one stable between them or two stable ones and one unstable between them. The appearence of such singularity triggers points out to a possibility of the appearence of the body stochastic behaviour under the determined conditions. However this does not happen in finite time.

The value of the energy parameter $h = 4g/l_r$ can be considered a bifurcational value where the movement separation takes place, the stochastic one after an infinite passage of time. The period of the rigid body motion around the stationary axis esentially changes when the constant $h \to 4g/l_r$ and the oscillation amplitude is approaching π.

Time periods of the body rotation around the stationary axis

The time intervals in which the body rotation around the stationary axis takes place is determined from the equation (18) solved by time t:

$$t = \frac{1}{\sqrt{h}} \int_0^{\varphi} \frac{d\varphi}{\sqrt{1 - \left(4\frac{g}{l_r h}\right)\sin^2\frac{\varphi}{2}}} \tag{22}$$

and if we introduced the notation:

$$\varepsilon^2 = 4\frac{g}{l_r h} \tag{22*}$$

and the integral is defined in the zone of real numbers for $\varepsilon < 1$. This is always satisfied for $h > 4g/l_r$. But for $h < 4g/l_r$ the subroot function is positive only if the elongations are in the following intervals:

$$\varphi \in \left[2n\pi - 2\arcsin\frac{1}{2}\sqrt{\frac{l_r h}{g}},\ 2n\pi + 2\arcsin\frac{1}{2}\sqrt{\frac{l_r h}{g}}\right] \tag{23}$$

When the subroot expression is always positive so that angle φ constantly increases and there is no angle φ for which the angular velocity is equal to zero so that the rigid body motion is progressively periodic but it is not oscillatory.

For the case when $\varepsilon < 1$, that is, when $h = 4g/l_r$ the integral (23) gets the form:

$$t = \frac{1}{\sqrt{h}} \int_0^{\varphi} \frac{d\varphi}{\cos\frac{\varphi}{2}} = \sqrt{\frac{l_r}{g}} \ln \operatorname{tg} \frac{\varphi + \pi}{4} \tag{22**}$$

For $\varphi \to \pi$ it follows that $t \to \infty$. Hence we conclude that it is a asymptotic approximation of the representational point along the separating curve to the singularity of the saddle type. In the Earth gravitational field the rigid body approaches asymptotically to the unstable equilibrium position and it reaches it in an infinite period of time.

In the case that $\varepsilon > 1$ the zone of the angle φ change is determined by the interval (23), while the integral (22) represents a complete elliptical integral of the first kind. Euler has given its solution by developing it into order so that we can write for the oscillation period:

$$T = 4\frac{\pi}{\sqrt{h}}\left\{1 + \sum_{k=1}^{k=\infty}\left[\frac{(2k-1)!!}{(2k)!!}\right]^2 \varepsilon^{2k}\right\} \tag{24}$$

The progressive motion condition is that $h > 4\,g/l_r$, that is, the kinetic energy at the initial moment has to have a greater value in the stable equilibrium position, that is, in the highest position of the rigid body masses center on the rotation axis.

For each value of the constant $h < 4\,g/l_r$ we have the initial energy levels - the kinetic and the potential one one as well as during the system motion which the closed phase trajectories correspond to the motion is periodic. These phase curves are closed within the separating trajectory (homoclinic trajectory).

The separating trajectories do not give any information about the motion of the representational point after it has got to the singular point of the saddle type (homocinic type). If the time parameter is excluded in which the motion is performed evrything points to the appearence of stochacity (stochasticaly like) since it cannot be foreseen whether the motion is going to remain oscillatory and thus it is going to repeat an infinite number of times or if it is going to be progresive periodic one or even a combination of these two in many variants. But due to the infinity of time, this is not possible.

A question of the phenomenon of the observed motion properties is present if an infinite period of time is necessary for the representational point motion along the separating (homoclinic) curve - separatrice to get to the unstable saddle - homoclinic point, for the energy communicated to the body which $h = 4\,g/l_r$ corresponds to and when, on the other hand, for slightly different initial conditions - initial energy we obtain the oscillatory motion of the rigid body or its progresive periodical rotation around the rotation axis under the effect of the Earth gravitational field for a definite time intervale?

Extreme value of the bifurcational parameter

The extreme value of the total system energy at which asymptotic motion of the rigid body appears in its rotation around the stationary axis, that is, at which the representational point moves along the separating phase trajectory is the one which corresponds to the extreme value of the constant h. The constant extreme value is maximal, that is, $h_{\max} = 4\,g/l_{r\min} = 4\Omega^2_{\max}$ and when l_r is minimal. Since the expression (17) is known then by its differentiation with respect to ($\varrho_C \sin\beta$) and by its identification with zero we obtain the extremum condition:

$$\varrho_C \sin\beta = i_n^{(C)} = \sqrt{\frac{J_n^{(C)}}{M}} \tag{25}$$

so that the minimal value of the reduced length $l_{r(\min)}$ and the maximal value of the frequent parameter $\Omega^2_{\max}$

$$l_{r(\min)} = 2\frac{i_n^{(C)}}{\cos\alpha}; \qquad \Omega^2_{\max} = \frac{g\,\cos\alpha}{2i^{(C)}} \tag{25*}$$

In this case since the bifurcational (homoclinic) value of the energy parameter h:

$$h_{\text{bif}} = 4\frac{g}{l_r} = 4\Omega^2 = 4\frac{M\,g\,\varrho_C \sin\beta\cos\alpha}{J_n^{(A)}} \tag{26}$$

its extreme value is:

$$h_{\text{bif}(\max)} = 2\frac{g\,\cos\alpha}{i_n^{(C)}} \tag{26*}$$

The maximal bifurcational value of the total system energy as well of the initial energy communicated to the rigid body at the beginning of the motion is:

$$\mathbb{E}_{o\text{bif}(\max)} = \frac{1}{2}J_n^{(C)}\overset{\circ}{\varphi}{}^2_{o\text{bif}} + M\,g\,i_n^{(C)}\cos\alpha\,(1-\cos\varphi_{o\text{bif}}) = M\,g\,i_n^{(C)}\cos\alpha \tag{27}$$

The maximal values of the initial energy at which asymptotic motion appears is when the masses center is distant from the rotation axis for $i_n^{(C)} = \varrho_C \sin\beta$.

Rotator

In the expressions for the kinetic pressures there is a vector $\vec{\mathfrak{R}}$ that we have named rotator and whose intensity square is in the form of the generalized coordinate function φ:

$$\mathfrak{R}(\varphi)^2 = \overset{\circ}{\omega}{}^2 + \omega^4 = \left(\frac{g}{l_r}\sin\varphi\right)^2 + \left[h + 2\frac{g}{l_r}(\cos\varphi - 1)\right]^2 \tag{28}$$

that is,

$$\mathfrak{R}(\varphi)^2 = \left(\frac{g}{l_r}\right)^2\left[\sin^2\varphi + 4(\cos\varphi-1)^2\right] + h\left[h + 4\frac{g}{l_r}(\cos\varphi - 1)\right] \tag{29}$$

For different value of the elongation the rotator has the following values:

$$\mathfrak{R}(0)^2 = h^2 = \left[\overset{\circ}{\varphi}{}^2_o + 2\frac{g}{l_r}(1-\cos\varphi_o)\right]^2 \tag{30}$$

$$\mathfrak{R}(\pi/2)^2 = \left[h + 4\frac{g}{l_r}\right]^2, \quad \mathfrak{R}(\pi)^2 = h^2 - 4h\frac{g}{l_r} + 5\left(\frac{g}{l_r}\right)^2 \tag{30*}$$

At the bifurcational value of the energy parameter:

$$h = 4\frac{g}{l_r} = 4\Omega^2 = 4\frac{M\, g\, \varrho_C \sin\beta \cos\alpha}{J_n^{(A)}}$$

the rotator has the following form:

$$\mathfrak{R}(\varphi)^2 = \left(\frac{g}{l_r}\right)^2 \left[\sin^2\varphi + 4\left(\cos\varphi + 1\right)^2\right] \tag{31}$$

and for different values of the elongation the rotator at the bifurcational values has the values

$$\mathfrak{R}(0)^2 = 16\left(\frac{g}{l_r}\right)^2 \quad \mathfrak{R}(\pi/2)^2 = 5\left(\frac{g}{l_r}\right)^2 \quad \mathfrak{R}(\pi)^2 = 0 \tag{32}$$

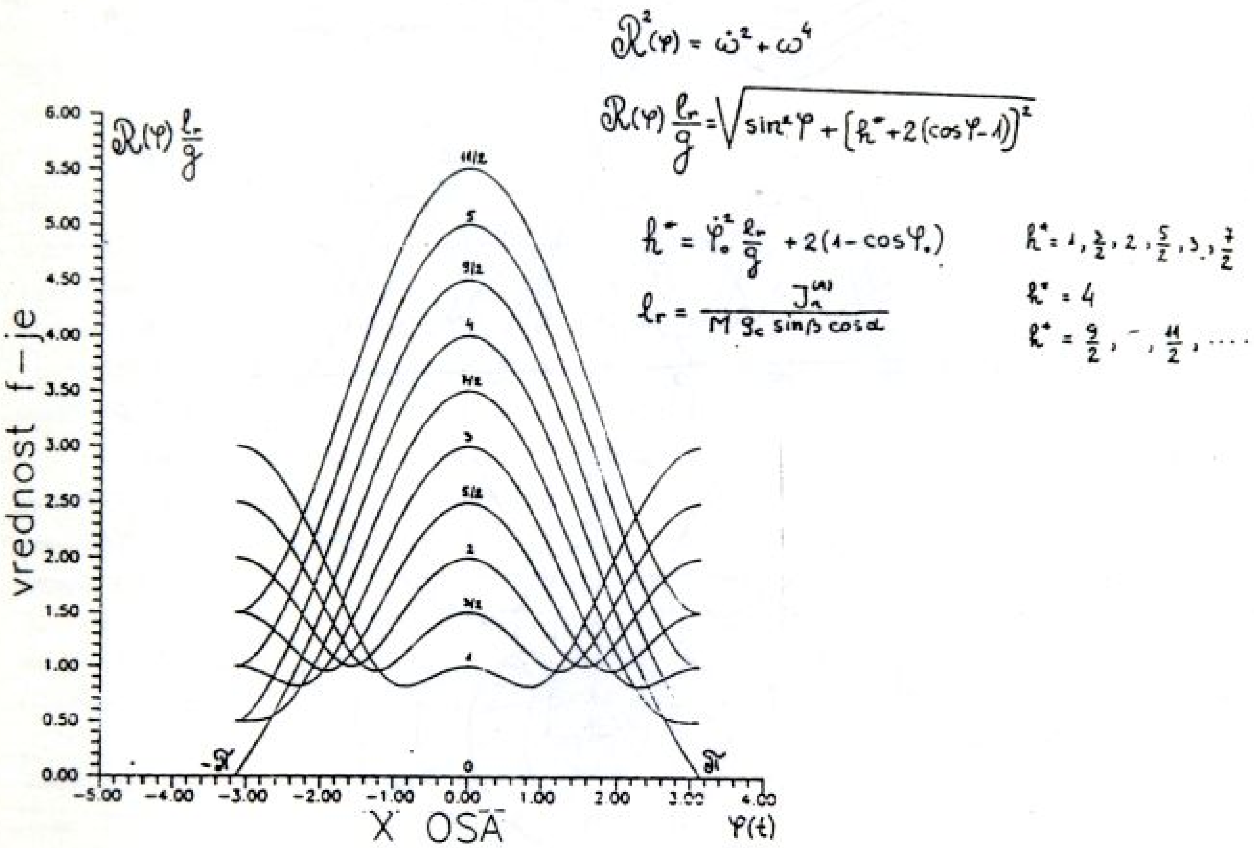


Figure N° 4

Since in the case when the energy parameter $h < 4g/l_r$ the motion is oscillatory and defined for the values of the generalized coordinate φ in the zone defined by the interval (23) it stands for that the rotator is defined this interval.

The *Fig.* N° 4 shows dependence of the rotator intensity as the function of the elongation φ and for different values of the initial parameter h of the energy, that is, of the initial system energy for which the families of the curve rotators are obtained. The next *Figure* N° 5 shows dependence $\mathfrak{R}(\varphi)$ in the polar coordinate system so that $\mathfrak{R}$ is shown as a versor.

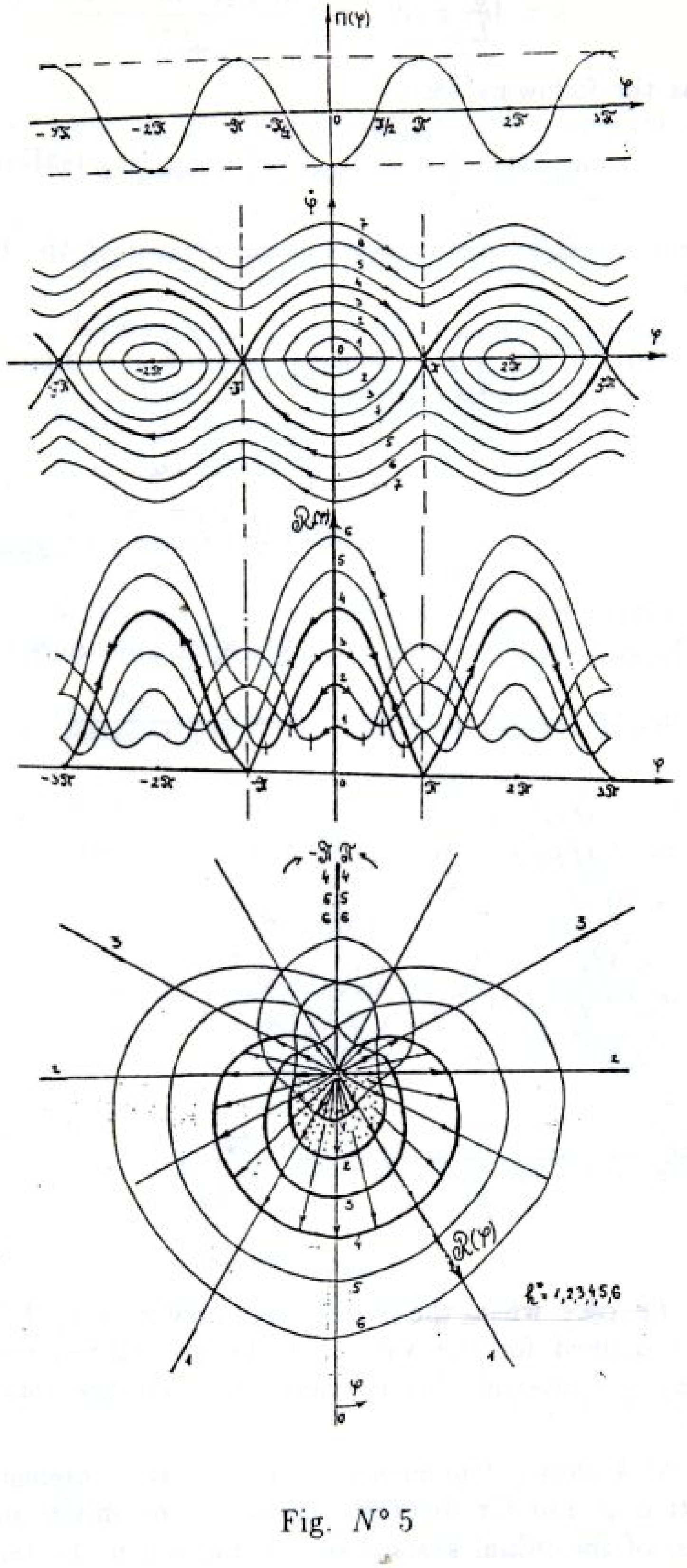


Fig. N° 5

Concluding remarks

The rotator is equal to zero only in the case when $h = 4g/l_r$ whereas in all other cases of the dynamically unbalanced rigid body rotation around a stationary axis is different from zero so that the dynamic pressures on the bearings are different from zero. The smallest values of the rotator are in the rigid body positions on the shaft corresponding to the position of the unstable static equilibrium whereas the greatest values are in the positions corresponding to the position of the stable static body equilibrium. We conclude that the dynamic pressures have extreme values in the positions corresponding to the positions of the rigid body static equilibrium on the shaft. In the position in which the potential energy is maximal the rotator has minimal value whereas in the pozitions with minimal potential energy the rotator has maximal value.

The folowing conclusion is that the smallest kinetic pressures are upon the bearings while the body is such a position that its masses center is at the highest level and vice versa it is the greatest when its masses center is the lowest.

In the rotation axis is the central inertia axis and the main inertia axis for the pole in the stationary bearing then it is a rigid body which is dynamically balanced and the members in the kinetic presures depending on the vector $\vec{\mathfrak{S}}_n^{(A)}$ and $\vec{\mathfrak{D}}_n^{(A)}$ are equal to zero and are not influenced by the rotator change. Then there are only the components of the bearing resistance arising from the bearing static resistances (pressures) in the definite position of the active forces system and the reactive forces system during the body rotation.

If the rotation axis is the axis of the inertia asymmetry for the referential point in the stationary bearing then the dynamic pressures are greatest both on moveable and stationary bearing. Since at each point on the rigid body there are three pairs of such mutually perpendicular axis which are in pair prependicular to one main inertia direction and they form with the other an angle of 45° each so that the inertia asymmetry axis which are perpendicular to the second main inertia direction forming angles of 45° each with the first and the third main inertia directions as the rotation axes will be the greatest vector of the deviation load and at the same time the greatest kinetic pressures on both the bearings. The kinetic pressure on the stationary bearing depends on the masses center position with respect to the rotation axis and this can be adjusted by the choice of the inertia asymmetry axis in pair as well as by the choice of the moveable bearing position with respect to the stationary one on the definite axis of inertia asymmetry. The inertia asymmetry axis should be avoided as the rotation axis in order to reduce the dynamic pressures upon the bearing.

For a pair of inertia asymmetry axes as the rotation axis as the rotation axis the axial moment of the masses inertia is identical so that depending on the masses center position with respect to one axis or another and on the choice of the moveable bearing an increase, that is, decrease of the kinetic pressure at a given constant value of the initial energy communicated to the rotating body.

There are four (that is, eight) axes through each point of the body which we have chosen as a stationary bearing for which the axial inertia moments are of the same value and the vectors $\vec{\mathfrak{D}}_n^{(A)}$ of the deviation load by the body mass inertia moment are proportional to the sum of the three deviation load vectors by the body mass inertia moment of the inertia asymmetry axes. For these octahedral axes the dynamic pressures on both the stationary and moveable bearings are the same while the pressures on the stationary bearing are different and by choosing one of the octahedral axes minimization or maximization of their value can be performed. By displacing the moveable bearing from one to another octahedral axis through the stationary bearing the kinetic pressure on the stationary bearing can be adjusted while retaining the share in the pressure on both the bearings of the part that corresponds to the deviation load vector although the ratator is going to change as well (but this can also be adjusted). The smallest pressures would appear an octahedral axis is chosen so the body masses center is closest to the rotation axis, that is, the most favourable of all the octahedral axes for the rotation axis is the one which the body masses center is closest to.

A general conclusion would be that if we cannot in the design method choose the main central axis of the rigid body rotation as the rotation axis when the system is dynamically balanced then we have to make an analysis of the inertia moment state at each possible point for the stationary bearing positioning and according to the design requirements we can make a choice of the stationary bearing as well as of the rotation axis according to the given analysis.

These conclusions are very important if the designer cannot change the stationary bearing but if he can change a moveable bearing and choose it freely in a rigid body; then it is surely important for him to choose it in such a way as to make the dynamic pressures as small as possible. The special rotation cases of a heavy rigid body around a stationary axis are when the axis is horizontal or vertical.

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ИНТЕРПРЕТАЦИЯ ДВИЖЕНИЯ ТЯЖЕЛОГО ТЕЛА ВРАШАЮШЕГОСЯ ОКО НЕПОДВИЖНОЙ ОСЫ И ДИНАМИЧЕСКИЕДАВЛЕНИЯ НА ОПОРАХ, С ПОМОШЮ ВЕКТОРОВМОМЕНТОВ МАССЫ ДЛЯ ОСЫ И ДЛЯ ПОЛЮСА

Для составления кинетических уравнений движения абсолютно жэсткого тела вращающегося око неподвижной осы воспользуемся интерпретациой кинетических параметров тела с помощю нововведенных векторов: $\vec{\mathfrak{S}}_n^{(A)}$ вектора линейного момента массы для осы ротации орентированой единичним вектором $\vec{n}$ и для полюса A в неподвижной опоры и вектора $\vec{\mathfrak{J}}_n^{(A)}$ момента инерции массы тела для осы ротации и для полюса A и его девиационой часты $\vec{\mathfrak{D}}_n^{(A)}$, девиационого нагружения моментом инерции массы тела для осы ротации и полюсу в неподвижной опоры.

Кинетические давления на опорах вала выражении с помощю вектора $\vec{\mathfrak{D}}_n^{(A)}$ девиационнога нагружения моментом инерции массы тела осы ротации и для полюса в неподвиженой опоры.

На основании выражениях для динамические давления введено понятие вектора $\vec{\mathfrak{R}}$ ротатора и на основании его интерпретированы кинетические давления на опорах. Представление графические илюстрации изменения модиля вектора $\vec{\mathfrak{R}}$ ротатора для часного случая свободной ротации колебания тела око неподвиженой осы.

INTERPRETACIJA KRETANJA TEŠKOG TELA OKO NEPOKRETNE OSE I DINAMIČKIH PRITISAKA NA LEŽIŠTA, POMOĆU VEKTORA MOMENATA MASA ZA POL I OSU

Kinetičke jednačine kretanja krutog tela oko nepokretne ose su interpretirane pomoću uvedenih vektora: $\vec{\mathfrak{S}}_n^{(A)}$ linearnog momenta masa za pol u nepokretnom ležištu i osu rotacije orijentisane jediničnim vektorom $\vec{n}$; $\vec{\mathfrak{J}}_n^{(A)}$ momenta inercije mase i njegovog devijacionog dela vektora $\vec{\mathfrak{D}}_n^{(A)}$ devijacionog opterećenja momentom inercije mase tela za osu rotacije i pol u nepokretnom ležištu.

Kinetički pritisci na ležišta vratila su izraženi pomoću vektora linearnog momenta mase tela i vektora $\vec{\mathfrak{D}}_n^{(A)}$ devijacionog opterećenja momentom inercije mase tela za osu rotacije i pol u nepokretnom ležištu. Na osnovu izraza za dinamičke pritiske uveden je pojam vektora $\vec{\mathfrak{R}}$ rotatora i pomoću njega isti interpretirani. Dati su grafički prikazi promene modula vektora rotatora za specijalni slučaj slobodne rotacije oko nepokretne ose.

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