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SECOND–ORDER ANALYSIS OF FRAME STRUCTURES

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1. Introduction

The members of light slender structures, under increasing external load, often undergo considerable elastic deformation. Elastic instability of this type of structures may occur before they exibit any nonlinear material behaviour. For this reason, geometrically nonlinear analysis for these structures has great importance. The problem of geometrically nonlinear analysis of frame structures has been extensively studied by numerous authors [6], [8], [11], [13], [18], [20].The approach is mainly based on finite element method and the analysis may be formulated in either Lagrangian or Euler description. Both these types of formulation have been extensively used for nonlinear and stability analysis of structures.

The basic purpose of this paper is matrix formulation of the second–order elastic analysis of frame structures which is much more simple than general geometrically nonlinear analysis, and, at the same time, accurate enough for practical purpose of structural design. The stress–strain and buckling analysis of structures based on the second–order theory has been studied by various authors [2], [7], [9], [14]. A review of earlier literature on the subject may be found in the article by Goto and Chen [9]. They presented three different kinds of stiffness equations that can be used according to the value of axial force of the member. The simplified form of those equations can be derived from finite element procedure assuming polynomial displacement functions [1].

Further extension of the former considerations with regard to formulation of stiffness matrices and nodal force vectors of a member for various boundary conditions, and the practical procedure of buckling analysis of frame structures have been presented in this paper. The stiffness matrices and nodal force vector are derived for various beam end connections (rigid–rigid, rigid–hinged, rigid–free, rigid–symmetrical).

The stiffness and stability matrices are derived using two different approachs: analytical solutions of the governing differential equations and finite element method assuming polynomial interpolation functions. The stiffness matrices derived from the analytical solutions are first expressed in series expansion and than reduced by some simplifications. The convergence of power series and the accuracy of the obtained solutions are examined.

The governing differential equations

The governing equations for the second-order analysis of a prismatic member can be easily derived using the direct approach. Herein, the variational approach is employed, as much more general. The governing differential equations are formulated through the principle of stationary property of potential energy.

A plane member - i, k - subjected to distributed force, $p(x)$, acting perpedicular to the member axis before deformation, is shown in Fig. 1.

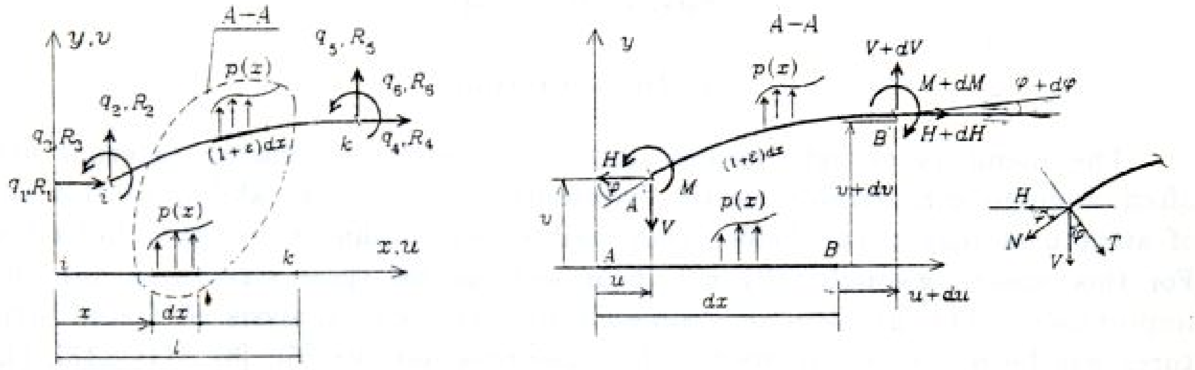


Fig.1. Member notation (end actions, displacements, internal forces)

A cartesian coordinate system, (x, y), and displacement components, (u, v), are defined at the initial configuration of the member. The notation for end actions, displacements and internal forces is given in Fig. 1.

By using the usual assumptions of classical beam theory, the displacement field of a beam can be given by

$$u(x,y) = u - yv_{,x} \ , \qquad (1)$$
$$v(x,y) = v \ ,$$

in which: $u, v=$ the displacement components on the centroidal axis, both are the function of x; and $(\ )_{,x}$ denotes differentiation with respect to x.

The nonzero component of the strain tensor (extensional strain) is expressed by the displacement components as

$$\varepsilon(x,y) = \varepsilon + \kappa y \ , \qquad (2)$$

where

$$\varepsilon = u_{,x} + \frac{1}{2} v_{,x}^2 \ , \qquad (3)$$
$$\kappa = v_{,xx} \ ,$$

in which; ε, $\kappa=$ one-dimensional stretching and bending deformation measures. The introduction of the nonlinear term $\frac{1}{2}v_{,x}^2$ in (3) reflects the couplng between the axial and bending deformations.

The axial force, N, and the bending moment, M, are related to the deformations by

$$N = EA(\varepsilon - \varepsilon_0) = EA(u_{,x} + \frac{1}{2}v_{,x}^2 - \varepsilon_0) , \quad (4)$$
$$M = EI(\kappa - \kappa_0) = EI(v_{,xx} + \kappa_0) ,$$

in which $E=$ Young's moduls; $A=$ cross–section area; $I=$ moment of inertia about z axis, and ε_0, $\kappa_0=$ the initial deformation measures. In case of temperature influence, it is defined by

$$\varepsilon_0 = \alpha t , \quad (5)$$
$$\kappa_0 = \alpha \frac{\Delta t}{h} ,$$

in which $\alpha=$ coefficient of thermal expansion, $t=$ temeperature change on the centroidal axis, $\Delta t=$ temperature difference between the upper and lower sides, $h=$ height of the member.

The transverse shear force, T, is related to, M, by

$$T = -M_{,x} = -V + H\varphi . \quad (6)$$

The principle of stationary property of total potential energy is applied to establish the governing differential equations and appropriate boundary conditions, i.e.

$$\delta\Pi = \delta U - \delta W ,$$

in which

$$\delta U = \int_o^l (N\delta\varepsilon + M\delta\kappa)dx , \quad (8)$$
$$\delta W = \int_o^l p(x)\delta v dx - \sum_{i=1}^{6} R_i q_i .$$

In the preceding expressions, Π is the total potential energy, U is the strain energy, while W is the work of external forces (distributed force $p(x)$ and generalized forces R_i corresponding to the generalized displacements $g_i, i = 1...6$ at the ends of the member); δ is variational symbol.

Taking variation of Eq. 3, the following can be given:

$$\delta\varepsilon = \delta u_{,x} + v_{,x}\delta v_{,x} , \quad (9)$$
$$\delta\kappa = -\delta v_{,xx} .$$

Substituting (4) and (9) into (8) and (7) leads to:

$$\int_o^l \left\{ EA \left[(u_{,x} + \frac{1}{2}v_{,x}^2 - \varepsilon_0)\delta u_{,x} + (u_{,x} + \frac{1}{2}v_{,x}^2 - \varepsilon_0)v_{,x}\delta v_{,x} \right] + \right.$$
$$\left. + EI(v_{,xx} + \kappa_0)\delta v_{,xx} \right\} dx - \int_o^l p(x)\delta v dx - \sum_{i=1}^{6} R_i \delta q_i = 0 . \quad (10)$$

Carrying out partial integration of (10) and taking care about (4) lead to:

$$N\delta u\,|_o^l + Nv_{,x}\delta v\,|_o^l - M\delta v_{,x}\,|_o^l + M_{,x}\delta v\,|_o^l - \sum_{i=1}^{6} R_i\delta q_i -$$
$$- \delta u \int_o^l \left[EA(u_{,x} + \frac{1}{2}v_{,x}^2 - \varepsilon_0)\right]_{,x} dx + \tag{11}$$
$$+\delta v \int_o^l \left\{[EI(v_{,xx} + \kappa_0)]_{,xx} - \left[EA(u_{,x} + \frac{1}{2}v_{,x}^2 - \varepsilon_0)v_{,x}\right]_{,x} - p(x)\right\} dx = 0\,.$$

As any variation of displacement unequal zero the next differential equations follow from (11):

$$\delta u \neq 0 : \quad \left[EA(u_{,x} + \frac{1}{2}v_{,x}^2 - \varepsilon_0)\right]_{,x} = 0\,, \tag{12}$$
$$\delta v \neq 0 : \quad [EI(v_{,xx} + \kappa_0)]_{,xx} - \left[EA(u_{,x} + \frac{1}{2}v_{,x}^2 - \varepsilon_0)v_{,x}\right]_{,x} - p(x) = 0\,.$$

and the corresponding boundary conditions at nodes i and k:

$$\text{for} \quad x = 0 \begin{cases} N + H_i = 0 \\ Nv_{,x} + M_{,x} + V_i = 0 \\ M - M_i = 0 \end{cases}$$
$$\tag{13}$$
$$\text{for} \quad x = l \begin{cases} N - H_k = 0 \\ Nv_{,x} + M_{,x} - V_k = 0 \\ M + M_k = 0 \end{cases}$$

By using the relations between the internal forces and strains, equations (12) can be transformed into:

$$N_{,x} = 0 \tag{14}$$
$$M_{,xx} - (N\varphi)_{,x} - p(x) = 0\,.$$

The first of equations (14) represents the equilibrium equation in x direction, while the second represents the combination of two other equilibrium equations, i.e.

$$V_{,x} - p(x) = 0\,, \tag{15}$$
$$M_{,x} - V + N\varphi = 0\,.$$

It derives from the first equations (12) and (14) that the axial strain and the axial force along x-direction of the member are constant because there are no distributed forces in the axial direction. Therefore, the internal force component N can be shown as

$$N = EA(\varepsilon - \varepsilon_0) = \frac{A}{I}(u_{,x} + \frac{1}{2}v_{,x}^2 - \varepsilon_0) = k^2 EI\,. \tag{16}$$

where k^2 is the constant ($k^2 = N/EI$).

Substitution of (17) into (12b), in case of the uniform prismatic member ($EI = const.$), leads to:

$$v_{,xxxx} - k^2 v_{,xx} = \frac{p(x)}{EI} - (\alpha \frac{\Delta t}{h})_{,xx} \,. \tag{17}$$

The coupling between axial and bending deformations, respectively between axial and transverse stiffnesses is evident from (16) and (17). The nonlinear term $\frac{1}{2}v_{,x}^2$ which exists in (16) is the reason for that. By neglecting this term, the axial and transverse deformations and corresponding differential equations become separate.

The basic difference between these two types of equations is whether or not thay consider the bowing deformation (the effect of flexure and chord rotation) in the calculation of axial displacement. As the bowing effect generally has little influence on the final results, the simplified form of the governing equations will be used.

Analytical solution of the governing equations

In general, the analytical solution of the differential equations (12) can not be found for nonuniform beams. It is possible to find analytical solutions for the simplified form of the governing differential equations (16) and (17) which are well known from numerous texts. The solution for axial displacement u is the same as the solution of classical (linear) theory. The solution of equation (17) with respect to transverse displacement v consists of the two parts:

$$v(x) = v_g(x) + v_p(x) \,, \tag{18}$$

where $v_g(x)$ and $v_p(x)$ are general and particular solutions, respectively. The general solution of this differential equation depends on whether parametar k^2 (or axial force N) in the governing equations produces compression ($k^2 < 0$), tension ($k^2 > 0$), or zero ($k^2 = 0$) axial strain, can be written as

$$\begin{aligned} v_g(x) &= \alpha_1 + \alpha_2 kx + \alpha_3 \sin kx + \alpha_4 \cos kx, && N < 0 \,, \\ v_g(x) &= \bar{\alpha}_1 + \bar{\alpha}_2 kx + \bar{\alpha}_3 kx + \bar{\alpha}_4 chkx && N > 0 \,, \\ v_g(x) &= \hat{\alpha}_1 + \hat{\alpha}_2 x + \hat{\alpha}_3 x^2 + \hat{\alpha}_4 x^3 && N = 0 \,, \end{aligned} \tag{19}$$

in which $\alpha_i, \bar{\alpha}_i$ and $\hat{\alpha}_i$, $i = 1...4$, are generalized coordinates (unknown constants) which can be determined from the boundary conditions at ends of the member. In the case when the boundary conditions are given on the left end, i.e.

$$\text{for} \quad x = 0 \begin{cases} v(x) = v_0 \\ \varphi(x) = \varphi_0 \\ M(x) = M_0 \\ V(x) = V_0 \end{cases} \tag{20}$$

after elimination of the constants α_i, it follows from (17):

$$v_g(x) = v_0 + \varphi_0 \frac{\sin kx}{k} - M_0 \frac{1-\cos kx}{N} - V_0 \frac{kx - \sin kx}{kN} = \sum_{i=1} C_i F_i(x) , \quad (21)$$

where

$$\begin{aligned} F_1(x) &= 1 , \\ F_2(x) &= \frac{1}{k} \sin kx , \\ F_3(x) &= \frac{-1}{N}(1-\cos kx) , \\ F_4(x) &= \frac{-1}{kN}(kx - \sin kx) . \end{aligned} \quad (22)$$

The constants $C_i, i = 1...4$, have the clear mechanical meaning (displacement, v_0, rotation, φ_0, bending moment, M_0, and transverse force, V_0, at the start or left end of the member respectively). For this reason, these constants are called the starting parameters. The functions $F_i(x), i = 1...4$, also, have quite clear geometrical meaning [16].

The particular solution $v_p(x)$ depends on external loads and the temperature. For distributed external load $p(x)$, Fig. 2, $v_p(x)$ can be expressed as [7]:

$$v_p(x) = \int_o^x \frac{1}{kN} \left[k(x-\xi) - \sin k(x-\xi)\right] p(\xi)\, d\xi . \quad (23)$$

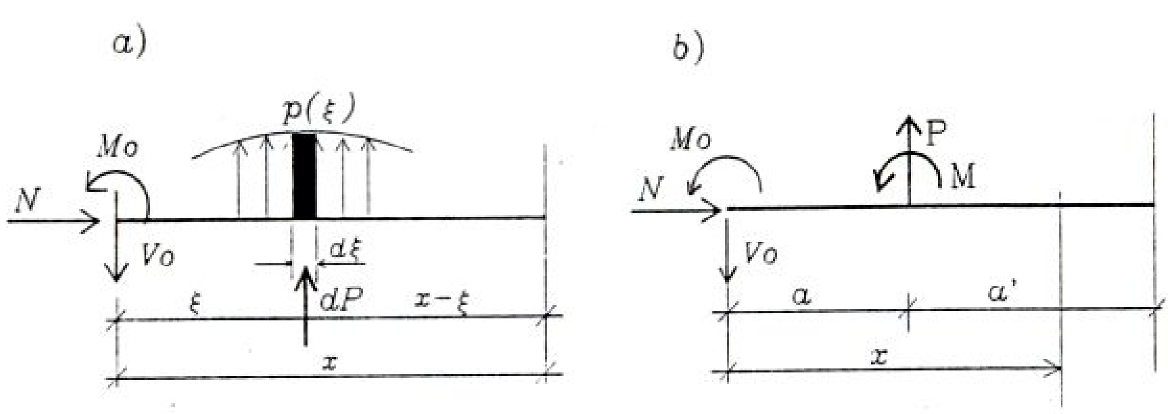


Fig. 2. Member loads: distributed and concentrated

For further considerations, especially for defining the stiffness equations, it is convenient to have the analytical solution transformed in the following form:

$$v(x) = \mathbf{N}\ \mathbf{q} = \sum_{i=1}^{n} N_i(x) q_i , \quad (24)$$

where $N_i(x), i = 1...4$, are interpolation (shape) functions. When the axial force is compresive ($N < 0$), the shape functions are:

$$\begin{aligned} N_1(\xi) &= \Delta^{-1}\left[1 - \cos\omega - \omega\sin\omega + (\omega\sin\omega)\xi - \sin\omega\sin\omega\xi + \right. \\ &\quad \left. + (1-\cos\omega)\cos\omega\xi\right], \\ N_2(\xi) &= l(\Delta\omega)^{-1}\left[\omega\cos\omega - \sin\omega + \omega(1-\cos\omega)\xi + \right., \\ &\quad \left. + (1-\cos\omega - \omega\sin\omega)\sin\omega\xi + (\sin\omega - \omega\cos\omega)\cos\omega\xi\right], \\ N_3(\xi) &= \Delta^{-1}\left[1-\cos\omega - (\omega\sin\omega)\xi + \sin\omega\sin\omega\xi - (1-\cos\omega)\cos\omega\xi\right], \\ N_4(\xi) &= l(\Delta\omega)^{-1}\left[\sin\omega - \omega + \omega(1-\cos\omega)\xi - (1-\cos\omega)\sin\omega\xi + \right. \\ &\quad \left. + (\omega - \sin\omega)\cos\omega\xi\right], \quad \xi = \frac{x}{l}, \quad \omega = kl. \end{aligned} \tag{25}$$

The shape functions for the tensile member ($N > 0$) can be obtained from (25) replacing $\omega = i\omega$ and using the relations $sh\omega = i\sin\omega$ and $ch\omega = \cos i\omega$. The geometrical meaning of the shape functions is analogous to their meaning in the classical theory. However, the difference between them exists. In classical theory, the shape functions are polynomial, while in the second–order theory the shape functions are trigonometric and hyperbolic functions, for compression and for tension, respectively. Besides, these shape functions depend on axial force of the member.

Stiffness equations

A stiffness equation of a member has usual matrix form:

$$\mathbf{R} = \mathbf{k}\ \mathbf{q} - \mathbf{Q},$$

where

$$\begin{aligned} \mathbf{R}^T &= [-H_i,\ V_i,\ M_i,\ H_k,\ V_k,\ M_k], \\ \mathbf{q}^T &= [u_i,\ v_i,\ \varphi_i,\ u_k,\ v_k,\ \varphi_k], \\ \mathbf{Q}^T &= [Q_1,\ Q_2,\ Q_3,\ Q_4,\ Q_5,\ Q_6], \\ \mathbf{k} &= [\ k_{ij}\ ]_{(6x6)}. \end{aligned}$$

and signify: the vector generalized forces, the vector generalized displacements, the vector equivalent nodal forces and the stiffness matrix of the member, respectively.

The stiffness matrix and the nodal force vector can be derived from analytical solution obtained in the preceding section.

Starting from analytical solution (24), (25) and taking into account (4) and (6), based on well–known mechanical meaning of the elements of stiffness matrix, the next form of stiffness matrix (Sekulovic 1978), can be given:

$$
\mathbf{k} = \frac{EI}{l^3\Delta}\begin{bmatrix}
\frac{Al^2\Delta}{I} & 0 & 0 & -\frac{Al^2\Delta}{I} & 0 & 0 \\
 & \omega^3\sin\omega & l\omega^2(1-\cos\omega) & 0 & -\omega^3\sin\omega & l\omega^2(1-\cos\omega) \\
 & & l^2\omega(\sin\omega-\omega\cos\omega) & 0 & -l\omega^2(1-\cos\omega) & l^2\omega(\omega-\sin\omega) \\
 & symmetric & & \frac{Al^2\Delta}{I} & 0 & 0 \\
 & & & & \omega^3\sin\omega & -l\omega^2(1-\cos\omega) \\
 & & & & & l^2\omega(\sin\omega-\omega\cos\omega)
\end{bmatrix}
$$

$$\omega = kl, \quad \Delta = 2(1-\cos\omega) - \omega\ \sin\omega$$

It is suitable to transform the stiffness matrix (28) into the form which is given by Goto and Chen [9]:

$$
\mathbf{k} = \frac{EI}{l^3}\begin{bmatrix}
\frac{Al^2}{I} & 0 & 0 & -\frac{Al^2}{I} & 0 & 0 \\
 & 12\phi_1 & 6l\phi_2 & 0 & -12\phi_1 & 6l\phi_2 \\
 & & 4l^2\phi_3 & 0 & -6l\phi_2 & 2l^2\phi_4 \\
 symmetric & & & \frac{Al^2}{I} & 0 & 0 \\
 & & & & 12\phi_1 & -6l\phi_2 \\
 & & & & & 4l^2\phi_3
\end{bmatrix} \tag{29}
$$

in which $\phi_i, i = 1...4$, are the functions-multipliers of the corresponding elements of classical stiffness matrix. The analytical form of these functions is given in Table 1.

Function	Axial force		
	compressive	$N=0$	tensile
ϕ_1	$\frac{\omega^3\sin\omega}{12\Delta_c}$	1	$\frac{\omega^3 \mathrm{sh}\,\omega}{12\Delta_t}$
ϕ_2	$\frac{\omega^2(1-\cos\omega)}{6\Delta_c}$	1	$\frac{\omega^2(\mathrm{ch}\,\omega-1)}{6\Delta_t}$
ϕ_3	$\frac{\omega(\sin\omega-\omega\cos\omega)}{4\Delta_c}$	1	$\frac{\omega(\omega\,\mathrm{ch}\,\omega-\mathrm{sh}\,\omega)}{4\Delta_t}$
ϕ_4	$\frac{\omega(\omega-\sin\omega)}{2\Delta_c}$	1	$\frac{\omega(\mathrm{sh}\,\omega-\omega)}{2\Delta_t}$

Table 1.

$$\omega = l\sqrt{\frac{|N|}{EI}} = \pi\sqrt{\omega^*}, \quad \omega^* = \frac{N}{P_E},$$

$$\Delta_c = 2(1-\cos\omega) - \omega\sin\omega, \quad \Delta_t = 2(1-ch\,\omega) + \omega sh\,\omega$$

The dependence ϕ_i from the ration $\omega^*, (\omega^* = \frac{N}{P_E}, P_E =$ Euler buckling load), is shown in Fig. 3.

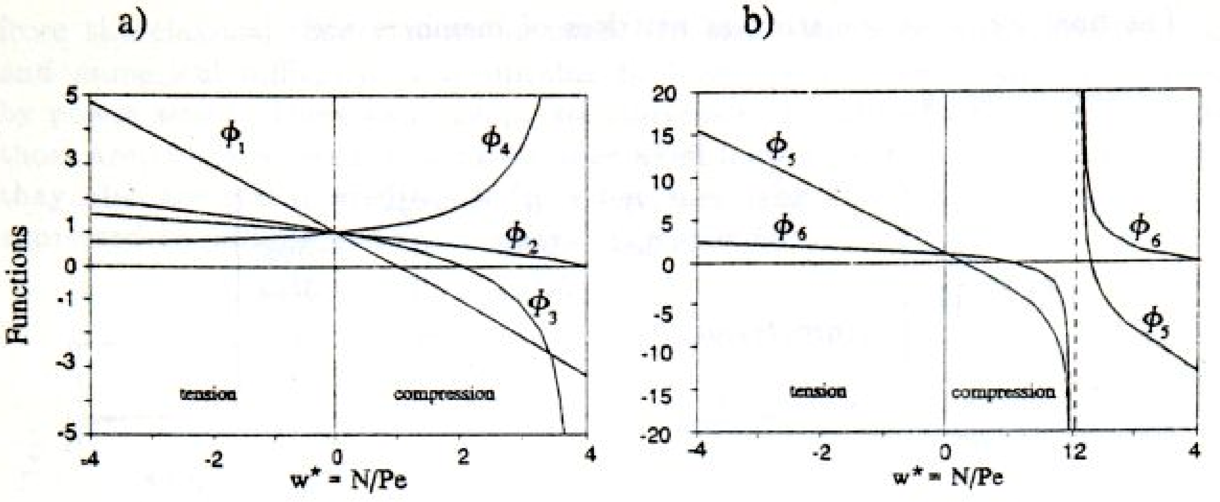


Fig. 3. Functions ϕ_i

The stiffness matrix (29) is derived for the member whose boundary conditions correspond to rigid connections in frame joints. Moreover, other different boundary conditions at member ends can exist. Four types of the member, which are nemed according to the marks of the right end of member, are shown in Table 2.

The stiffness matrices are derived for all types of members by using boundary conditions on the right end of member.

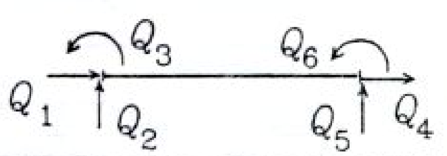
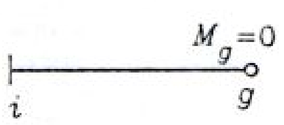
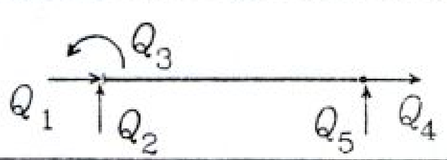
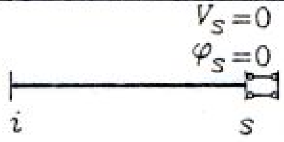
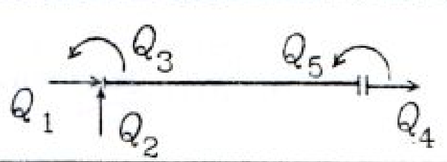
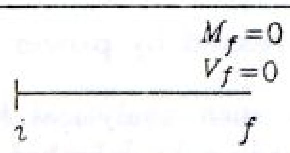
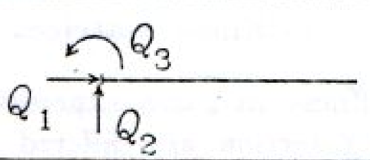
Type of member	End connections	Generalized displacements (forces)
k	 i, k	 Q_1, Q_2, Q_3, Q_4, Q_5, Q_6
g	 $M_g=0$; i, g	 Q_1, Q_2, Q_3, Q_4, Q_5
f	 $V_s=0$, $\varphi_s=0$; i, s	 Q_1, Q_2, Q_3, Q_4, Q_5
s	 $M_f=0$, $V_f=0$; i, f	 Q_1, Q_2, Q_3

Table 2.

The final forms of the stiffness matrices of members are:

Member of type g

$$\mathbf{k}_g = \frac{EI}{l^3}\begin{bmatrix} \frac{Al^2}{I} & 0 & 0 & -\frac{Al^2}{I} & 0 \\ & 3\phi_5 & 3l\phi_6 & 0 & -3\phi_5 \\ & & 3l^2\phi_6 & 0 & -3l\phi_6 \\ symmetrical & & & \frac{Al^2}{I} & 0 \\ & & & & 3\phi_5 \end{bmatrix},$$

Member of type s

$$\mathbf{k}_s = \frac{EI}{l}\begin{bmatrix} \frac{Al^2}{I} & 0 & 0 & -\frac{Al^2}{I} & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 \\ & & \phi_7 & 0 & 0 & -\phi_8 \\ & & & \frac{Al^2}{I} & 0 & 0 \\ symmetrical & & & & 0 & 0 \\ & & & & & \phi_7 \end{bmatrix},$$

Member of type f

$$\mathbf{k}_f = \frac{EI}{l}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\omega tg\omega \end{bmatrix}.$$

The functions $\phi_i, i = 5...8$ are given in Table 3.

Table 3.

Function	Axial force		
	compressive	$N = 0$	tensive
ϕ_5	$\frac{\omega^3 \cos\omega}{3\Delta_c}$	1	$\frac{-\omega^3 \mathrm{ch}\,\omega}{3\Delta_t}$
ϕ_6	$\frac{\omega^2 \sin\omega}{3\Delta_c}$	1	$-\frac{\omega^2 \mathrm{sh}\,\omega}{3\Delta_t}$
ϕ_7	$\omega tg\omega$	1	$-\omega cth\omega$
ϕ_8	$\frac{\omega}{\sin\omega}$	1	$-\frac{\omega}{sh\omega}$

$$\omega = l\sqrt{\frac{|N|}{El}}, \quad \Delta_c = \sin\omega - \omega\cos\omega, \quad \Delta_t = \mathrm{sh}\,\omega - \omega\mathrm{ch}\,\omega.$$

Stiffness matrices expressed by power series

The stiffness matrices expresed by such analytical functions as shown in the preceding section are differed according to whether the axial force N is positive or negative. Moreover, the coefficients of the stiffness matrices become indefinite when the axial force approaches zero. In that case, the stiffness matrix from the classical theory must be used. To avoid the mentioned disadvantages and numerical difficulty, it is suitable to have the stiffness matrices expressed by power series. These expressions are convenient for numerical analysis because those are the same regardless of whether axial force is tensile or compressive, and thay also are stable when axial force becomes zero. The functions ϕ_i $i = 1...8$ expressed by infinite series are shown in Appendix 1.

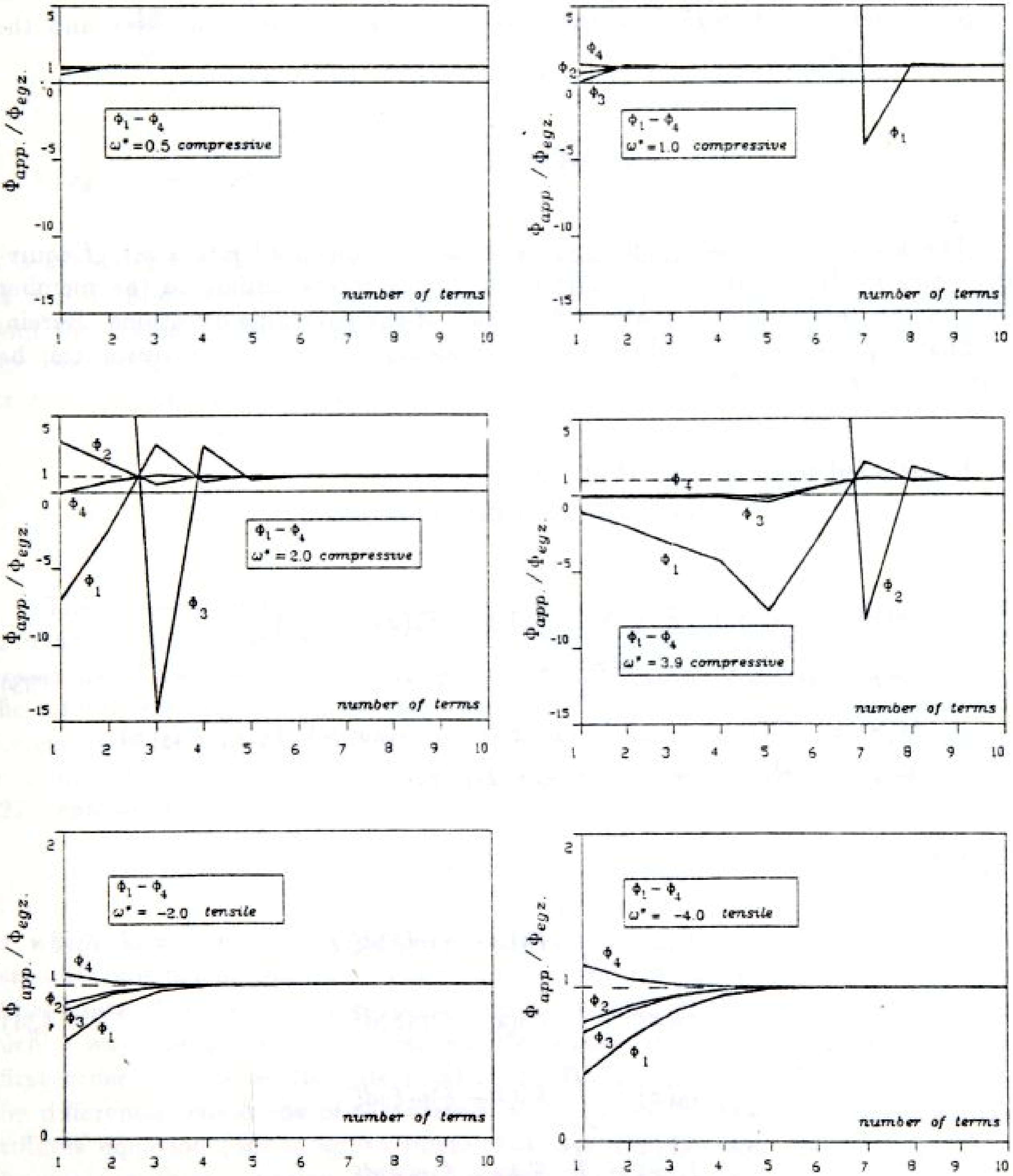


Fig. 4. Convergence of power series

The convergence of the infinite power series is very important for efficient numerical analysis. Therefore, this problem is especially examined and some of the results are shown in Fig. 4. As ϕ_i, $i = 1\ldots8$, are the functions of N, the necessary number of the series terms also depends on the axial force N. It is evident from Fig. 4. that the convergence of series is slower for the larger values of N. Moreover, the convergence of series is faster for tensile than for compressive axial force. For this reason, the necessary number of terms for numerical computation is necessary determined by examining the convergence of the power series with regard to the maximum compressive axial force and the required accuracy.

Nodal force vector

The loads applied along the member must be converted into a set of equivalent joint loads (nodal forces). The nodal forces corresponding to the member loads can be obtained from analytical solution of the governing equations. Herein, the final expressions from which the components of nodal force vector can be determined are given [16]:

Distributed member load (Fig. 2a)

$$\begin{aligned}
v(x) &= v_o + \varphi_o F_2(x) + M_o F_3(x) + V_o F_4(x) - I_4(x) , \\
\varphi(x) &= \varphi_o \cos kx - M_o \frac{k \sin kx}{N} + V_o F_3(x) - I_3(x) , \\
M(x) &= -EIv_{,xx} = \varphi_o EIk \sin kx + M_o \cos kx + V_o F_2(x) - I_2(x) , \\
V(x) &= -EIv_{,xxx} + Nv_{,x} = V_o - I_1(x) ,
\end{aligned} \tag{33}$$

in which

$$\begin{aligned}
I_1(x) &= \int_o^x F_1(x-\xi)p(\xi)d\xi , \\
I_2(x) &= \int_o^x F_2(x-\xi)p(\xi)d\xi , \\
I_3(x) &= \int_o^x F_3(x-\xi)p(\xi)d\xi , \\
I_4(x) &= \int_o^x F_4(x-\xi)p(\xi)d\xi .
\end{aligned} \tag{34}$$

Concentrated member load (Fig. 2b)

$$
\begin{aligned}
v(x) &= v_o + \varphi_o F_2(x) + M_o F_3(x) + V_o F_4(x) + \mid M F_3(x-a) - P F_4(x-a) , \\
\varphi(x) &= \varphi_o \cos kx - \frac{M_o}{N} k \sin kx + V_o F_3(x) + \mid - \\
&- \frac{M}{N} k \sin k(x-a) - P F_3(x-a) , \qquad (35) \\
M(x) &= \varphi_o EIk \sin kx + M_o \cos kx + V_o F_2(x) + \\
&+ \mid M \cos k(x-a) - P F_2(x-a) , \\
V(x) &= V_o + \mid -P .
\end{aligned}
$$

It is necessary to take into consideration the terms after vertical line in (35) only for $x > a$. End actions of the member can be obtained from (33) and (35) and from the corresponding boundary conditions. The end actions are transformed into the nodal forces by a simple change in sign. The componentes of the nodal force vector for various member loads are shown in Appendix 2.

Simplified stiffness equation

The stiffness matrices obtained in the preceding section can be accepted as exact as they have been derived from the analytical solution of differential equations of the second-order theory. Consiqently, the matrices expressed by the infinite expansions, can be used whatever the quantity of axial force my be. Except that form of the stiffness equation, another simplified form exists which is commonly used. In such a case, the matrix stiffness equation analogous to (27), can be written as

$$
\mathbf{R} = (\mathbf{k}_0 + \mathbf{k}_G)\,\mathbf{q} - \mathbf{Q} \qquad (36)
$$

in which $\mathbf{k}_0$= conventional stiffness matrix, $\mathbf{k}_G$= geometric striffness matrix (stability coefficient matrix). The geometric matrix can be obtained following usual finite element procedure, assuming polynomial displacement functions. In such a way, using the polynomials satisfying the differential equations of linear (first-order) theory as the interpolation functions, the approximate solution of the differential equations of the second-order theory is obtained. Therefore, the stiffness equation (36) is approximate, but it is simplier and easier to use than the exact stiffnes equation. The procedure for finding the geometrical matrix is herein omitted. Only the final expressions for various types of members are given:

Member of type k

$$\mathbf{k}_{Gk} = \frac{N}{30l}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ & 36 & 3l & 0 & -36 & 3l \\ & & 4l^2 & 0 & -3l & -l^2 \\ & & & 0 & 0 & 0 \\ & \text{symmetrical} & & & 36 & -3l \\ & & & & & 4l^2 \end{bmatrix} \tag{37}$$

Member of type g

$$\mathbf{k}_{Gg} = \frac{N}{5l}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ & 6-\lambda & (1-\lambda)l & 0 & -(6-\lambda) \\ & & (1-\lambda)l^2 & 0 & -(1-\lambda)l \\ \text{symmetrical} & & & 0 & 0 \\ & & & & 6-\lambda \end{bmatrix} \tag{38}$$

$$\lambda = \frac{3\omega^2}{8(30+\omega^2)}$$

Member of type s

$$\mathbf{k}_{Gs} = \frac{N}{6l}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 \\ & & 2-\bar{\lambda} & 0 & 0 & (1-\bar{\lambda})l \\ & \text{symmetrical} & & 0 & 0 & 0 \\ & & & & 0 & 0 \\ & & & & & 2-\bar{\lambda} \end{bmatrix} \tag{39}$$

$$\bar{\lambda} = \frac{5\omega^2}{4(10+\omega^2)}$$

Member of type f

$$\mathbf{k}_{Gf} = Nl\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1-\lambda}{6-\lambda} \end{bmatrix}, \quad \lambda = \frac{3\omega^2}{8(30+\omega^2)} \, . \tag{40}$$

Geometric matrices given by (37-40), can also be obtained from the corresponding stiffness matrices expressed by power series retaining the first two terms (Allen and Bulson, 1980). Eq. 36 has quite a simple form and is suitable for the eigenvalue analysis of frame buckling. In the stress- strain frame analysis Eq. 36 can be used only for moderate value of the axial force, N, and lateral load, $p(x)$. When, N, or, $p(x)$, become large, Eq. 36 loses its accuracy and, in such a case, a member must be divided into smaller elements. The division of a member has a disadvantage as it considerably increases the member of equations (degrees of freedom) in structural analysis.

Solution procedure

The structural stiffness equation can be obtained using the usual assembling procedure. This equation is nonlinear algebraic equation because the member stiffness equations are nonlinear. The axial force is present in both the member stiffness matrix and the nodal force vector. An iterative procedure is necessary to solve such problems. There are more than one solution procedure for geometrically nonlinear structural analysis, which have been described in many references (see, for example [10]).

Herein, a simplified procedure may be applied by utilizing the two following qualities: 1) the member stiffness equation is nonlinear only in terms of axial force, which is the linear function of the nodal displacements 2) the member axial force calculated by the second-order theory usually does not differ greatly from the corresponding force calculated by the first-order theory. It has been made posible to reduce the number of cycles in the iteration procedure to just one. The value of member axial force may be obtained from the first-order analysis (using the same second-order formulation with $N = 0$). After the axial forces are obtained it is posible to calculate stiffness matrix and force vector for each member, and to attain structural stiffness equation. In the next, final step, it is necessary to solve stiffness equation and to determine the nodal displacements and member forces. Regarding applicability to structural design analysis, the procedure with only one cycle is the simplest and, at the same time, mainly accurate enough for practical purposes.

Bifurcation stability analysis

Bifurcation stability of frame structures may be examined by employing the homogeneous structural equation of the second-order theory. According to the well-known stability criterion, the structure becomes unstable at the load intensity ω (bifurcation load) when structural (tangent) stiffness matrix becomes singular. The critical buckling load is determined by finding the smallest value of ω for which the determinant of the structural stiffness matrix is equal to zero, i.e.

$$det\ \mathbf{K}(\omega) = 0\ . \tag{41}$$

This procedure gives exact values for critical load. However, a large quantity of caluclation is generally involved because the elements of stiffness matrix are transcendental functions of ω. Therefore, for investigation of the bifurcation stability problem, the simplified form of stiffness equation is much more convinient to use. Then, according to the stability criterion, it follows:

$$det\ [\mathbf{K}_0 + \omega_c \mathbf{K}_G] = 0\ . \tag{42}$$

in which ω_c is the critical (least) intensity of load. Eq. 42 has the form of the classical eigenvalue problem. The eigenvector corresponding to the eigenvalue ω_c gives the mode shape of the system and can be determined from the following equation

$$[\mathbf{K}_0 + \omega_c \mathbf{K}_G]\, \bar{\mathbf{q}} = 0 \; ,$$

in which $\bar{\mathbf{q}}$ is the generalized displacement eigenvector. The critical load obtained from (42) is approximate, but usually accurate enough for a wide range of frame structures.

Computer program and numerical results

On the basis of teoretical consideration presented in the preceding sections, the programe (NAK–2) for PC has been developed. The program is intended for static and buckling analyses of plane frame structures subjected to external load and the temperature influences. This program creates opportunity for two kinds of the second–order analysis: "exact" and "simplified". The first– order analysis as a special case of the second–order analysis is contained, too. The critical (bifurcation) load and the corresponding buckling form can be determined.

Using this program two of typical plane frames have been analyzed.

Portal frame - The structure–load system and load–deflection curves are shown in Fig. 5. The solution is compared with the solutions obtained using finite element analysis (FEA) and the geometrically nonlinear analysis. It is obvious that it is well in agreement with these solutions. The load- deflection curves for various values of lateral (horizontal) load is shown in Fig. 5b. and compared with the solution given by Conor et al. [6] (the second–order theory with taking into consideration the bowing effects). It is evident that these two solutions are practically the same, thus, neglecting the bowing effects has been justified. The bifurcation load for the frame is determined.

Two–story frame - The second analyzed example is two–story and two– bay frame with uniformly distributed load and concentrated lateral loads (Fig. 6). The load–deflection curve for the horizontal displacement on the top of frame is shown in Fig. 6. The critical load is determined and the corresponding buckling form is shown in Fig. 6. The comparative results obtained by the second–order theory and the first–order theory for maximum column moment (at node 1), according to different load levels are given in Table 4.

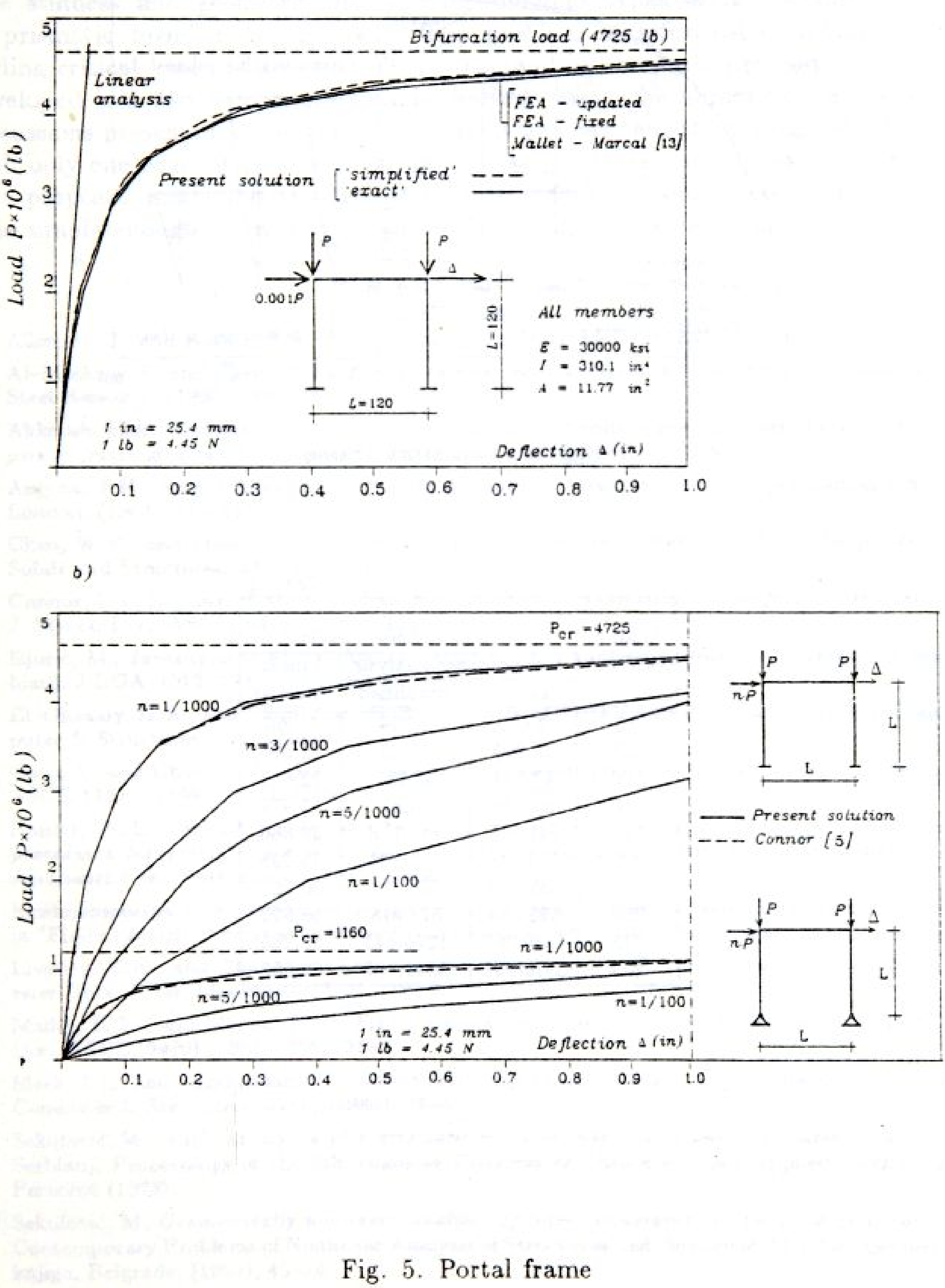


Fig. 5. Portal frame

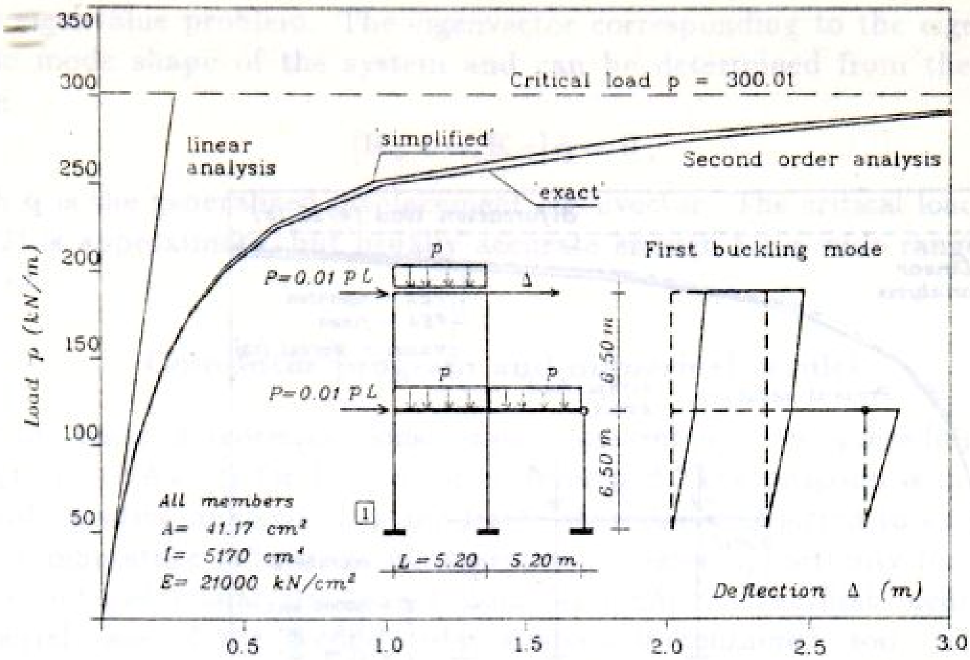


Fig. 6. Two story frame

Table 4.

Load p (kN/m)	Moment at node 1 (kNm)		
	Second order analysis		Linear analysis
	'exact'	'simplified'	
10	2.401	2.399	2.265
50	15.299	15.23	11.325
100	42.834	42.741	22.649
150	95.689	95.42	33.974
175	141.681	141.178	39.636
200	213.743	212.104	45.298
225	339.359	333.614	50.961
250	598.064	577.018	56.622
275	1389.267	1275.843	62.285
290	3810.537	3031.167	65.682

Conclusion

The elastic second-order analysis of the plane frames has been considered. The matrix formulation for two basic kinds of the second-order analysis: the stability matrix approach and the geometric matrix approach are presented. The stiffness and geometric matrices for different types of nodal connections of prismatic member are derived. The problem of bifurcation stability and finding critical load is discussed. The corresponding computer program has been developed and two numerical examples solved. From the numerical results and discussions presented in the preceding text, it appears that the proposed method, with only one element per member, may be used with reasonably good accuracy. The proposed method is strongly theoretically founded and, at the same time, it is simple enough, so it may be recommended for design analysis.

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ТЕОРИЯ ВТОРОГО ПОРЯДКА ДЛЯ АНАЛИЗА РАМНЫХ КОНСТРУКЦИЙ

Целью настоящеы работы является консистентное изложение матричной формулировки Теории второго порядка с применением в практическом анализе и расчете рамных конструкций. Изложена вариационная формулировка Теории второго порядка, как специальный случай общего геометрического нелинейного анализа.

Пренебрегая эффектом сопряжения аксиальных и попречнах деформаций (bowing effect) из общих выиражений получена упрощенная форма дифференциальных уравнений задачи. Выведены матрицы жесткости, геометрическая матрица и вектор эквивалентных сил в узлах для различнных случаев краевых условий на концах стержня. В качестве интерполяционных функций использованы полиномы и тригонометрические (гиперболические) функции, соотвецтвующие решению линеаризованной Теории второго порядка.

Разработана общая программа для компьютера, применение которой илюстрировано на нескольких конкретнах примерах. Решения сопоставлены соотвецтвующими решениями общего геометрически нелинейного анализа и с решениями классической линейной теории.

TEORIJA DRUGOG REDA ZA ANALIZU OKVIRNIH KONSTRUKCIJA

Cilj ovog rada je konzistentan prikaz matrične Teorije drugog reda sa primenom u praktičnoj analizi i proračunu okvirnih konstrukcija. Izložena je varijaciona formulacija Teorije drugog reda kao specijalan slučaj opšte geometrijski nelinearne analize.

Zanemarujući efekte sprezanja aksijalnih i transverzalnih deformacija (bowing effect) iz opštih izraza dobijen je pojednostavljen oblik diferencijalnih jednačina problema. Izvedene su matrica krutosti, geometrijska matrica i vektor ekvivalentnih sila u čvorovima za različite slučajeve konturnih uslova na krajevima štapa. Za interpolacione funkcije korišćene su polinomi i trigonometrijske (hiperboličke) funkcije koje odgovaraju rešenju linearizovane teorije drugog reda.

Razvijen je opšti program za računar čija je primena ilustrovana na nekoliko konkretnih primera. Rešenja su upoređena sa odgovarajućim rešenjima opšte geometrijske nelinearne analize i sa rešenjima klasične linearne teorije.

APPENDIX I.

- NOTATION

The following symbols are used in this paper:

ε = extension in x direction

κ = bending deformation

ε_0 = initial extension in x direction

κ_0 = initial bending deformation

A = cross–section area

I = moment of inertia

E = Young's modulus

α = coefficient of thermal expansion

ω = axial force parameter

$\mathbf{R}$ = generalized force vector

$\mathbf{q}$ = generalized displacement vector

$\mathbf{Q}$ = generalzied loads vector

$\mathbf{k}$ = member stiffness matrix

$\mathbf{k}_G$ = member geometric stiffness matrix

$\mathbf{k}_g$ = stiffness matrix for member of type q

$\mathbf{k}_f$ = stiffness matrix for member of type f

$\mathbf{k}_s$ = stiffness matrix for member of type s

U = strain energy

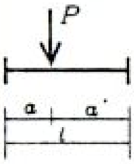
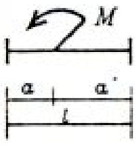
W = work of external forces

Π = total potential energy

APPENDIX II.

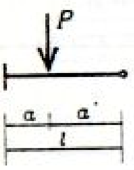
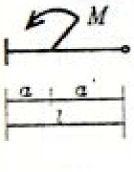
- NODAL FORCE VECTOR FOR DIFFERENT MEMBER TYPES

II. 1. - Member of type 'k'

Load	Components of nodal force vector	
	compressive	tensile
Distributed loads and temperature changes	$Q_1 = 0$ $Q_2 = \frac{N}{\Delta}[\ \frac{\omega}{l}\sin\omega I_4 - (1 - \cos\omega)I_3\ ]$ $Q_3 = \frac{-N}{\Delta}[\ \frac{l}{\omega}(\omega - \sin\omega)I_3 - (1 - \cos\omega)I_4\ ]$ $Q_4 = 0$ $Q_5 = \frac{-N}{\Delta}[\ \frac{\omega}{l}\sin\omega I_4 - (1 - \cos\omega)I_3\ ] + I_1$ $Q_6 = \frac{N}{\Delta}[\ \frac{l}{\omega}(\omega\cos\omega - \sin\omega)I_3 + (1 - \cos\omega)I_4\ ] - I_2$	$Q_1 = 0$ $Q_2 = \frac{N}{\Delta}[\ \frac{\omega}{l}\mathrm{sh}\omega I_4 - (1 - \mathrm{ch}\omega)I_3\ ]$ $Q_3 = \frac{-N}{\Delta}[\ \frac{l}{\omega}(\omega - \mathrm{sh}\omega)I_3 - (1 - \mathrm{ch}\omega)I_4\ ]$ $Q_4 = 0$ $Q_5 = \frac{-N}{\Delta}[\ \frac{\omega}{l}\mathrm{sh}\omega I_4 - (1 - \mathrm{ch}\omega)I_3\ ] + I_1$ $Q_6 = \frac{N}{\Delta}[\ \frac{l}{\omega}(\omega\mathrm{ch}\omega - \mathrm{sh}\omega)I_3 + (1 - \mathrm{ch}\omega)I_4\ ] - I_2$
	$Q_1 = 0$ $Q_2 = \frac{P}{\Delta}[(ka' - \sin ka')\sin\omega$ $- (1 - \cos ka')(1 - \cos\omega)]$ $Q_3 = \frac{Pl}{\Delta\omega}[(1 - \cos\omega)(ka' - \sin ka')$ $- (\omega - \sin\omega)(1 - \cos ka')]$ $Q_4 = 0$ $Q_5 = \frac{P}{\Delta}[(ka - \sin ka)\sin\omega$ $- (1 - \cos ka)(1 - \cos\omega)]$ $Q_6 = \frac{Pl}{\Delta\omega}[(1 - \cos\omega)(ka - \sin ka)$ $- (\omega - \sin\omega)(1 - \cos ka)]$	$Q_1 = 0$ $Q_2 = \frac{P}{\Delta}[(ka' - \mathrm{sh}ka')\mathrm{sh}\omega$ $- (1 - \mathrm{ch}ka')(1 - \mathrm{ch}\omega)]$ $Q_3 = \frac{Pl}{\Delta\omega}[(\mathrm{ch}\omega - 1)(ka' - \mathrm{sh}ka')$ $+ (\omega - \mathrm{sh}\omega)(1 - \mathrm{ch}ka')]$ $Q_4 = 0$ $Q_2 = \frac{P}{\Delta}[(ka - \mathrm{sh}ka)\mathrm{sh}\omega$ $- (1 - \mathrm{ch}ka)(1 - \mathrm{ch}\omega)]$ $Q_3 = \frac{Pl}{\Delta\omega}[(\mathrm{ch}\omega - 1)(ka - \mathrm{sh}ka)$ $+ (\omega - \mathrm{sh}\omega)(1 - \mathrm{ch}ka)]$
	$Q_1 = 0$ $Q_2 = \frac{Ml}{\Delta\omega}[(\sin\omega(1 - \cos ka') -$ $- (1 - \cos\omega)\sin ka']$ $Q_3 = \frac{M}{\Delta}[(1 - \cos\omega)(1 - \cos ka')$ $- (\omega - \sin\omega)\sin ka']$ $Q_4 = 0$ $Q_5 = \frac{Ml}{\Delta\omega}[(\sin\omega(1 - \cos ka) -$ $- (1 - \cos\omega)\sin ka]$ $Q_6 = \frac{M}{\Delta}[(1 - \cos\omega)(1 - \cos ka)$ $- (\omega - \sin\omega)\sin ka]$	$Q_1 = 0$ $Q_2 = \frac{Ml}{\Delta\omega}[(\mathrm{sh}\omega(\mathrm{ch}ka' - 1) -$ $+ (1 - \mathrm{ch}\omega)\mathrm{sh}ka']$ $Q_3 = \frac{M}{\Delta}[(1 - \mathrm{ch}\omega)(1 - \mathrm{ch}ka')$ $- (\omega - \mathrm{sh}\omega)\mathrm{sh}ka']$ $Q_4 = 0$ $Q_5 = \frac{Ml}{\Delta\omega}[(\mathrm{sh}\omega(\mathrm{ch}ka - 1) -$ $+ (1 - \mathrm{ch}\omega)\mathrm{sh}ka]$ $Q_6 = \frac{M}{\Delta}[(1 - \mathrm{ch}\omega)(1 - \mathrm{ch}ka)$ $- (\omega - \mathrm{sh}\omega)\mathrm{sh}ka]$

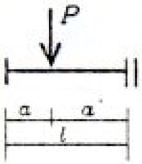
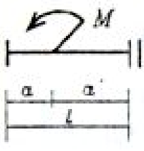
$\Delta = 2(1 - \cos\omega) - \omega\sin\omega$ $\qquad$ $\Delta = 2(1 - \mathrm{ch}\omega) + \omega\mathrm{sh}\omega$

II. 2. - Member of type 'g'

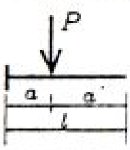
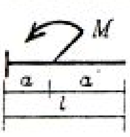
Load	Components of nodal force vector	
	compressive	tensile
Distributed loads and temperature changes	$Q_1 = 0$ $Q_2 = \frac{\omega \sin\omega}{\Delta_g l}[I_2(1 - \cos\omega) + N I_4 \cos\omega]$ $Q_3 = \frac{-1}{\Delta_g}[N I_4 \sin\omega + I_2(\omega - \sin\omega)]$ $Q_4 = 0$ $Q_5 = \frac{\omega \sin\omega}{\Delta_g l}[I_2(1 - \cos\omega) + N I_4 \cos\omega] - I_1$	$Q_1 = 0$ $Q_2 = \frac{\omega \,\mathrm{sh}\omega}{\Delta_g l}[I_2(1 - \mathrm{ch}\omega) + N I_4 \mathrm{ch}\omega]$ $Q_3 = \frac{-1}{\Delta_g}[N I_4 \mathrm{sh}\omega + I_2(\omega - \mathrm{sh}\omega)]$ $Q_4 = 0$ $Q_5 = \frac{\omega \,\mathrm{sh}\omega}{\Delta_g l}[I_2(1 - \mathrm{ch}\omega) + N I_4 \mathrm{ch}\omega] - I_1$
	$Q_1 = 0$ $Q_2 = \frac{P}{\Delta}[(1 - \cos\omega)\sin ka' -$ $- (ka' - \sin ka')\cos\omega]$ $Q_3 = \frac{Pl}{\Delta\omega}[(\omega - \sin\omega)\sin ka'$ $- (ka' - \sin ka')\cos\omega]$ $Q_4 = 0$ $Q_5 = \frac{P}{\Delta}[(1 - \cos\omega)\sin ka' -$ $- (ka' - \sin ka')\cos\omega] - P$	$Q_1 = 0$ $Q_2 = \frac{P}{\Delta}[(\mathrm{ch}\omega(ka' - \mathrm{sh}ka') +$ $+ \mathrm{ch}\omega(ka' - \mathrm{sh}ka')]$ $Q_3 = \frac{Pl}{\Delta\omega}[(\omega - \mathrm{sh}\omega)\mathrm{sh}ka')$ $- \mathrm{sh}\omega(ka' - \mathrm{sh}ka')]$ $Q_4 = 0$ $Q_5 = \frac{P}{\Delta}[(\mathrm{ch}\omega(ka' - \mathrm{sh}ka') +$ $+ \mathrm{ch}\omega(ka' - \mathrm{sh}ka')] - P$
	$Q_1 = 0$ $Q_2 = \frac{M\omega}{\Delta l}[(1 - \cos\omega)\cos ka'$ $- \cos\omega(1 - \cos ka')]$ $Q_3 = \frac{M}{\Delta}[\cos ka'(\omega - \sin\omega)$ $- (\sin\omega(1 - \cos ka')]$ $Q_4 = 0$ $Q_5 = \frac{-M\omega}{\Delta l}[(1 - \cos\omega)\cos ka'$ $- \cos\omega(1 - \cos ka')]$	$Q_1 = 0$ $Q_2 = \frac{M\omega}{\Delta l}[(1 - \mathrm{ch}\omega)\mathrm{ch}ka'$ $- \mathrm{ch}\omega(1 - \mathrm{ch}ka')]$ $Q_3 = \frac{M}{\Delta}[\mathrm{sh}\omega(1 - \mathrm{ch}ka')$ $- (\omega - \mathrm{sh}\omega)\mathrm{ch}ka')]$ $Q_4 = 0$ $Q_5 = \frac{-M\omega}{\Delta l}[(1 - \mathrm{ch}\omega)\mathrm{ch}ka'$ $- \mathrm{ch}\omega(1 - \mathrm{ch}ka')]$

$\Delta_g = \omega \cdot \cos\omega - \sin\omega$ $\qquad$ $\Delta_g = \mathrm{sh}\omega - \omega\,\mathrm{ch}\omega$

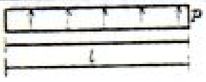
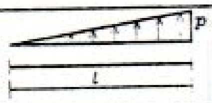
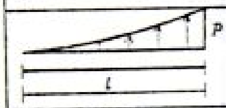
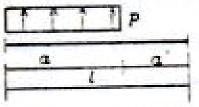
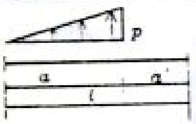
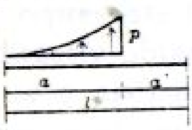
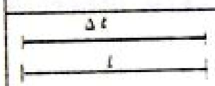
II. 3. - Member of type 's'

Load	Components of nodal force vector	
	compressive	tensile
Distributed loads and temperature changes	$Q_1 = 0$ $Q_2 = I_1$ $Q_3 = \frac{-l}{\omega \sin\omega}[N I_3 + I_1(1 - \cos\omega)]$ $Q_4 = 0$ $Q_6 = \frac{l}{\omega \sin\omega}[N I_3 \cos\omega + I_1(1 + \cos\omega)] - I_2$	$Q_1 = 0$ $Q_2 = I_1$ $Q_3 = \frac{-l}{\omega sh\omega}[N I_3 + I_1(1 - ch\omega)]$ $Q_4 = 0$ $Q_6 = \frac{l}{\omega sh\omega}[N I_3 ch\omega + I_1(1 + ch\omega] - I_2$
	$Q_1 = 0$ $Q_2 = P$ $Q_3 = \frac{Pl}{\sin\omega\omega}[(\cos\omega - \cos ka')$ $Q_4 = 0$ $Q_6 = \frac{Pl}{\omega \sin\omega}[1 - \cos(\omega - ka')]$	$Q_1 = 0$ $Q_2 = P$ $Q_3 = \frac{Pl}{\sin\omega\omega}[(ch\omega - ch ka')$ $Q_4 = 0$ $Q_6 = \frac{Pl}{sh\omega\omega}[(1 - ch(\omega - ka')]$
	$Q_1 = 0$ $Q_2 = 0$ $Q_3 = \frac{M \sin ka'}{\sin\omega}$ $Q_4 = 0$ $Q_6 = \frac{-M \sin ka}{\sin\omega}$	$Q_1 = 0$ $Q_2 = 0$ $Q_3 = \frac{M sh ka'}{sh\omega}$ $Q_4 = 0$ $Q_6 = \frac{-M sh ka}{sh\omega}$

II. 4. - Member of type 'f'

Load	Components of nodal force vector	
	compressive	tensile
Distributed loads and temperature changes	$Q_1 = 0$ $Q_2 = I_1$ $Q_3 = \frac{-1}{\cos\omega}[I_2 - I_1 \frac{\sin\omega}{k}]$	$Q_1 = 0$ $Q_2 = I_1$ $Q_3 = \frac{1}{ch\omega}[I_2 - I_1 \frac{sh\omega}{k}]$
	$Q_1 = 0$ $Q_2 = P$ $Q_3 = \frac{-Pl}{\omega\cos\omega}[\sin ka' - \sin\omega]$	$Q_1 = 0$ $Q_2 = P$ $Q_3 = \frac{Pl}{\omega ch\omega}[sh\omega - sh ka']$
	$Q_1 = 0$ $Q_2 = 0$ $Q_3 = \frac{M \cos ka'}{\cos\omega}$	$Q_1 = 0$ $Q_2 = 0$ $Q_3 = \frac{M ch ka'}{ch\omega}$

II. 5. - Integrals I_i, i=1,..,4 ($x = l$, end of member)

Load	I_1	I_2	I_3	I_4
	pl	$\frac{pl^2}{\omega^2}(1-\cos\omega)$	$\frac{-pl}{N\omega}(\omega - \sin\omega)$	$\frac{pl^2}{N\omega^2}(1-\cos\omega - \frac{\omega^2}{2})$
	$\frac{pl}{2}$	$\frac{pl^2}{\omega^3}(\omega - \sin\omega)$	$\frac{-pl}{N\omega^2}(\frac{\omega^2}{2} - 1 + \cos\omega)$	$\frac{pl^2}{N\omega^3}(\omega - \sin\omega - \frac{\omega^3}{6})$
	$\frac{pl}{3}$	$\frac{pl^2}{\omega^4}(\omega^2 - 2 + 2\cos\omega)$	$\frac{-pl}{N\omega^3}(\frac{\omega^3}{3} - 2\omega + 2\sin\omega)$	$\frac{pl^2}{N\omega^4}(\omega^2 - 2 + 2\cos\omega - \frac{\omega^4}{12})$
	pa	$\frac{p}{k^2}[\cos k(l-a) - \cos\omega]$	$\frac{-pl}{N\omega}[ka + \sin k(l-a) - \sin\omega]$	$\frac{-pl^2}{N\omega^2}[wka - \frac{k^2a^2}{2} - \cos k(l-a) + \cos\omega]$
	$\frac{pa}{2}$	$\frac{p}{k^3a}[ka\cos k(l-a) + \sin k(l-a) - \sin\omega]$	$\frac{-p}{aNk^2}[\frac{k^2a^2}{2} + ka\sin k(l-a) - \cos k(l-a) + \cos\omega]$	$\frac{-p}{k^3aN}[\frac{wk^2a^2}{2} - \frac{k^3a^3}{3} - ka\cos k(l-a) - \sin k(l-a) + \sin\omega]$
	$\frac{pa}{3}$	$\frac{p}{k^4a^2}[k^2a^2\cos k(l-a) + 2ka\sin k(l-a) - 2\cos k(l-a) + 2\cos\omega]$	$\frac{-p}{k^3a^2N}[\frac{k^3a^3}{3} + k^2a^2\sin k(l-a) - 2ka\cos k(l-a) - 2\sin k(l-a) + 2\sin\omega]$	$\frac{-p}{k^4a^2N}[\frac{wk^3a^3}{3} - \frac{k^4a^4}{4} - k^2a^2\cos k(l-a) - 2ka\sin k(l-a) + 2\cos k(l-a) - 2\cos\omega]$
	0	$\frac{EI\alpha\Delta t}{h}(1 - \cos\omega)$	$\frac{\alpha\Delta t}{hk}\sin\omega$	$\frac{\alpha\Delta t}{hk^2}(1 - \cos\omega)$

APPENDIX III.

- FUNCTIONS $\phi_i, i = 1...8.$ EXPRESSED IN THE FORM OF POWER SERIES

$$\phi_1 = \frac{1}{12\phi}\left[1 + \sum_{n=1}^{\infty}\frac{1}{(2n+1)!}(\pm\omega^2)^n\right] ,$$

$$\phi_2 = \frac{1}{6\phi}\left[\frac{1}{2} + \sum_{n=1}^{\infty}\frac{1}{(2n+2)!}(\pm\omega^2)^n\right] ,$$

$$\phi_3 = \frac{1}{4\phi}\left[\frac{1}{3} + \sum_{n=1}^{\infty}\frac{2(n+1)}{(2n+3)!}(\pm\omega^2)^n\right] ,$$

$$\phi_4 = \frac{1}{2\phi}\left[\frac{1}{6} + \sum_{n=1}^{\infty}\frac{1}{(2n+3)!}(\pm\omega^2)^n\right] ,$$

$$\phi_5 = \frac{1}{3\phi_g}\left[1 + \sum_{n=1}^{\infty}\frac{1}{(2n)!}(\pm\omega^2)^n\right] ,$$

$$\phi_6 = \frac{1}{3\phi_g}\left[1 + \sum_{n=1}^{\infty}\frac{1}{(2n+1)!}(\pm\omega^2)^n\right] ,$$

$$\phi_7 = \frac{1}{\phi_s}\left[1 + \sum_{n=1}^{\infty}\frac{1}{(2n)!}(\pm\omega^2)^n\right] ,$$

$$\phi_8 = \frac{1}{\phi_5} ,$$

where

$$\phi = \frac{1}{12} + \sum_{n=1}^{\infty}\frac{2(n+1)}{(2n+4)!}(\pm\omega^2)^n ,$$

$$\phi_g = \frac{1}{3} + \sum_{n=1}^{\infty}\frac{2(n+1)}{(2n+1)!}(\pm\omega^2)^n ,$$

$$\phi_s = 1 + \sum_{n=1}^{\infty}\frac{1}{(n+1)!}(\pm\omega^2)^n .$$

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