TEORIJSKA I PRIMENJENA MEHANIKA 20, pp 259–267, 1994. UDK 539

A NOETHER'S THEOREM IN GRANULAR MATERIALS

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(Received 19.09.1994.)

1. Introduction

Conservation laws (or balance laws) have been the subject of considerable research in recent years. One of these laws, the J–integral, has been applied extensively to the fracure mechanics problems with much success. In this paper, we examine a similar type of integrals for granular solids. Their investigations were based on a continuum theory of granular materials, developed by Cowin and Goodman [1,2,3]. Central to their theory is the concept of distributed body which leads naturally to the introduction of an independent kinematical quantity called the volume distribution function ν. This quantity represents the fact that the granules do not occupy the entire volume of material.

The importance of fracture mechanics for practice is well-known, as well as the importance of the J integral concept in fracture mechanics, J integral derivation, based on Noether's theorem, for linearized and finite classical elastostatics, was given by Gunther [4], and independently, by Knowles and Sternberg [5]. The latter work was extended to linear elastodynamics by Fletcher [6].

In this paper the intention is to derive conservation laws (or balance laws) using the invariant characteristic of the variational principle in relation to the Euclidean group of transformation. Using the Euclidean group of transformation, the equivalence between the conservation law and the Euclidean invariance is demonstrated. As a consequence a nowel result for the conservation law (or balance laws) for granular materials has been obtained. Finaly, one of the laws is an example used to ilustrate its application.

2. The symmetrical theorem

In this section, we give Noether's symmetrical theorem of granual continuum.

Let $\xi = (\xi_a) \in R$, $a = 1, 2, \ldots, n$, be the independent and $u = (u_i) \in R$, $i = 1, 2, \ldots, m$, v and $w \in R$ be arbitrary vector and scalar variables, which describe the behaviour of the material system under consideration.

We suppose that these fields are twice continuously differentiable in R.

We consider a continuous transformation group G with one single parameter

$$G: \quad \begin{aligned} \overline{\xi}_a &= \xi_a + \alpha_a \eta + 0(\eta^2) \\ \overline{\psi} &= \psi + b\,\eta + 0(\eta^2) \\ \psi = (u_k, v, w), &\quad b = (\beta_k, \gamma, \delta) \end{aligned} \tag{2.1}$$

where η is a paremeter, $a = 1,2,3,4$; $k = 1,2,3$ and the quantities

$$\alpha_a = \left.\frac{\mathrm{d}\overline{\xi}_a}{\mathrm{d}\eta}\right|_{\eta=0}, \quad b = \left.\frac{\mathrm{d}\overline{\psi}}{\mathrm{d}\eta}\right|_{\eta=0} \tag{2.2}$$

It is obvious that we can obtain some kind of transformation group G by assigning α_a, β_k, γ, and δ some special values. Further, suppose that

$$L = L(Y) \tag{2.3}$$

is a real function of Y is defined and differentiable for all values of its arguments:

$$Y = Y(\xi_a,\, u_k,\, u_{k,a},\, v,\, v_{,a},\, w,\, w_{,a}) \tag{2.4}$$

Now, we define an action functional A for given field $(u_k,\, v,\, w)$ by the formula

$$A(\psi) = \int_T \int_B L\, \mathrm{d}V\, \mathrm{d}t = \int_R L(Y)\, \mathrm{d}\xi \tag{2.5}$$

The functional A in (2.5) is said to be invariant at $(u_k,\, v,\, w)$ under the transformation (2.1) if

$$\int_{\overline{R}} L(\overline{Y})\, \mathrm{d}\overline{\xi} = \int_R L(Y)\, \mathrm{d}\xi \tag{2.6}$$

for all sufficiently small values of $|\eta|$. If for a given $(u_k,\, v,\, w)$

$$\left.\left\{ \frac{\mathrm{d}}{\mathrm{d}\eta} \int_{\overline{R}} L(\overline{Y})\, \mathrm{d}\overline{\xi} \right\}\right|_{\eta=0} = 0 \tag{2.7}$$

then A is said to be infinetisemally invariant at $(u_k,\, v,\, w)$.

Now we can state a restricted version of Noether's theorem.

If the action functional A has infinitesimal invariance under the group G there exists

$$B_{a,a} - \{Q,\, p\} + \{p,\, E(L)_\psi\} = 0 \tag{2.8}$$

where

$$B_a = \left\{ \frac{\partial L}{\partial \psi_{,a}},\, p \right\} \quad E(L)_\psi = \frac{\partial}{\partial \xi_a} \frac{\partial L}{\partial \psi_{,a}} - \frac{\partial L}{\partial \psi} - Q = 0, \quad Q = (F_k,\, G,\, R) \tag{2.9}$$

and

$$p = b - \psi_{,a}\alpha_a, \quad p = (p_k, q, r) \tag{2.10}$$

It was conventient to use abbreviated notation sugested by Ericksen,

$$\{(a,b,c)(a,b,c)\} = aa + bb + cc.$$

Noether's symmetrical theorem plays an important role in modern field theory. From it we can obtain the field equations, the conservation laws and the dynamical criterion of a singularity motion for the material system under consideration.

The proof of this theorem can be found in [7].

3. Thermodynamic theory of granular solids

We follow the approach of Godman and Cowin [1] and assign to the solid the mathematical structure of a distributed body. The motion of such body is described by the functions:

$$x_k = x_k(X_k, t) \tag{3.1}$$

An important consequence of the motion of a distributed body is the fact that at any point $x(X_k)$ and time t the density ϱ can be decomposed as

$$\varrho = \nu\gamma \tag{3.2}$$

where $\gamma = \gamma(X_k, t)$ is the density of granules, and $\nu = \nu(X_k, t)$ $(0 < \nu < 1)$ is called the volume distribution function. This distribution function represents the ratio of the volume of granules dV_g, to the volume of the material dV i.e.

$$dV_g = \nu dV \tag{3.3}$$

The local balance laws in the reference frame X are listed below for thermodynamic granular materials [2]:

The balance of linear momentum

$$T_{Kk,K} + \varrho_0 f_k = \varrho_0 \dot{v}_k \tag{3.4}$$

The balance of moment of momentum

$$T_{Kk}\, x_{l,K} = T_{Kl}\, x_{k,K} \tag{3.5}$$

The balance of energy

$$\varrho_0 \dot{\varepsilon} = T_{Kk}\, \dot{v}_{k,K} + Q_{K,K} + H_K \nu_{,K} - \varrho_0 g \dot{\nu} + \varrho_0 h \tag{3.6}$$

The balance of equilibrated force

$$\varrho_0 k \ddot{\nu} = \varrho_0 (l + g) + H_{K,K} \tag{3.7}$$

where the above given quantities are

T_{Kk} – the first Piola-Kirchhof stress tensor,

Q_K – the heat flux,

ϱ_0 – the initial mass density,

ε – the internal energy density,

$\frac{\mathrm{d}}{\mathrm{d}t} = \overline{(\ )}$ the material derivative,

H_K – the equilibrated stres vector,

l – the external equilibrated,

h – the heat supply per unit mass,

g – the intrinsic equilibrated body force,

η – the entropy density.

Finaly we write the constitutive equations

$$T_{Kk} = \varrho_0 \frac{\partial \psi}{\partial x_{k,K}} = \varrho_0 \frac{\partial \varepsilon}{\partial x_{k,K}}, \quad \eta = -\frac{\partial \psi}{\partial \theta}, \quad g = -\frac{\partial \psi}{\partial \nu}, \quad H_K = \varrho_0 \frac{\partial \psi}{\partial \nu_{,K}} \tag{3.8}$$

where θ is the absolute temperature and

$$\psi = \psi(x_{k,K},\, \nu,\, \nu_{,K},\, \theta) = \varepsilon - \theta\eta \tag{3.9}$$

the free energy.

From (3.8) and (3.6) we obtain

$$\varrho_0 \theta \dot{\eta} = Q_{K,K} + \varrho_0\, h. \tag{3.10}$$

In order to write the differential form of the balance law (3.4), (3.7) in more compact form it is convenient to introduce some additional notation. We set

$$\xi_a = \begin{cases} X_K & a = K \\ t & a = 4 \end{cases} \quad \frac{\mathrm{d}}{\mathrm{d}t}(\ ) = (\ )_{,4} \tag{3.11}$$

and

$$L = \varrho_0(\psi - \tfrac{1}{2} v_k\, v_k - \tfrac{1}{2} k \dot{\nu} \dot{\nu}) = L(x_{k,a}; \nu; \nu_{,a}; \theta) \tag{3.12}$$

It may be verified that

$$\begin{gathered} L_{,x_{k,K}} = T_{Kk}, \quad L_{,\dot{x}_k} = -\varrho_0 \dot{x}_k, \quad L_{,x_k} = 0 \\ L_{,\nu_{,K}} = H_K, \quad L_{,\dot{\nu}} = -\varrho_0\, k\dot{\nu}, \quad L_{,\nu} = \varrho_0\, g \end{gathered} \tag{3.13}$$

holds, and the laws (3.4 and 3.7) are

$$\frac{\partial}{\partial \xi_a} L_{,x_{k,a}} + \varrho_0\, f_k = 0, \quad \frac{\partial}{\partial \xi_a} L_{,\nu_{,a}} + \varrho_0\,(g + l) = 0 \tag{3.14}$$

Then, Noether's theorem can be applied to our case. To confirm this we chose

$$\begin{gathered} L = L, \quad u_k = x_k, \quad v = \nu, \quad w = \theta, \quad F_k = \varrho_0\, f_k, \quad G = \varrho_0\, l, \quad R = 0, \\ X = (\xi,\, x,\, \nu,\, \theta), \quad \psi = (x_k,\, \nu,\, \theta), \quad Y = (x_{k,a},\, \nu,\, \nu_{,a},\, \theta), \\ p_k = \beta_k - x_{k,a}\, \alpha_a, \quad q = \gamma - \nu_{,a}, \alpha_a, \quad r = \delta - \theta_{,a}\, \alpha_a. \end{gathered} \tag{3.15}$$

Then, from relations (2.8), together with (3.15) and the divergence theorem, it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V (L_{,\dot{x}_k}\, p_k + L_{,\dot{\nu}}\, q + L\alpha_4)\,\mathrm{d}v + \int_S (L_{,x_{k,K}}\, p_k + L_{,\nu,K}\, q + L\alpha_K)\, N_K\,\mathrm{d}s - \\ - \int_V L_{,\theta}\, r\,\mathrm{d}v + \int_V \varrho_0 (f_k\, p_k + l\, q)\,\mathrm{d}v = 0 \tag{3.16}$$

4. Invariance and conservation

Following R. A. Toupin [8], we postulate that the eqivalence of Euclidean invariance of the action density and certain conservation laws.

The action density L is invariant under the group of Euclidean displacements if

$$L(\xi_a, x_k, \nu, \theta) = L(\bar{\xi}_a, \bar{x}_k, \bar{\nu}, \bar{\theta})$$

where storred and corresponding undstorred quantities are related by

$$\begin{aligned} \bar{\xi}_a &= \xi_a + C_a\,\eta \\ \bar{x}_k &= x_k + (R^j_k\, x_j + D_k)\eta \\ \bar{\phi} &= \phi + E\,\eta \quad E = (E_0,\, E_1) \end{aligned} \tag{4.1}$$

and where R^j_k is a constant antisymmetric tensor, $R^j_k = -R^k_j$, and C_a, D_k and E are arbitraru constants.

By taking all of the arbitrary constants in (4.1) to be equal to zero except the one in turn, we obtain the corresponding conservation laws. They are the following three transformations under which the functional is infinitesimally invariant:

$$(I) \quad D_k \neq 0, \qquad \beta_k = D_k, \quad p_k = D_k, \quad q = 0, \quad r = 0$$

This transformation represents rigid body translations. The corresponding conservation law (3.16) now reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \varrho_0\, v_k\,\mathrm{d}v - \int_S T_{Kk}\, N_K\,\mathrm{d}s - \int_V \varrho_0\, f_k\,\mathrm{d}v = 0 \tag{4.2}$$

$$(II) \quad R^j_k \neq 0, \qquad \beta_k = 0, \quad p_k = R^j_k\, x_j, \quad q = 0, r = 0.$$

This transformation represents rigid body rotation, and the corresponding conservation law reads

$$R^j_k \left\{ \frac{\mathrm{d}}{\mathrm{d}t}\int_V \varrho_0\, v_k\, x_j\,\mathrm{d}v - \int_S T_{Kk}\, x_j\, N_K\,\mathrm{d}s - \int_V \varrho_0\, x_j\, f_k\,\mathrm{d}v \right\} = 0 \tag{4.3}$$

$$(III) \quad \alpha_4 = C_0 \neq 0, \qquad p_k = -v_k\, C_0, \quad q = -\dot{\nu} C_0, \quad r = -\dot{\theta} C_0$$

This transformation represents a shift of time, and the corresponding law read

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V E\,\mathrm{d}v - \int_S [T_{Kk}\,v_k + H_K\,\dot\nu]\,N_K\,\mathrm{d}s - \int_V \varrho_0(f_k\,v_k + l\,\dot\nu)\,\mathrm{d}v = 0 \qquad (4.4_1)$$

where

$$E = W + \tfrac{1}{2}\varrho_0\,v_k\,v_k + \tfrac{1}{2}\varrho_0\,k\,\dot\nu\dot\nu = W + K$$

Upon using (3.8-3.12) this reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V E\,\mathrm{d}v - \int_S [T_{Kk}\,v_K + H_k\,\dot\nu + Q_k]\,N_k\,\mathrm{d}s - \int_V \varrho_0(f_k\,v_k + l\,\dot\nu + h)\,\mathrm{d}v = 0 \quad (4.4_2)$$

The above conservation laws (4.2)–(4.4) represent the conservation of linear momentum, angular momentum, and energy, respectively. Thus, we have established the basic theorem of equivalence between conservation and invariance [8]:

Linear momentum, angular momentum, and energy are conserved in a granular medium if the action A is invariant under the group of Euclidean displacement.

Now we consider the case when

$$(IV)\quad \alpha_K = C_K \neq 0, \qquad p_k = -x_{k,K}\,C_K, \quad q = -\nu_{,K}\,C_K, \quad r = -\theta_{,K}\,C_K$$

This transformation represents the family of coordinate translations, and the corresponding conservation law reads

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_V (\varrho\,v_k\,x_{k,K}\,\varrho\,k\,\dot\nu\,\nu_{,K})\,\mathrm{d}v - \\ &\int_S \left[(W-K)\delta_{KL} - T_{Lk}\,x_{k,K} - H_L\,\nu^1_{,K}\right]\,N_L\,\mathrm{d}s - \qquad (4.5)\\ &-\int_V L_{,\theta}\,\theta_{,K}\,\mathrm{d}v - \int_V \varrho\,(f_k\,x_{k,K} + l\,\nu_{,K})\,\mathrm{d}v = 0\end{aligned}$$

The last case follows from

$$(V)\quad \gamma = E_0 \neq 0, \qquad p_k = 0, \quad q = 0, \quad r = E_0$$

This transformation represent a family of scale change, and corresponding law reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \varrho\,k\,\dot\nu\,\mathrm{d}v - \int_S H_K\,N_K\,\mathrm{d}s - \int_V \varrho\,l\,\mathrm{d}v = 0 \qquad (4.6)$$

which denote the balance of equilibrated force.

In the local form the integral (4.2) and (4.6) give the balance equations (3.4) and (3.7).

DISCUSION

Classical granular materials

In the absence of heat conditions the Lagrange function does not depend on θ, so that $L_{,\theta}$, vanishes and expression (4.5) reduces to the two-D version for the J integral for granular materials.

Material without voids

In that case $\nu = \nu_0 = 1$, $\varrho_0 = \gamma_0$, the Lagrange function does not depend on ν thus H_k vanishes as it may be seen from (3.9). Expressions (4.2-4.6) reduce to expressions as obtained in [9].

5. The case of steady crack propagation

Let us consider a two-D granular deformation field for which the displacement vector x and the volume distribution function ν depend on the coordinates ξ, η.

Theorem. *The integral*

$$J = \int_{t_0}^{t} \left[\int_{\Gamma} (W - K - F_k x_k - G\nu)\, d\eta - \left(T_k \frac{\partial x_k}{\partial \xi} + H \frac{\partial \nu}{\partial \xi} \right) dl \right] dt +$$
$$\left[\int_{S} \left(\varrho \dot{x}_k \frac{\partial x_k}{\partial \xi} + \varrho k \dot{\nu} \frac{\partial \nu}{\partial \xi} \right) ds \right] \Bigg|_{t_0}^{t}$$

is path–independent for any path around the crack tip (Fig.1) and for any $t_0 > t > 0$.

Here W and K are the strain energy density and kinetic energy density respectively. The domain S is a region bounded by Γ and the crack surfaces (Fig.1). The proof may be found in [7].

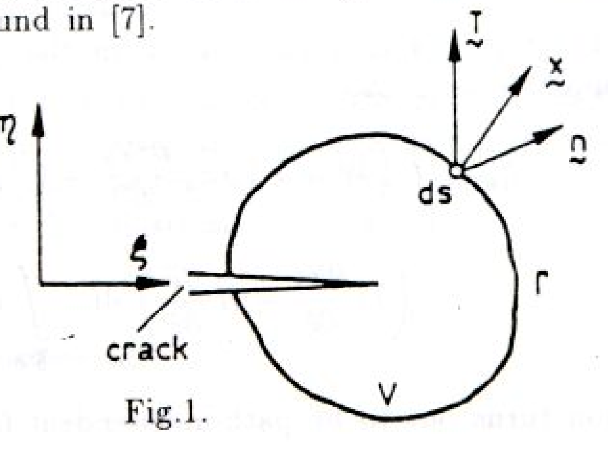

Fig.1.

Example:

Below we consider an interesting case in granular fracture dynamics. It is the case of the steady-state crack propagation.

Let the propagating velocity for the crack tip be V. Consider paths $\overline{\Gamma}$ moving with the same velocity V in a new coordinate system $\overline{\xi}$, $\overline{\eta}$ moving together with the crack tip (Fig.2).

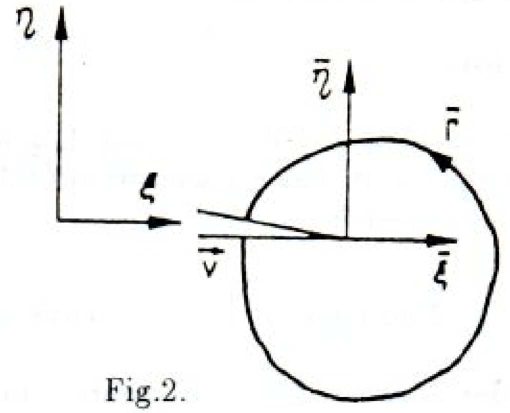

Fig.2.

The transformation from ξ, η, to $\overline{\xi}$, $\overline{\eta}$, $\overline{t}$ is

$$\begin{aligned} \overline{\xi} &= \xi - V\,t, \quad V = \text{const.} \\ \overline{\eta} &= \eta \\ \overline{t} &= t \end{aligned} \tag{5.2}$$

and

$$\begin{aligned} x_k(\xi, \eta, t) &= x_k(\overline{\xi}, \overline{\eta}, \overline{t}) \\ \nu(\xi, \eta, t) &= \nu(\overline{\xi}, \overline{\eta}, \overline{t}) \end{aligned} \tag{5.3}$$

For a steadily moving crack, we have $\frac{\partial}{\partial t} = 0$, that, from (5.3), the expression for velocity ($\dot{x}_k$, $\dot{\nu}$) and acceleration ($\ddot{x}_k$, $\ddot{\nu}$) we get

$$\dot{x}_k = -V\frac{\partial \overline{x}_k}{\partial \xi}, \quad \dot{\nu} = -V\frac{\partial \overline{\nu}}{\partial \xi}, \quad \ddot{x}_k = V^2\frac{\partial^2 \overline{x}_k}{\partial \xi^2}, \quad \ddot{\nu} = V^2\frac{\partial^2 \overline{\nu}}{\partial \xi^2} \tag{5.4}$$

Thus, recognizing a constant path Γ in the ξ, η system from Eq.(5.1) we have for granular elastic case

$$F_\xi = \int\limits_{\overline{\Gamma}} \left(W + V^2\,\varrho\,x_k\,\frac{\partial^2 \overline{x}_k}{\partial \xi^2} + V^2\,\varrho\,k\,\nu\,\frac{\partial^2 \overline{\nu}}{\partial \xi^2} - F_k\,x_k - G\,\nu \right) \mathrm{d}\overline{\eta} - \\ - \left(T_k\frac{\partial \overline{x}_k}{\partial \xi} + H\,\frac{\partial \overline{\nu}}{\partial \xi} \right) \mathrm{d}\overline{l} - \int\limits_{\overline{S}} V^2 \left(\varrho\,x_k\frac{\partial^3 \overline{x}_k}{\partial \xi^3} + \varrho\,k\,\nu\frac{\partial^3 \overline{\nu}}{\partial \xi^3} \right) \mathrm{d}\overline{s}$$

The expression turns out to be path-independent for different $\overline{\Gamma}$.

Acknowledgement. This work was supported by a grant from the National Science Foundation of Yugoslavia, Grant No. 0402, and partially by SMNTR thorough the grant number OSI262.

REFERENCES

[1] Goodman M.A., Cowin S.C.: *A continuum theory for granular materials*, Arch. Rat. Mech. Anal. **44** (1972), 249-.

[2] Cowin S.C.: *Thermodynamic model for porous materials with vacuous pores*, J.App.Phys. **43** (1972), 2495-.

[3] Cowin S.C., Goodman M.A.: *A variational principle for granular materials*, ZAMM **56** (1976), 281-.

[4] Gunter, W.: *Uber einige Randintegrale der Elastomechanik*, Abh. Brunschw. Wiss. Ges. **14** (1962), 54-.

[5] Knowles, J.K., Sternberg, E.: *On a class of conservation laws in linearized and finite elastostatics*, Arch. Rat. Mech. Anal. **44** (1972), 187-211.

[6] Fletcher, D.C.: *Conservation laws in linear elastodynamics*, Arch. Rat. Mech. Anal. **60** (1976), 329-353.

[7] Vukobrat, Dj.M.: *Conservation laws in micropolar elastodynamics and path-independent integrals*, Int. J. Engng. Sci. **27** (1989), 1093-1106.

[8] Toupin, R.A.: *Theories of elasticity with couple-stress*, Arch. Rat. Mech. Anal. **17** (1964), 85-113.

[9] Jaric, J.: *On conservation laws in thermoelasticity*, Theor. and Appl. Mech. **5** (1979), 20-34.

NOETHERS THEOREM IN DEN GRANULIERTEN STOFFEN

Das Ziel dieser Arbeit ist darin, dass die Untersuchung der Erhaltungsgesetze auf granulierte Stoffe erweitert wird und dadurch gezeigt wird, wie dieselben mittels Noethers Theoreme folgen.

Benützend Euclideans Gruppe der infinitesimalen Transformationen wurde die bestimmte Klasse der Erhaltungsgesetze fur dieses Materialtyp gewonnen und danach eines von diesen Gesetzen als ilustriertes Beispiel im Falle der Bruschweiterung mit der konstanten Geschwindigkeit ausgenützt.

O TEOREMI NETEROVE U GRANULARNIM MATERIJALIMA

Poznato je da se zakoni konzervacije, koji dovode do integrala nezavisnih od putanje, mogu dobiti na razne načine. Jedan od tih načina zasnovan je na invarijantnosti datog akcionog integrala u odnosu na odgovarajuću grupu infinitezimalnih transformacija (teorema Neterove).

Svrha ovog rada upravo je u tome da se porširi ispitivanje dobijanja zakona održanja na granularne materijale i da se pokaže kako oni slede primenom teorẹme Neterove. Koristeći Euklidovu grupu infinitezimalnih transformacija dobijena je određena klasa zakona održanja za ovaj tip materijala, a potom je jedan od tih zakona iskorišćen kao ilustrativni primer u slučaju širenja prsline konstantnom brzinom.

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