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A NOTE ON THE RITZ APPROACH TO THE EXTREMAL CURVES

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1. Introduction

We consider functional that depends on function and its derivative. In the calculus of variations different classes of functions are candidates for the extremum value of the functional [1]-[3]. Usually, the following three classes of functions are in use: 1) Class $\mathbf{D}_1$, smooth functions, which are continuous function with continuous first derivatives; 2) Class $\mathbf{D}_2$, piecewise–smooth functions, which are all continuous functions but with corner points, at which the first derivative is not continuous; 3) Class, $\mathbf{D}_3$ are piecewise continuous functions. It is obvious that the following inclusion holds $\mathbf{D}_1 \subset \mathbf{D}_2 \subset \mathbf{D}_3$. If, for some problem, we have two extremals of the different classes then the extremum on the extremal from the wider class is better (smaller or larger depends on the type of extremum). If the candidates for extremal curve belong to the classes $\mathbf{D}_1$ and $\mathbf{D}_2$, then the functional is the Riemann integral, while in the case of the class $\mathbf{D}_3$ it is the Lebesque integral.

Pars, considering two examples in [1] p.47 and p.70, show that by an ad hoc selected extremal of the class $\mathbf{D}_1$ it is possible to approach, close as it is necessary, to the extremal of the class $\mathbf{D}_2$. However, the functional does not have stationary value on this ad hoc extremal curve.

Practical determination of the extremal requires solution of the Euler equation, generally nonlinear differential equation and very inconvenient for solving. Therefore, the calculus of variations suggests approximate methods for finding the extremals. One of the most convenient methods is the Ritz method, developed in 1908. The crux idea of the Ritz method is that the value of a functional calculates on a curve that is linear combination with constant coefficients of some finite number of known smooth functions of the independent variable. The functions satisfy the corresponding boundary conditions identically. As a rule, the Ritz curves approximate the extremals of the $\mathbf{D}_1$ type.

We devote the paper to approximation of the extremal curves by the Ritz method. The corresponding extremal curves can be of $\mathbf{D}_1$, $\mathbf{D}_2$ or $\mathbf{D}_3$ type.

We stress that the Ritz method may approximate extremals belonging to the different classes.

Obviously, the study is the most clear if we explain them on particular problems. In this course we elaborate three particular examples.

2. The basic theory

Let us consider extremity of the following functional

$$I = \int_{t_A}^{t_B} F(t,\, y,\, \dot{y})\, \mathrm{d}t, \tag{1}$$

which depend on a function $y(t)$, where t is independent variable, $\dot{y}$ is derivative of y concerning t, t_A and t_B are fixed initial and terminal values of t, F is an arbitrary function of t, y and $\dot{y}$. The function F is continuous and has continuous partial derivatives to second order inclusive.

Here, the permissible curves must pass through known boundary points A and B

$$y(t_A) = y_A, \quad y(t_B) = y_B, \tag{2}$$

where y_A and y_B are known constants. Also, let us suppose that the extremal belongs to the class $\mathbf{D}_2$ and has only one corner point at C (see Fig.1). If there are several corner points, then the same argument applies to each one.

It is obvious that the separate smooth arcs which make up the broken-line extremal must be integral curves of the Euler equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial F}{\partial \dot{y}} - \frac{\partial F}{\partial y} = 0. \tag{3}$$

Because the broken-line extremal has one corner point C, we can consider (1) as

$$I = \int_{t_A}^{t_C} F(t,\, y,\, \dot{y})\, \mathrm{d}t + \int_{t_C}^{t_B} F(t,\, y,\, \dot{y})\, \mathrm{d}t. \tag{4}$$

where t_C is the abscissa of the corner point (Fig.1). Taking that AC and CB are integral curves of the Euler equation passing through the boundary points A and B and that the point C can move in arbitrary fashion, we get

$$\begin{aligned}\delta I = &\left[\left(F - \dot{y}\frac{\partial F}{\partial \dot{y}}\right)_{t=t_{C-0}} - \left(F - \dot{y}\frac{\partial F}{\partial \dot{y}}\right)_{t=t_{C+0}}\right]\delta t_C + \\ &+ \left[\left(\frac{\partial F}{\partial \dot{y}}\right)_{t=t_{C-0}} - \left(\frac{\partial F}{\partial \dot{y}}\right)_{t=t_{C+0}}\right]\delta y_C.\end{aligned} \tag{5}$$

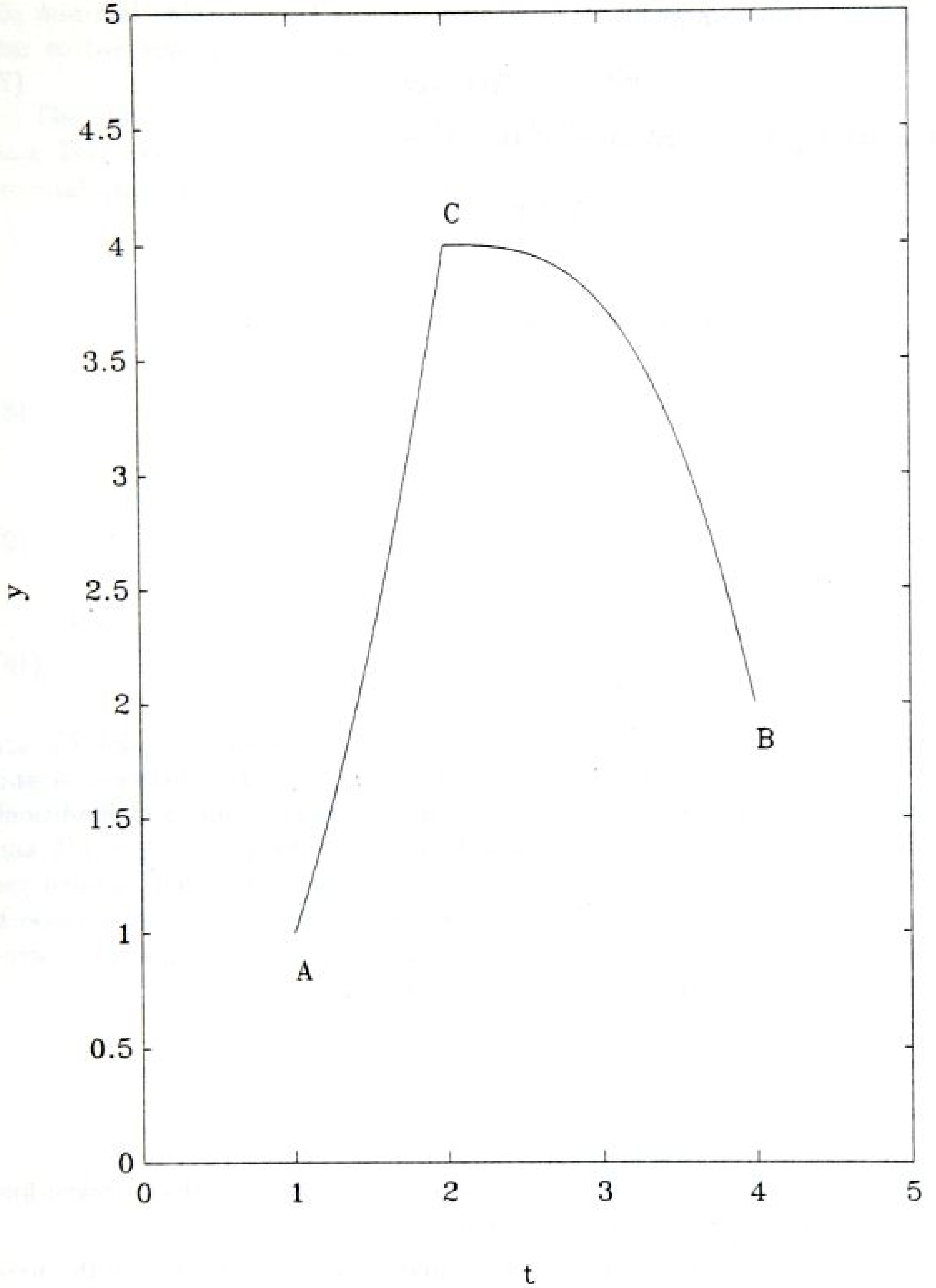

Figure 1

The necessary condition for extremity of (1) $\delta I = 0$, and since δt_C and δy_C are independent, yields

$$\left(F - \dot{y}\frac{\partial F}{\partial \dot{y}}\right)_{t=t_{C-0}} = \left(F - \dot{y}\frac{\partial F}{\partial \dot{y}}\right)_{t=t_{C+0}}, \quad \left(\frac{\partial F}{\partial \dot{y}}\right)_{t=t_{C-0}} = \left(\frac{\partial F}{\partial \dot{y}}\right)_{t=t_{C+0}}. \quad (6)$$

This conditions, together with the following continuity condition of the desired extremals at the corner point C

$$y(t_{C-0}) = y(t_{C+0}), \tag{7}$$

permits determining the coordinates of the corner point.

3. Applications

Example 1.

Let us consider minimization of the functional

$$I = \int_{-1}^{1} y^2(1-\dot{y})^2 \, \mathrm{d}t, \tag{8}$$

where

$$y(-1) = 0, \quad y(1) = 1. \tag{9}$$

In this case the Euler equation (3) is

$$y\left[1 - \frac{\mathrm{d}}{\mathrm{d}t}(y\dot{y})\right] = 0, \tag{10}$$

whose solutions are $y = 0$ or $y = (t^2 + C_1 t + C_2)^{1/2}$, where C_1 and C_2 are integration constants. Let us assume a smooth curve (class $\mathbf{D}_1$) between A and B as an extremal. Then, the first solution does not satisfy boundary conditions (9), while for the second solution the boundary conditions yields $C_1 = 1/2$ and $C_2 = -1/2$. The second smooth curve between A and B is not defined for $t \in (-1, 1/2)$. Hence, in this problem an extremum can not be on a smooth curve (class $\mathbf{D}_1$) between A and B. Therefore, consider the extremal as broken-line at the point C (class $\mathbf{D}_2$). For $t \in [-1, 0]$ the extremal is

$$y = 0,$$

while for $t \in [0, 1]$ it is

$$y = (t^2 + C_1 t + C_2)^{1/2}.$$

Satisfying (6), (7) and (9) we have $C_1 = 0$ and $C_2 = 0$. On that broken-line extremal the functional (8) has absolute minimum $I = 0$.

Let us suppose trial solution for the extremal curve of (8) as a smooth curve (class $\mathbf{D}_1$)

$$y = \frac{(1+t)^2}{4} + (a_0 u + a_1 u^2 + a_2 u^3 + a_3 u^4)u^3, \quad u = 1 - t^2, \tag{11}$$

where $a_0, \ldots a_3$ are unknown constants. The solution satisfies the boundary conditions (9) for arbitrary $a_0, \ldots a_3$. Substituting (11) into (8) and minimizing that functional concerning $a_0, \ldots a_3$ we have optimal values if these constants:

$$a_0 = -0.87883, \quad a_1 = 1.29369, \quad a_2 = -0.13584, \quad a_3 = -0.48980.$$

The minimal value of (11) is $I = 1.87831 \cdot 10^{-4}$. Obviously the minimum is very close to the absolute minimum $I = 0$ of the functional (8).

The curves on the Fig.2 are the approximate (class $\mathbf{D}_1$) and the broken-line (class $\mathbf{D}_2$) extremals. The Ritz's approximate extremal follows the broken-line extremal quite well.

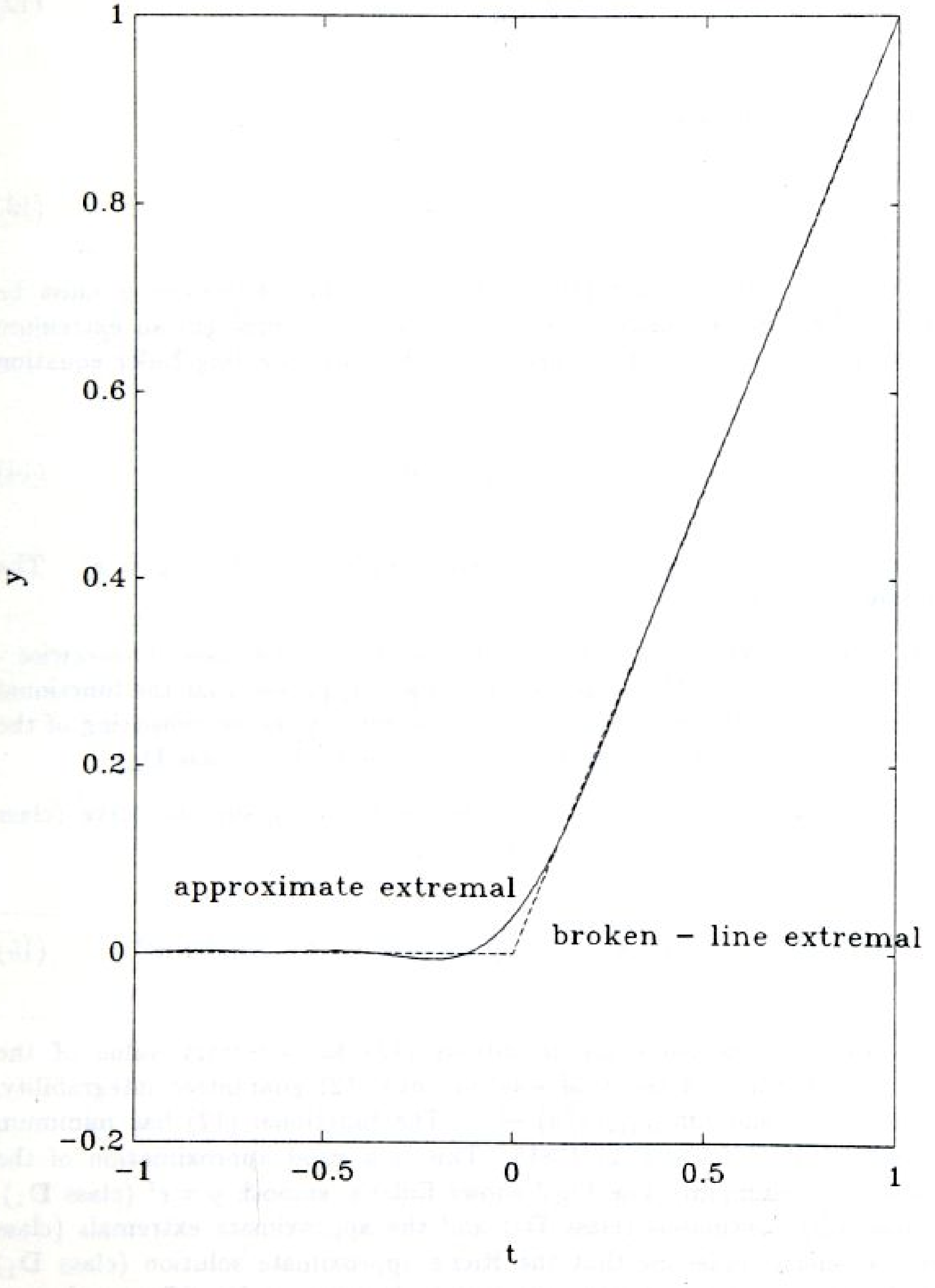

Figure 2

Example 2.

Find the minimum of the functional (see [2] p.110)

$$I = \int_0^1 \left(\frac{y}{\dot y}\right)^2 \mathrm{d}t, \tag{12}$$

for the boundary conditions

$$y(0) = 1, \quad y(1) = e. \tag{13}$$

From the relations (6), (7) and (12) it is obvious that derivative $\dot y$ must be continuous at the possible corner point C. Hence, in the problem an extremum may exist only on a smooth curve (class $\mathbf{D}_1$). The corresponding Euler equation (3) is

$$\frac{y}{\dot y^2}\left(1 - \frac{y}{\dot y^2}\ddot y\right) = 0. \tag{14}$$

The only solution of the equation (14), which satisfies (13), is $y = e^t$. The corresponding value of (12) is $I = 1$.

Let us seek the extremum of the functional (12) in the class of piecewise - continuous functions (class $\mathbf{D}_3$). Krotov (see [2] p.105) proved that the functional (12) achieves the absolute minimum, zero, on a composite curve consisting of the segment $y = 0$ and the vertical segments at $t = 0$ and $t = 1$ (class $\mathbf{D}_3$).

Let us consider trial extremal of (12) as the following smooth curve (class $\mathbf{D}_1$).

$$y = 1 + \left(e - 5 - \frac{a}{4}\right)t + \left(8 - 4e + \frac{5a}{4}\right)t^2 + (4e - 4 - 2a)t^3 + at^4. \tag{15}$$

The curve (15) satisfies boundary conditions (13) for arbitrary value of the constant a. Substitution of the trial solution into (12) guarantees integrability, because $\dot y(1/2) = 0$ and $\lim_{t\to 1/2}(y/\dot y) \to 0$. The functional (12) has minimum value $I = 5.8616 \cdot 10^{-3}$ for $a = 27.41818$. This is a good approximation of the Krotov's absolute minimum. The Fig.3 shows Euler's, smooth $y = e^t$ (class $\mathbf{D}_1$), Krotov's, piecewise-continuous (class $\mathbf{D}_3$) and the approximate extremals (class $\mathbf{D}_1$). Here, we must underline that the Ritz's approximate solution (class $\mathbf{D}_1$) of the problem can converge to the extremals belonging to the different class of curves.

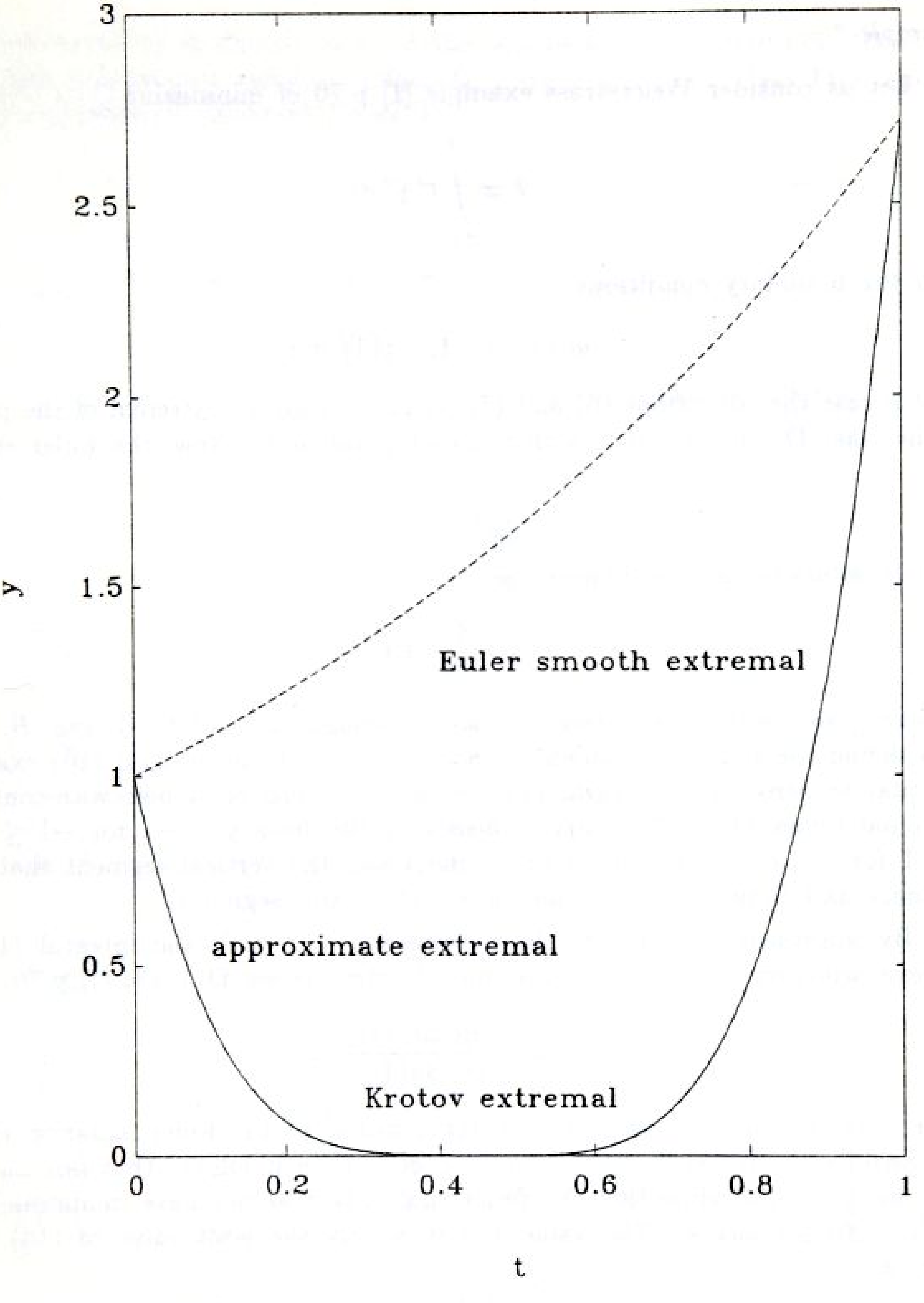
3
2.5
2
1.5
1
0.5
0
y
Euler smooth extremal
approximate extremal
Krotov extremal
0
0.2
0.4
0.6
0.8
1
t

Figure 3

Example 3.

Let us consider Weierstrass example [1] p.70 of minimizing

$$I = \int_{-1}^{1} t^2\,\dot{y}^2\,\mathrm{d}t, \tag{16}$$

with the boundary conditions

$$y(-1) = -1, \quad y(1) = 1. \tag{17}$$

In this case the conditions (6) and (7) do not permit an extremal of the problem of the class $\mathbf{D}_2$, i.e. a curve with a corner point at C. Now, the Euler equation is

$$\frac{\mathrm{d}}{\mathrm{d}t}(t^2\,\dot{y}) = 0, \tag{18}$$

and its solutions are the hyperbolas

$$y = \frac{C_1}{y} + C_2, \tag{19}$$

however, no continuous extremal passes through the points A and B, where (17) define the points. Meanwhile, the minimum of the integral (16) exist, and is equal to zero. The integral (16) is equal to zero on a piecewise-continuous extremal (class $\mathbf{D}_3$). The curve consists of the lines $y = -1$ for $-1 \leq t \leq 0$, $y = 1$ for $0 \leq t \leq 1$ ($\dot{y} = 0$ on these lines) and the vertical segment that is the ordinate axis between $y = -1$ and $y = 1$ (t on this segment).

By appropriate choice of the ε, where $0 < \varepsilon < 1$, the integral (16) can achieve arbitrary small value on a smooth curve (class $\mathbf{D}_1$), see [1] p.70,

$$y = \frac{\arctan(t/\varepsilon)}{\arctan(1/\varepsilon)}, \tag{20}$$

This is an ad hoc extremal that is not solution to the Euler equation (18). If we consider the integral (16) as function of ε, then it follows that $\lim_{\varepsilon \to 0} I \to 0$. But, we must underline that the functional (16) does not have minimum on the family (20) for any ε. The value $I = 0$ is only the least value of (16), where $\delta I(\varepsilon) \neq 0$.

Let us consider trial solution of the problem (class $\mathbf{D}_1$)

$$y = \sin(\pi t/2) + \sum_{k=0}^{k=3} a_k \sin[(k+1)\pi t], \tag{21}$$

where a_k are arbitrary constants. Minimizing (16) with respect to those constant we have their optimal values

$$a_0 = 0.3220826, \quad a_1 = 0.21820, \quad a_2 = 0.08130, \quad a_3 = 0.03373.$$

The corresponding minimum value of the functional is $I = 0.08486$. The Fig.4 shows the approximate extremal (class $\mathbf{D}_1$), the curve (20) (class $\mathbf{D}_1$), and the piecewise-continuous extremals (class $\mathbf{D}_3$).

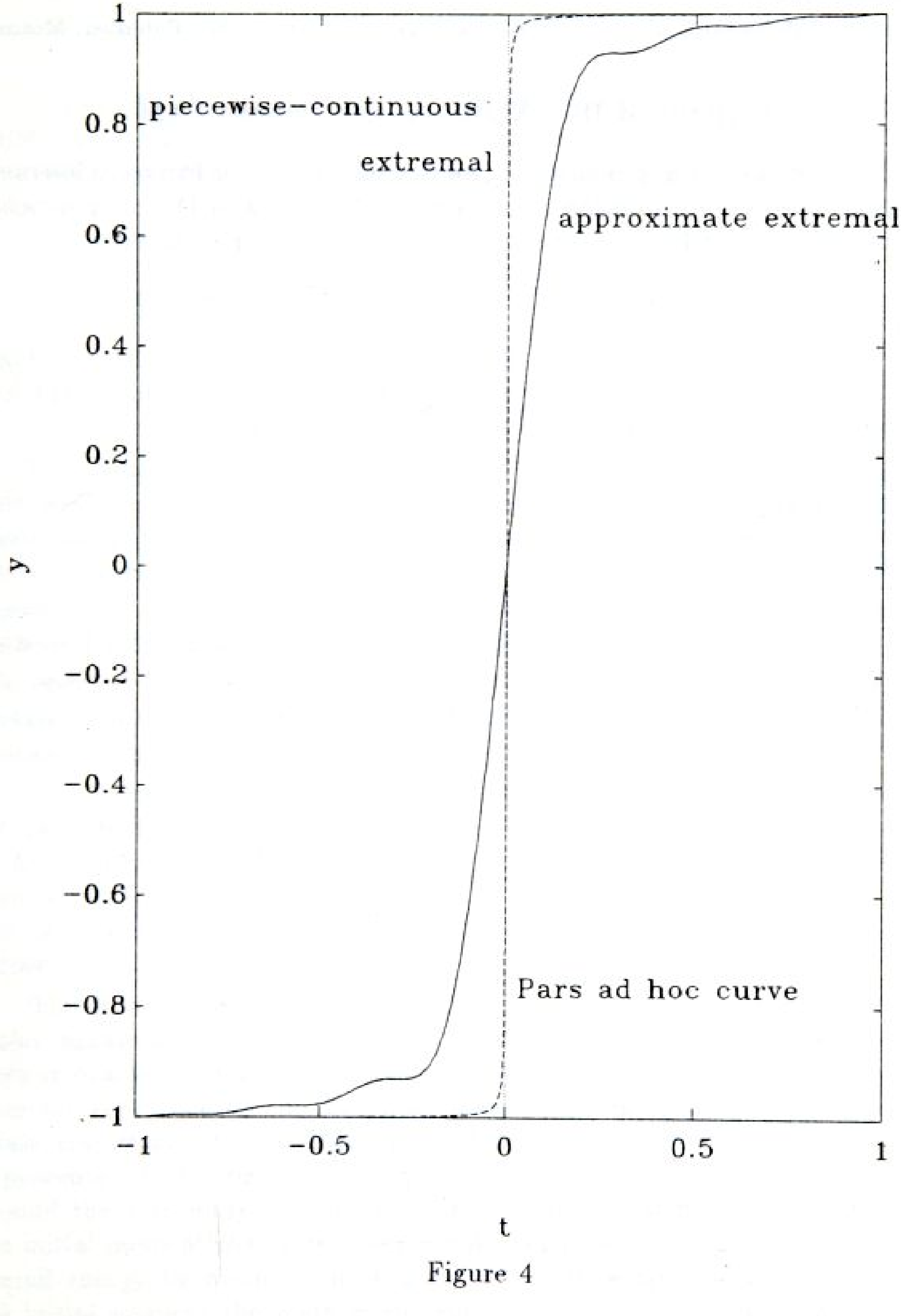

Figure 4

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О РИТЦОВОМ ПОДХОДУ К ЭКСТРЕМАЛАМ

Рассмотриваается апроксимация экстремал методой Ритца. Соотвеетвующие экстремалиы принадлежают классе $\mathbf{D}_1$, $\mathbf{D}_2$ или $\mathbf{D}_3$. Апроксимативные линии классы $\mathbf{D}_1$ использовани для решения при задачи.

O RICOVOM PRISTUPU EKSTREMALNIM KRIVIM

Rad je posvećen aproksimaciji ekstremalnih kriva metodom Rica. Odgovarajuće ekstremalne krive pripadaju klasi $\mathbf{D}_1$, $\mathbf{D}_2$ ili $\mathbf{D}_3$ krivih linija. Aproksimativne krive klase $\mathbf{D}_1$ su upotrebljene za rešavanje tri problema.

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