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APPLICATION OF COMPLEX FUNCTIONS IN FLUID MECHANICS

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Introduction

It is well known that the dynamic equations for a potential flow of an incompressible fluid are automatically satisfied (the energy of fluid particles does not depend on their position in the flow field, as well as on the time when the stationary flow is concerned) and that the kinematic part of the problem is reduced to the solution of the following two equations:

$$\mathrm{div}\vec{V} = 0, \quad \mathrm{rot}\vec{V} = 0, \tag{1}$$

where $\vec{V}$ represents the flow velocity. The pressure distribution in the flow field is then directly obtained from the solution of dynamic equations. If in addition the flow is a planar one, the solution of equations (1) is then reduced to the determination of two harmonic functions: the potential function $\varphi(x, y)$ and the stream function $\psi(x, y)$, which are related by Cauchy-Riemann's conditions:

$$u = \frac{\partial\varphi}{\partial x} = \frac{\partial\psi}{\partial y}, \quad v = \frac{\partial\varphi}{\partial y} = -\frac{\partial\psi}{\partial x}, \tag{2}$$

where u and v are velocity projections on x and y axis, respectively. The conditions (2), in fact, show that from the expression $\varphi + i\psi$, the variables x and y can be eliminated by using one single complex variable $z = x + iy$. Due to the physical importance of the function $\varphi(x, y)$, the following expression

$$\varphi(x, y) + i\psi(x, y) = w(z) \tag{3}$$

is usually called the complex potential, whereas its derivative

$$\frac{\mathrm{d}w}{\mathrm{d}z} = u - iv = \overline{V}(z) \tag{4}$$

is known as the complex velocity. Both of them are analytical, for they depend only on $z = x+iy$ variable. The analytical complex function have very significant properties which enable a very broad application of them in fluid mechanics.

Namely, it is possible, using the conditions (2), to show that the functions φ and ψ are harmonic and that the curves $\varphi(x,y)$ =const. and $\psi(x,y)$ =const. are mutually perpendicular, so they form a square net, for it is grad$\varphi\perp$gradψ and $|\text{grad}\varphi| = |\text{grad}\psi|$. The following two properties of analytical complex functions which also result from the conditions (2), play a very important role. The derivative of a complex analytical function does not depend on the direction in which the increment of variable z is observed, for conditions (2) show that $\mathrm{d}w/\mathrm{d}z = \partial w/\partial x = \partial w/\partial(iy)$; and finally, that conformal mapping of an analytical function leads again to an analytical one. The previously mentioned property of the complex function derivative is connected with complex velocity (4), which, as a physical quality, must have a finite value in the whole flow field, whereas the conclusion about conformal mapping provides that each potential flow of an incompressible fluid, determined by complex potential (3), can be transformed in as many potential flows as is the number of available transformations. In this way, fluid mechanics has gained a mathematical power which enables a successful solution of a great number of important and complex problems in the potential flow theory.

Nonanalytical complex functions and their deflection from the analicity

First of all it is to be noted that only inviscid fluids flow with the velocity potential i.e. fluids which in fact do not exist. In case of flow of real (viscous) fluids, we always have rot$\vec{V} \neq 0$. The vorticity exist and it is only expressed in a weaker or a stronger form. This means that the study of real fluids velocity field by means of a Laplace field (which corresponds to the potential flow of an inviscid fluid) can only be accepted as an approximation of the reality. Sometimes this approximation can offer acceptable results for technical practice, but may also lead to some wrong conclusions about physical qualities concerned with the flow. According to this, it is concluded that the real fluid velocity field deflects, more or less, from the Laplace field which is valid for a potential flow of an inviscid fluid [2]. Observing the fact that for the Laplace field investigation the analytical complex functions are used, it is natural to conjecture the existence of such complex functions which could be also used for real fluid velocity field studying [6]. These complex functions are called nonanalytical complex functions. Namely, when the real fluid flow is concerned, the velocity potential does not exist ($\vec{V} \neq$ gradφ). The stream function $\psi(x,y)$ itself, always exists, due the continuity of a flow field (div$\vec{V} = 0$). However, it is still possible, together with the stream function $\psi(x,y)$, to observe some arbitrary real function $\varphi(x,y)$ (but now $\vec{V} \neq$ gradφ), whereby the expression $\varphi(x,y) + i\psi(x,y)$ will now represent the nonanalytical complex function:

$$w(z,\overline{z}) = \varphi(x,y) + i\psi(x,y), \tag{5}$$

in which the $\overline{z} = x - iy$ is the conjugated value of a complex variable $z = x + iy$. In this case, functions $\varphi(x,y)$ and $\psi(x,y)$ do not satisfy the Cauchy-Riemann's conditions, for gradφ is not perpendicular to gradψ and also $|\text{grad}\varphi| \neq |\text{grad}\psi|$. It is possible, according to Bilimovitch [1], to define the deflection of a nonanalytical complex function (5) from the analicity such as:

$$\vec{B} = \text{grad}\varphi - [\text{grad}\psi, \vec{k}], \tag{6}$$

where $\vec{k} = [\vec{i}, \vec{j}]$ is the z-axis unit vector. If the vector $\vec{B}$ is equal to zero, the function (5) becomes (3), for the equation (6) giver the vector form of the Cauchy-Riemann's conditions:

$$\text{grad}\varphi - [\text{grad}\psi, \vec{k}] = 0, \tag{7}$$

since, in this case, $\vec{V} = \text{grad}\varphi$. In the paper [6], the geometric interpretation of vector $\vec{B}$ has also been given, which is for that purpose shown in fig.1.

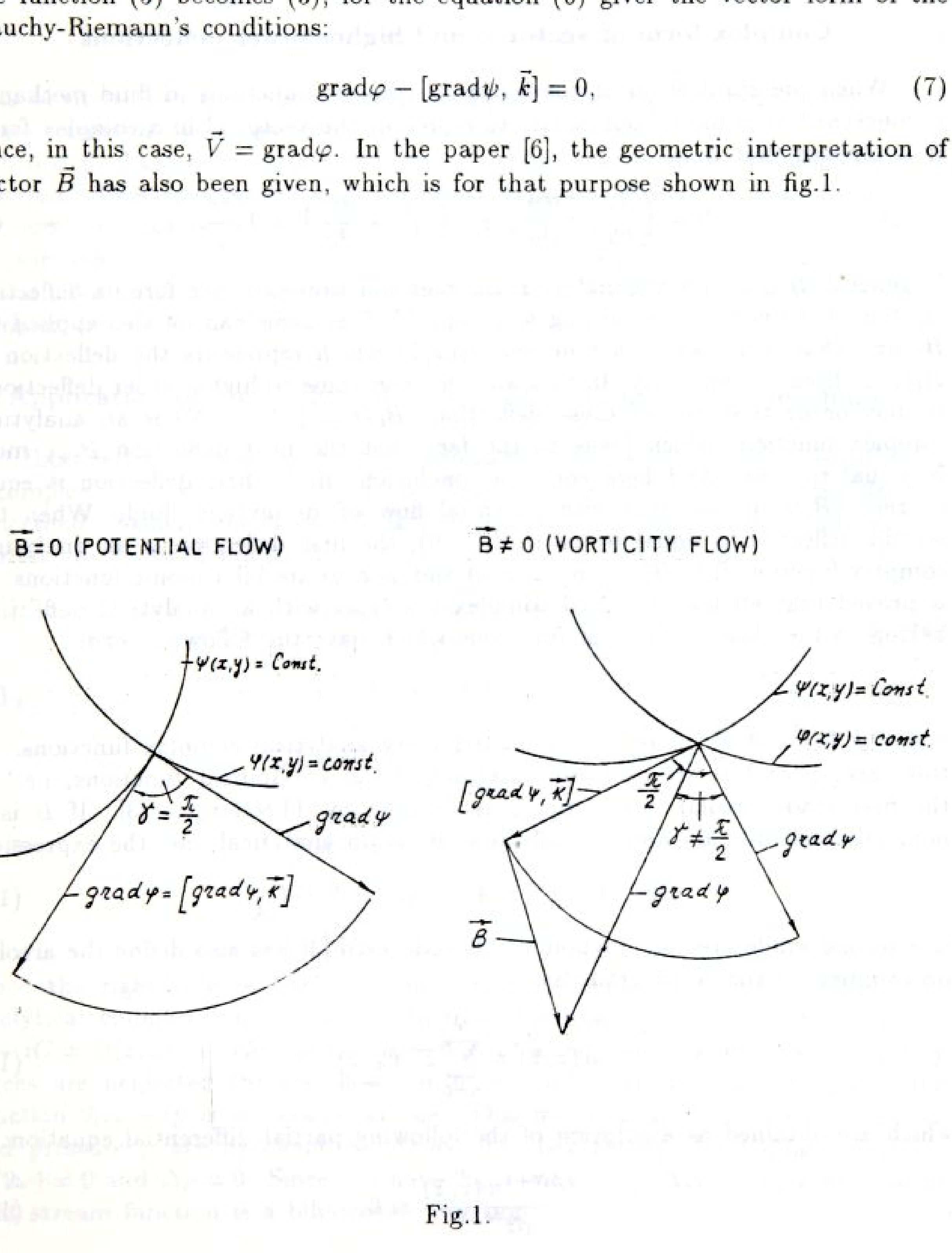

Fig.1.

It is obvious that the vector $\vec{B}$ changes itself according to the position of point in a flow field. It can always be projected on two arbitrary perpendicular directions, thus in x and y directions also. In some or even in all points of a flow field, one of the vector $\vec{B}$ projections on coordinate axis may be equal to zero. In that case, function $w(z, \overline{z})$ belongs to the class of para-analytical functions and gradφ or gradψ have then a permanent direction in a flow field.

Complex form of vector $\vec{B}$ and higher order deflections

When the application of nonanalytical complex functions in fluid mechanics is concerned, it is more appropriate to represent the vector $\vec{B}$ in a complex form. It is obvious that it is:

$$B = \left(\frac{\partial \varphi}{\partial x} - \frac{\partial \psi}{\partial y}\right) + i\left(\frac{\partial \varphi}{\partial y} + \frac{\partial \psi}{\partial x}\right) = 2\frac{\partial w}{\partial \overline{z}}. \tag{8}$$

In general, B is also a nonanalytical complex function and therefore its deflection B_1 can be defined too, according to Fempl [5]. The same can be also applied for B_1 and that will lead to a function $B_2(z, \overline{z})$ which represents the deflection of $B_1(z, \overline{z})$ from the analicity. In this way, one can come to higher order deflections. It may occur that one of these deflections B_i $(i = 1, 2, \dots, N)$ is an analytical complex function, which leads to the fact that the next deflection B_{i+1} must be equal to zero. And here comes a conclusion: if the first deflection is equal to zero $(B = 0)$, we deal with potential flow of an inviscid fluid. When the second deflection is equal to zero $(B_1 = 0)$, the first deflection is an analytical complex function $B = B(z)$, and $\varphi(x, y)$ and $\psi(x, y)$ are biharmonic functions. It is proved that all nonanalytical complex functions with an analytical deflection belong to the class of Goursat functions which have the following form:

$$w(z, \overline{z}) = F_o(z) + \overline{z}\, F_1(z), \tag{9}$$

where $F_o(z)$ and $F_1(z)$ represent the arbitrary analytical complex functions. In this case, φ and ψ are real and imaginary parts of Goursat functions, i.e. of the first grade areolar polynomials, as denoted by Théodoresco [3]. If B is a nonanalytical function and its deflection B_1 is an analytical one, the expression

$$w(z, \overline{z}) = F_o(z) + \overline{z}\, F_1(z) + \overline{z}^2\, F_2(z), \tag{10}$$

is a second grade areolar polynomial. Théodoresco [3] has also define the areolar polynomials of the N-th grade as:

$$w(z, \overline{z}) = \sum_{n=0}^{N} \overline{z}^n\, F_n(z), \tag{11}$$

which are obtained as a solution of the following partial differential equation:

$$\frac{\partial^{N+1} w(z, \overline{z})}{\partial \overline{z}^{N+1}} = 0. \tag{12}$$

Since the order of the derivative of an areolar polynomial at the same time represents the order of its deflection from analicity, these polynomials may be described as nonanalytical complex functions whose deflection from the analicity is an analytical function of the very same grade as the polynomial itself.

One must note that the vector $\vec{B}$ field is, in general, a complex field, for from the equation (6) it is obtained:

$$\operatorname{div}\vec{B} = \Delta\varphi, \quad \operatorname{rot}\vec{B} = \Delta\psi\,\vec{k}. \tag{13}$$

It is obvious that the vector $\vec{B}$ field is going to be Laplace field when φ and ψ are harmonic functions. However, this does not mean that the velocity field will also be Laplace field and that sum $\varphi + i\psi$ must be an analytical complex function. This comes from the fact that functions φ and ψ, even if they are harmonic, do not necessarily satisfy the Cauchy–Riemann's conditions. In order for the velocity field be a Laplace field too, the equations (13) must have a trivial solution $\vec{B} = 0$, which means that function $\varphi + i\psi = w(z)$ is an analytical complex functions.

Application of nonanalytical complex functions in fluid mechanics

Bilimovitch [1] was the first to demonstrate the application of vector $\vec{B}$ in a complex form when the slow plane flow of an incompressible viscous fluid is concerned. Namely, the Navier–Stokes' equations for incompressible flow may be written in the following form:

$$A + iC = \left[\frac{\partial(2\mu\omega)}{\partial x} - \frac{\partial p}{\partial y}\right] - i\left[\frac{\partial(2\mu\omega)}{\partial y} + i\frac{\partial p}{\partial x}\right], \tag{14}$$

where:

$$A = \varrho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} - F_y\right),$$

$$C = \varrho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - F_y\right),$$

$$2\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}.$$

Since the right side of the equation (14) represents a deflection of the nonanalytical complex function $2\mu\omega - ip$ from the analicity it may be written as $A + iC = B(z, \bar{z})$. In case of the slow flow of a very viscous fluid, when gravity forces are neglected (Stokes' flow), it is $A + iC = B = 0$, and therefore the function $2\mu\omega - ip$ is an analytical one. This means that both the vorticity 2ω and pressure p are harmonic functions, for they satisfy the Laplace equation $\Delta(2\omega) = 0$ and $\Delta p = 0$. Since, we have $2\omega = -\Delta\psi$, it is $\Delta\Delta\psi = 0$, which means that stream function is a biharmonic function.

Voronjec [2] has used the vector $\vec{B}$ for studying a stationary potential flow of a compressible fluid. Namely, the continuity equation and the nonvorticity condition

$$\operatorname{div}\left(\frac{\varrho}{\varrho_o}\right) = 0, \quad \operatorname{rot}\vec{V} = 0, \tag{15}$$

lead to following well known relations:

$$u = \frac{\partial \varphi}{\partial x} = \frac{\varrho}{\varrho_o}\frac{\partial \psi}{\partial y}, \quad v = \frac{\partial \varphi}{\partial y} = -\frac{\varrho}{\varrho_o}\frac{\partial \psi}{\partial x}. \tag{16}$$

The densities ϱ_0 and ϱ correspond to the velocities $\vec{V}_0$ and $\vec{V}$ and pressures p_0 and p, respectively. Although, we deal with the potential flow, complex function $\varphi + i\psi$ is not an analytical one, for equations (16) do not represent the Cauchy–Riemann's conditions. That is why a deflection of the nonanalytical function $\varphi + i\psi = w(z, \overline{z})$ from the analicity, can be formed:

$$\vec{B} = \left(1 - \frac{\varrho}{\varrho_o}\right)\vec{V} = \left[1 - f\left(V^2\right)\right]\vec{V}, \tag{17}$$

because from the dynamic equation follows that $\varrho/\varrho_0 = f(V^2)$. The form of the function $f(V^2)$ depends on the way the change of a thermodynamic state of a compressible fluid during the flow is made. Practically, only isothermal and adiabatic processes are to be taken into account. Now, the following expressions can now be evaluated:

$$\operatorname{div}\vec{B} = \Delta\varphi = -\frac{\varrho}{\varrho_o}\left(\vec{V}, \operatorname{grad}\frac{\varrho}{\varrho_o}\right), \tag{18}$$

$$\operatorname{rot}\vec{B} = \Delta\psi\,\vec{k} = \left[\vec{V}, \operatorname{grad}\frac{\varrho}{\varrho_o}\right]. \tag{19}$$

From these equations one may see that both φ and ψ can not be simultaneously harmonic functions. It is impossible for the scalar and vector product of two vectors to be equal to zero simultaneously. It is well known, however, that from the expression (16) the nonlinear partial differential equations for φ and ψ are derived. The solution of these equations in some particular cases may be of a great difficulty. That is the reason why Chapligin has proposed the velocity hodograf plane for studying the compressible fluid potential flow.

In the paper [6] one more approach of using vector $\vec{B}$ for studying real incompressible fluid plane flow has been suggested. In that case, the continuity is satisfied with the stream function $\psi = \psi(x, y)$ using the following relations: $u = \partial\psi/\partial y$, $v = -\partial\psi/\partial x$. The vorticity may be expressed as $2\omega = -\Delta\psi$ and therefore, Navier–Stokes equations are reduced to this form:

$$\frac{\partial \psi}{\partial y}\frac{\partial \Delta\psi}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \Delta\psi}{\partial y} = \nu\Delta\Delta\psi. \tag{20}$$

When the variables x and y by using z and $\overline{z}$ are eliminated from the stream function $\psi(x, y)$ the new function $\psi = \psi(z, \overline{z})$ is obtained, and therefore, equation

(20) becomes:

$$\frac{\partial\psi}{\partial\bar{z}}\frac{\partial^3\psi}{\partial z^2\partial\bar{z}}-\frac{\partial\psi}{\partial z}\frac{\partial^3\psi}{\partial z\partial\bar{z}^2}=2i\nu\frac{\partial^4\psi}{\partial z^2\partial\bar{z}^2}. \tag{21}$$

This equation suggests the use of complex velocity in the form

$$-i\overline{V}=-i(u-iv)=2\frac{\partial\psi}{\partial z}, \tag{22}$$

instead of function $w(z,\bar{z})$ defined by the expression (5). Since the complex velocity is a nonanalytical function, because the real (viscous) fluid incompressible flow is concerned, the deflection of this complex velocity from the analicity may be defined as:

$$B_1=2\frac{\partial(-i\overline{V})}{\partial\bar{z}}=-2\omega-i\,\mathrm{div}\vec{V}. \tag{23}$$

From this expression it is easily seen that B_1 can not be equal to zero (i.e. that complex velocity can not be an analytical function), because when we deal with real (viscous) fluid flow, we always have $2\omega\neq 0$. The other part in the equation (23) is, however, equal to zero because of incompressibility. In other words, the deflection B_1 would be equal to zero (i.e. the complex velocity would be an analytical function) if it were $2\omega=0$, which means if the flow were potential. The methods for stream function determination are, in this case, well known.

Since B_1 is, in general, a nonanalytical complex function, its deflection from the analicity may be defined also (that is the second order deflection from the analicity for a complex velocity):

$$B_2=2\frac{\partial B_1}{\partial\bar{z}}=\left[\frac{\partial(\mathrm{div}\vec{V})}{\partial x}+\frac{\partial(2\omega)}{\partial y}\right]+i\left[\frac{\partial(\mathrm{div}\vec{V})}{\partial y}-\frac{\partial(2\omega)}{\partial x}\right]. \tag{24}$$

In case of the real fluid incompressible flow ($\mathrm{div}\vec{V}=0$), the deflection B_2 may be equal to zero (i.e. B_1 may be an analytical function). Then the vorticity condition $2\omega=-\Delta\psi=$const. as well as the equation (21) is satisfied. As it is well known, the stream function in this case represents a complex flow which consists of an arbitrary potential flow and a vorticity flow along concentrated circles or straight lines.

The third order complex velocity deflection from the analicity for incompressible flow is:

$$B_3=2\frac{\partial B_2}{\partial\bar{z}}=\frac{\partial^2\Delta\psi}{\partial x^2}-\frac{\partial^2\Delta\psi}{\partial y^2}-2i\frac{\partial^2\Delta\psi}{\partial x\partial y}. \tag{25}$$

The condition $B_3=0$ shows that the second order deflection B_2 is an analytical complex function and leads to the following two partial differential equations:

$$\frac{\partial^2\Delta\psi}{\partial x^2}-\frac{\partial^2\Delta\psi}{\partial y^2}=0,\qquad\frac{\partial^2\Delta\psi}{\partial x\partial y}=0. \tag{26}$$

From the second equation (26) follows that

$$\Delta\psi=f_1(x)+f_2(y).$$

The first equation (26) determines these two functions in the form of a second order polynomial and, therefore, it can be written:

$$\Delta\psi = C_1(x^2 + y^2 + C_2 x + C_3 y + C_4).$$

This equation may be by using an appropriate position of a coordinate system and by introducing polar coordinates rewritten as follows:

$$\Delta\psi = C_1 r^2 + C_5.$$

The solution of this differential equation is:

$$\psi = \psi_1 + \frac{1}{16}C_1 r^4 + \frac{1}{4}C_5 r^2, \tag{27}$$

where ψ_1 is a harmonic function. The polar angle θ may appear in the expression for ψ_1 which is not the case for the remaining part on the right side of the equation (27). When the equation (27) is written in the following form:

$$\psi = \psi_1(z,\overline{z}) + \frac{1}{16}C_1 z^2\overline{z}^2 + \frac{1}{4}C_5 z\overline{z},$$

and then replaced into (21), it will be obtained:

$$\psi_1(z,\overline{z}) = -i\,\nu\ln(z/\overline{z}).$$

The coefficient C_5 may be equal to zero or proportional to the $\ln(z,\overline{z})/z\overline{z}$ form of function. If C_5 is taken to be proportional to this function, it will definitely be obtained that:

$$\psi = -i\,\nu\ln(z/\overline{z}) + \frac{1}{16}C_1 z^2\overline{z}^2 + C_6\ln(z\overline{z}),$$

and by returning to polar coordinates:

$$\psi = 2\nu\,\theta + \frac{1}{16}C_1 r^4 + 2C_6\ln r. \tag{28}$$

This solution satisfies the dynamic equation (21), i.e. (20). The very same solution has been attained by Jeffery [8], who started with the assumption that in the incompressible fluid flow field vorticity depends only on r. As it is easily seen, this assumption is equivalent to the condition that the third order complex velocity deflection from the analicity is equal to zero.

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DIE ANWENDUNG DER KOMPLEX-FUNKTIONEN IN DER FLUID MECHANIK

Vor allem behandelt diese Arbeit den Beitrag analytischer Komplex-Funktionen zur Untersuchung ebener potentiellen Strömungen des inkompressiblen Fluids. Weil der wirkliche Fluid nicht mit dem Geschwindigkeitspotential strömen kann, sucht man üblicherweise eine andere Klasse der Komplex-Funktionen um die ebene Strömung des viskosen Fluids zu erforschen. So erscheinen die areolar-Polynome, die aus allgemeiner Klasse der Komplex-Funktionen mittels von Bilimovitch [1] definierter "Abweichungstufe" unanalytischer Funktionen von der Analytizität abgesondert sind. Die Benutzung dinamischer Gleichungen und der "Abweichungstufe" des Geschwindigkeitsfeldes des wirklichen Fluids, Voronjec [2] definiert hat, hat uns eine neue Menge von Lösungen der Navier-Stokes'schen Gleichungen zu findem ermöglicht.

PRIMENA KOMPLEKSNIH FUNKCIJA U MEHANICI FLUIDA

U radu se najpre govori o ulozi analitičkih kompleksnih funkcija u proučavanju ravanskih potencijalnih strujanja nestišljivog fluida. Realan fluid ne može da struji sa potencijalom brzine, pa je bilo prirodno potražiti neku drugu klasu kompleksnih funkcija koje bi mogle da posluže za proučavanje ravanskih strujanja realnog (viskoznog) fluida. Pokazano je da su to areolarni polinomi koji su iz opšte klase kompleksnih funkcija izdvojeni korišćenjem "mere odstupanja" od analitičnosti neanalitičnih kompleksnih funkcija, koju je definisao Bilimović [1]. Korišćenjem dinamičkih jednačina strujanja realnog fluida i "mere odstupanja" brzinskog polja realnog fluida od Laplace-ovog polja, koju je definisao Voronjec [2], nađena su nova rešenja Navier-Stokes-ovih jednačina.

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