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A NUMERICAL SOLUTION OF THE DIFFERENTIAL EQUATION OF VIBRATION

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1. Introduction

The differential equation of vibration of a single degree of freedom system has been a subject of investigation by many authors. A particular problem represents the solution of this equation in the case of arbitrary external load, like earthquake ground motion, and the analysis of non-linear systems. In such cases, a numerical step-by-step method of solution has to be applied. For solution of this problem there are several numerical methods. The most of them are based on the assumption of constant or linear variation of the response acceleration during the time step of integration. Between the most well known methods of solution are the Newmark's β method, the Wilson's Θ method and the Clough's method of linear variation of the acceleration [1].

However, the application of these methods requires short time intervals of integration, resulting in large number of integrations, and sometimes the solution could be unstable. This is particularly true in the case of non-linear analysis and short period systems. The Willson's Θ method is considered as unconditionally stable, but its application implies an iteration process and long computational time of solution. A simple and unconditionally stable method of solution would have some advantages over the presently available methods of solution of this problem. Such a method should be the one which is going to be presented in this paper.

The method is based on the numerical solution of the differential equation of the problem. The procedure of solution is similar to the derivation of one-dimensional finite elements for analysis of beam problem, which recently also has been applied in the development of two- and three-dimensional elements [2,3]. The derivatives of the differential equations are approximated by the derivatives of the interpolation function, yielding the time finite element matrix. The one element solution gives the step solution. The analogous one-dimensional beam bending and stability element always gives excellent or even exact results, and consequently, the time finite element for solution of the vibration problem, also should give very good results.

2. The method of solution

As already has been mentioned, the method of solution is similar to the derivation of finite elements and therefore it could be called time finite element method. The time interval of integration is represented by one element and the derived element gives the solution of the problem at the time step.

2.1 Derivation of the solution

The differential equation of equilibrium of the forces acting on a mass of a single degree of freedom system is as follows,

$$m\ddot{u} + c\dot{u} + ku = p \tag{2.1}$$

where m is the mass of the system, c is damping coefficient, k is stiffness, p is external load, function of the time, u is relative displacement, and the dots mean derivatives on the time. In the case of multi-degree of freedom systems the differential equation in matrix form will be the same. The solution of such system can be represented in terms of the solution of single degree of freedom system. Thus, the solution of single degree of freedom system presented here, can be applied on the solution of multi-degree of freedom systems also.

Equation 2.1 with the first term on the left side only, is the same as the beam bending equation, with beam stiffness EI instead of m and distributed load p. With the third term on the left side added the differential equation defines the beam stability problem. The derivation of the time finite element should be similar to the derivation of the beam element.

By differentiation of eq.2.1 twice is derived,

$$m\frac{\mathrm{d}^4u}{\mathrm{d}t^4} + c\frac{\mathrm{d}^3u}{\mathrm{d}t^3} + k\frac{\mathrm{d}^2u}{\mathrm{d}t^2} = \frac{\mathrm{d}^2p}{\mathrm{d}t^2} \tag{2.2}$$

The main point in the numerical solution of differential equations by the applied method, which yields finite elements, is the approximation of the derivatives of the equation as a product of lower order derivatives of the interpolation function. The derivatives of this equation can be approximated as follows,

$$\begin{aligned}
\frac{\mathrm{d}^2u}{\mathrm{d}t^2} &= -\frac{\mathrm{d}\phi}{\mathrm{d}t}\frac{\mathrm{d}u}{\mathrm{d}t}\\
\frac{\mathrm{d}^3u}{\mathrm{d}t^3} &= -\frac{\mathrm{d}^2\phi}{\mathrm{d}t^2}\frac{\mathrm{d}u}{\mathrm{d}t}\\
\frac{\mathrm{d}^4u}{\mathrm{d}t^4} &= -\frac{\mathrm{d}^2\phi}{\mathrm{d}t^2}\frac{\mathrm{d}^2u}{\mathrm{d}t^2}
\end{aligned} \tag{2.3}$$

where ϕ is assumed interpolation function, $u = \phi\, u_i$ and u_i are nodal parameters. The best assumption for this function is the solution of the homogeneous portion of the differential equation, or the highest order polynomial which satisfies the homogeneous portion of the differential equation [4]. In this case that is the following third order polynomial,

$$u = a_1 + a_2\, t + a_3\, t^2 + a_4\, t^3 \tag{2.4}$$

The coefficients of this expression can be defined by the values of the displacement u and velocity (first derivative) at the beginning and the end of the time interval Δt ("nodal" values), which is equivalent to their defining in the case of stiffness beam element. Another approach could the assumption of the displacements and accelerations as primary unknowns, which is equivalent to the mixed beam element. The expression of the coefficients of the polynomial 2.4 in terms of the nodal displacements u and velocity $\dot{u}$ yields the well known beam function, or Hermitian polynomial, as follows,

$$\begin{aligned} u =& u_1\,(1 - 3t^2/\Delta t^2 + 2t^3/\Delta t^3) + u_2\,(3t^2/\Delta t^2 - 2t^3/\Delta t^3) + \\ &+ \dot{u}_1\,(t - 2t^2/\Delta t + t^3/\Delta t^2) + \dot{u}_2\,(-t^2/\Delta t + t^3/\Delta t^2) \end{aligned} \tag{2.5}$$

The interpolation function ϕ is defined by the associated terms to the nodal parameters of this polynomial.

The first term of eq. 2.2, approximated as eq. 2.3c provides, and integrated along the time Δt, yields,

$$Md = \frac{m}{\Delta t}\begin{bmatrix} 12/\Delta t^2 & -12/\Delta t^2 & 6/\Delta t & 6/\Delta t \\ & 12/\Delta t^2 & -6/\Delta t & 6/\Delta t \\ & & 4 & 2 \\ \text{Symm.} & & & 4 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ \dot{u}_1 \\ \dot{u}_2 \end{Bmatrix} \tag{2.6}$$

This is the same matrix as the well known beam stiffness matrix, with $EI = m$ and beam length $l = \Delta t$. In the standard slope-deflection method, or the finite element method, this matrix always yields exact results, regardless of the type of loading and the number of elements of subdivision.

The second term of eq. 2.2. which gives the contribution of the damping, approximated as eq. 2.3b provides, and integrated along the time Δt, yields the following matrix,

$$C = \frac{c}{\Delta t}\begin{bmatrix} 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & -\Delta t/2 & -\Delta t/2 \\ 1 & 1 & -\Delta t/2 & -\Delta t/2 \end{bmatrix} \tag{2.7}$$

The third term of eq. 2.2, approximated as eq. 2.3a provides, and integrated, yields the following matrix,

$$K = \frac{k}{\Delta t}\begin{bmatrix} -1.2 & 1.2 & -0.1\Delta t & -0.1\Delta t \\ & -1.2 & 0.1\Delta t & 0.1\Delta t \\ & & -2\Delta t^2/15 & \Delta t^2/30 \\ \text{Symm.} & & & -2\Delta t^2/15 \end{bmatrix} \tag{2.8}$$

This is also a well known matrix, the geometry stiffness matrix, used in the analysis of the stability of beams subjected to compression by an axial force N, which in this case is $N = k$. The use of the eqs. 2.6 and 2.8 in the analysis of the stability problem, yields critical force with error of the order of one percent.

It is interesting to note that matrix 2.8, usually derived in an energetic way, is with opposite signs to the signs of the matrix derived here (with positive diagonal terms). The signs of the matrix derived here are correct.

The initial conditions, at the beginning of the time interval of integration, $-u_1$ and $\dot{u}_1$, have to be known. The unknowns are u_2 and $\dot{u}_2$, at the end of the time interval. Therefore only two equations are necessary, which have to be defined by the second and fourth rows of the previously defined matrices. In matrix form these equations will be,

$$\frac{m}{\Delta t}\begin{bmatrix} a & -a & b & b \\ -b & b & c & d \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ \dot{u}_1 \\ \dot{u}_2 \end{Bmatrix} = \begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix} \tag{2.9}$$

where:

$$a = \frac{-12}{\Delta t^2} + \frac{1.2k}{m} \qquad b = \frac{-6}{\Delta t} + 0.1\Delta t\frac{k}{m} - \frac{c}{m}$$

$$c = 2 + \frac{\Delta t^2}{30}\frac{k}{m} - c\frac{\Delta t}{2m} \qquad d = 4 - 2\frac{\Delta t^2}{15}\frac{k}{m} - c\frac{\Delta t}{2m}$$

The integration of the differential equation yields nodal forces, i.e. forces at the beginning (1) and at the end (2) of the time interval. The nodal forces due to the external load p, according to eq. 2.3a will be defined as follows,

$$P = -\int \frac{\mathrm{d}\phi}{\mathrm{d}t}\frac{\mathrm{d}p}{\mathrm{d}t}\,\mathrm{d}t \tag{2.10}$$

In the case of linear variation of p,

$$p = p_1(1 - t/\Delta t) + p_2\, t/\Delta t$$

the load matrix becomes,

$$P = \begin{Bmatrix} (p_1 - p_2/\Delta t) \\ 0 \end{Bmatrix}$$

During the derivation of this matrix there was a dilemma how this matrix has to be defined. For easier understanding of the problem the beam analogy had to be used. In the case of the beam problem the second term of this matrix means nodal moments due to the distributed load. And since the moments (external load in this case) are of linear variation, there is no distributed load and consequently the load term is equal to zero. The present system: known initial conditions and unknown values at the end of the interval, is equivalent to a cantilever beam with left end fixed. In the beam problem the second equation defines zero total moments at the right hand end. However, in this case the equivalent to the moment is the end inertial force, which is not equal to zero, but it is $m\ddot{u}$. This means that the load vector has to be as follows,

$$P = \begin{Bmatrix} (p_1 - p_2/\Delta t) \\ m\ddot{u}_2 \end{Bmatrix} \tag{2.11}$$

The inertia force can be defined from eq. 2.1 as follows,

$$m\ddot{u}_2 = p_2 - c\dot{u}_2 - ku_2 \tag{2.12}$$

The second and the third portion of this value have to be transferred to the left side and the following equation will be derived,

$$\begin{bmatrix} -c_1 & c_1 & c_3 & c_3 \\ -c_3 & c_4 & c_5 & c_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ \dot{u}_1 \\ \dot{u}_2 \end{Bmatrix} = \frac{1}{m} \begin{Bmatrix} p_1 - p_2 \\ p_2\Delta t \end{Bmatrix} \tag{2.13}$$

where,

$$\begin{gathered} c_1 = 12/\Delta t^2 - 1.2k/m, \quad c_2 = 4 - 2k\Delta t^2/15m + c\Delta t/2m, \\ c_3 = -6/\Delta t + 0.1k\Delta t/m - c/m \\ c_4 = -6/\Delta t + 1.1k\Delta t/m + c/m, \quad c_5 = 2 + k\Delta t/30m - c\Delta t/2m \end{gathered} \tag{2.14}$$

For convenience, eq. 2.9 has been multiplied by $\Delta t/m$. The main contribution to the matrix equation (2.13) have the terms $12/\Delta t^2$ and $6/\Delta t$. When the time step is small, the contribution of these terms is much dominant over the contribution of the terms associated to $k/m = \omega^2$. In such case the resulting values of u_2 and $\dot{u}_2$ have to be the same as the same values derived by the method of linear acceleration variation. Thus, in the limit, when the time step becomes small enough, the both methods have to give same results, which means that the results derived by solution of eq. 2.13 should be good. More about this is given in subchapter 2.3.

Two ways of solution of eq. 2.13 can be applied: (a) incremental, by derivation of Δu_2 and $\Delta\dot{u}_2$, and (b) direct derivation of u_2 and $\dot{u}_2$.

2.2 Incremental solution

The general solution of the differential equation (2.1) can be represented as a sum of the solution of the homogeneous portion of the equation and the particular solution, as follows,

$$u = e^{-\xi\omega t}\left[\frac{\dot{u}_0 + \xi\omega u_0}{\omega}\sin\omega t + u_0\cos\omega t\right] + \Delta u \tag{2.15}$$

where u_0 and $\dot{u}_0$ are initial displacement and velocity. The first portion is the well known contribution of the initial conditions, and the second portion is the particular solution, contribution of the external load. This portion can be derived from eq. 2.13, by substitution $u_0 = \dot{u}_0 = 0$. In that way eq. 2.13 becomes,

$$\begin{bmatrix} c_1 & c_3 \\ c_2 & c_4 \end{bmatrix} \begin{Bmatrix} \Delta u \\ \Delta \dot{u} \end{Bmatrix} = \frac{1}{m} \begin{Bmatrix} p_1 - p_2 \\ p_2 \Delta t \end{Bmatrix} \tag{2.16}$$

The solution of this matrix equation yields,

$$\Delta u = \frac{1}{m} \frac{c_2(p_1 - p_2) - c_3 p_2 \Delta t}{c_1 c_2 - c_3 c_4} \tag{2.17}$$

$$\Delta \dot{u} = \frac{1}{m} \frac{c_1 p_2 \Delta t - c_4(p_1 - p_2)}{c_1 c_2 - c_3 c_4} \tag{2.18}$$

The total solution is,

$$\begin{aligned} u_2 &= u_2^0 + \Delta u \\ \dot{u}_2 &= \dot{u}_2^0 + \Delta \dot{u} \end{aligned} \tag{2.19}$$

where u_2^0 is transient response, derived by substitution $t = \Delta t$ into eq. 2.15, as follows,

$$u_2^0 = e^{-\xi\omega\Delta t} \left[\frac{\dot{u}_1 + \xi\omega u_1}{\omega} \sin\omega\Delta t + \cos\omega\Delta t \right] \tag{2.20}$$

where u_1 and $\dot{u}_1$ are the values of the previous step. The derivative of this expression defines the velocity,

$$\dot{u}_2^0 = -\xi\omega\, u_2^0 + e^{-\xi\omega\Delta t} \left[(\dot{u}_1 + \xi\omega u_1)\cos\omega\Delta t - u_1\omega\sin\omega\Delta t \right] \tag{2.21}$$

The last 2 expressions seem rather complex. However, the time step of integration Δt usually is taken as constant and once these expressions computed for $t = \Delta t$ they become simple.

The acceleration at the end of the step will be computed on the base of the previously computed displacement and velocity, according to eq. 2.12, as follows,

$$\ddot{u}_2 = (p_2 - \dot{u}_2 - k u_2)/m \tag{2.22}$$

Here the following question can be arisen: can the contribution of the transient response, represented by first portion of eq. 2.15, of the all previous history, be represented by the contribution of the initial conditions at the beginning of the time interval, as defined by eq. 2.20?

The answer to this question will be given by consideration of the undamped transient response, which at a time t will be defined as follows,

$$u^0 = \frac{\dot{u}_0}{\omega} \sin\omega t + u_0 \cos\omega t$$

At the time $t_2 = t_1 + \Delta t$ this expression yields,

$$u_2^0 = \frac{\dot{u}_0}{\omega} \sin\omega(t_1 + \Delta t) + u_0 \cos\omega(t_1 + \Delta t)$$

The following substitutions:

$$\sin\omega(t_1 + \Delta t) = \sin\omega t_1 \cos\omega\Delta t + \cos\omega t_1 \sin\omega\Delta t$$
$$\cos\omega(t_1 + \Delta t) = \cos\omega t_1 \cos\omega\Delta t - \sin\omega t_1 \sin\omega\Delta t$$

into previous equation yield,

$$u_2^0 = \left[\frac{\dot{u}_0}{\omega}\cos\omega t_1 - u_0 \sin\omega t_1\right]\sin\omega\Delta t + \left[\frac{\dot{u}_0}{\omega}\sin\omega t_1 + u_0\cos\omega t_1\right]\cos\omega\Delta t =$$
$$= \frac{\dot{u}_1}{\omega}\sin\omega\Delta t + u_1\cos\omega\Delta t$$

This means that the total transient response can be substituted by the contribution of the initial values at the beginning of the time interval. Thus, the procedure applied in this, called incremental method, is correct. And in addition, it is important to note that for constant Δt the contribution of the transient response is defined by a simple expression.

2.3 Direct solution

The displacement u_2 and velocity $\dot{u}_2$ at the end of the time step of integration can be computed directly from eq. 2.13. The values of the previous step u_1 and $\dot{u}_1$ as known can be transferred to the right hand side of the equation and the following solution of the equation derived,

$$u_2 = \frac{c_2\Delta p_1 - c_3\Delta p_2}{DT} \tag{2.23}$$

$$\dot{u}_2 = \frac{c_1\Delta p_1 - c_4\Delta p_2}{DT} \tag{2.24}$$

where

$$\begin{aligned} \Delta p_1 &= (p_1 - p_2)/m + c_1 u_1 - c_3 \dot{u}_1 \\ \Delta p_2 &= p_2\Delta t/m + c_3 u_1 - c_5 \dot{u}_1 \\ DT &= c_1 c_2 - c_3 c_4 \end{aligned} \tag{2.25}$$

The acceleration $\ddot{u}_2$ again is computed according to eq. 2.22.

It is interesting to compare the solution by the linear acceleration method and the solution presented here. The acceleration at time t_1 ($\ddot{u}_1$) can be defined as follows,

$$m\ddot{u}_1 = p_1 - c\dot{u}_1 - ku_1$$

The substitution of this value into the linear acceleration method solution yields,

$$\Delta u = \frac{p_1(2 + c\Delta t) + p_2 + \dot{u}_1 m(6/\Delta t - c^2\Delta t) - u_1(3k + 3kc\Delta t)}{k + 6m/\Delta t^2 + 3c/\Delta t} \tag{2.26}$$

The same increment derived from eq. 2.23 is as follows,

$$\Delta u = \frac{p_1\left(2+\frac{c}{2}\Delta t - 2k\Delta t^2\right) + p_2\left(1+\frac{k}{60}\Delta t^2 + \frac{c}{4}\Delta t\right)}{0.4k + 6m/\Delta t^2 + 3c/\Delta t + f_3(k,\,\Delta t,\,c)} + \tag{2.27}$$

$$+\frac{\dot{u}_1\left[\frac{6m}{\Delta t} + f_1(k,\,\Delta t,\,c)\right] - u_1[3k + f_2(k,\,\Delta t,\,c)]}{0.4k + 6m/\Delta t^2 + 3c/\Delta t + f_3(k,\,\Delta t,\,c)}$$

This expression is rather complex and therefore some terms of minor importance are not given explicitly, but they are stated as functions. The comparison of the eq. 2.26 with eq. 2.27 shows that some terms in eq. 2. 26 are not present. But the main contribution terms, associated with Δt in the denominator, are the same. Thus, in the case of very small time step interval the both equations will give approximately the same results. However, in the case of a long time step comparing to the system period, the results will not be the same. The linear acceleration method in such cases does not give good solution. The results of the analysis of the numerical examples which follow show that the method presented in this paper gives good and stable solutions even in the cases of long time steps of integration.

3. Numerical examples

For the testing of the accuracy of the presented method some examples, for which there is a theoretical solution, have been solved.

The stability of the solution is governed by the determinant DT(2.25). In the analogous beam stability problem (with matrix C=0) and $DT = c_1c_2 - c_3^2$ this determinant defines the critical force. In this case it is the "critical" k/m,

$$\frac{k}{m} = \omega^2 = \frac{\pi^2}{4\Delta t^2}$$

or $\omega\Delta t = \pi/2$. With the damping included and somewhat modified DT, as defined by eq. 2.25, there is no real "critical" value of $\omega\Delta t$, but the solution of the problem becomes unstable around the same $\omega\Delta t = \pi/2$. This is in the case of $\Delta t \doteq T/4$. Such a time step length is quite long. It seems reasonable to take a time step of $\Delta t = T/8$. The length of the step is limited by the external load. In the case of arbitrary loading, for instance like earthquake ground motion, it seems reasonable to take a maximum step length of $\Delta t = 0.05$ sec.

The examples following are solved with assumed step $\Delta t = 0.05$ sec. The period of the analyzed single degree of freedom system is $T = 0.4$ sec.

The system was subjected to a rectangular (step) impulse, half sinusoidal impulse and triangular impulse, for which there is a theoretical solution. The two cases of step and sinusoidal impulses were subdivided and integrated in two steps, and the triangular impulse case was integrated in one step. The results of the analysis are presented in table 1.

The load vector computed by application of eq. 2.10 and 2.13 is given in the first row. The analysis is carried out for undamped systems ($\xi = 0$), and damped systems with damping coefficient $\xi = 0.05$ of the critical. The results of the analysis by the incremental method are presented in columns 1. The increment of the displacement Δu and velocity $\Delta \dot{u}$ are computed by application of eq. 2.17–18. The contribution of the initial values (of the previous step) u^0 and $\dot{u}^0$ are computed by eq. 2.20–21. The total response is a sum of these two components (eq. 2.19).

Table 1. Response of a single degree of freedom system to impulsive loads; $\Delta t = T/8 = 0.05$ sec.; multiplication factor 10^2.

	Step impulse $t = 2\Delta t = 0.1$ Sec			Half sine impulse $t = 2\Delta t = 0.1$ Sec			Triangular impulse $t = \Delta t = 0.05/0.1$ Sec	
	1	2	3	1	2	3	1,2	3
$\xi = 0$								
Load vec.	{0 1}			{-1.04672 0.05801}			{1 0}	
u_1	0.1186	0.1186	0.1187	0.0557	0.0557	0.0558	0.0783	0.0784
$\dot{u}_1$	4.499	4.499	4.502	2.993	2.993	3.001	2.127	2.127
Load vec.	{0 1}			{1.04672 0.00569}			{1 0}	
u_2	0.4050	0.4049	0.4053	0.2676	0.2676	0.2702	0.1893	0.1815
$\dot{u}_2$	6.362	6.360	6.366	4.236	4.233	4.244	2.661	2.553
$\xi = 0.05$								
u_1	0.1156	0.1156	0.1159	0.0545	0.0545	0.0544		
$\dot{u}_1$	4.328	4.328	4.333	2.910	2.910	2.919		
u_2	0.3851	0.3876	0.3866	0.2604	0.2608	0.2592		
$\dot{u}_2$	5.877	5.852	5.900	3.915	3.939	3.911		

1) Column 1-incremental solution, 2-direct solution, 3-impulse solution.

The results of the analysis of the response by the direct method are presented in columns 2. In the first step, because of the zero initial conditions, the incremental and direct solution are the same. The displacement and velocity of the next step are computed by eq. 2.23, as a function of the external load at the step time and the previous step values.

The results of the analysis by the impulse solution are given in columns 3. The contribution of the previous step values are computed by eq. 2.20–21. That is the solution of the homogeneous portion of the differential equation. The theoretical solution of the response of the system to a step impulse is as follows,

$$\Delta u = \frac{p_0}{k}\left[1 - e^{-\xi\omega t}(\cos\omega t + \xi\sin\omega t)\right] \tag{3.1}$$

This is the particular solution of the differential equation of vibration for initial conditions $u = \dot{u} = 0$. The substitution of $t = \Delta t$ into this equation yields displacement increment Δu. The total solution at the end of the time interval will be,

$$u = u^0 + \Delta u \tag{3.2}$$

The particular solution of the velocity increment $\Delta\dot{u}$ is derived by differentiation of eq. 3.1 and substitution of $t = \Delta t$. The total velocity solution is,

$$\dot{u} = \dot{u}^0 + \Delta\dot{u} \tag{3.3}$$

By the use of the corresponding impulse solutions, in a similar way the theoretical results for the sine and triangular impulses were computed.

The response to a triangular impulse was analyzed for an undamped system, but for two cases of impulse length: $\Delta t = 0.05$ sec. and $\Delta t = 0.1$ sec. The time step of integration is the same as the impulse length, i.e. the integration is in one step. It was interesting to see what would be the accuracy of the computed response by a step length close to the "critical". Therefore the second time of integration and the impulse length were taken $\Delta t = 0.08$, close the "critical" value $\Delta t = 0.1$ sec. $(\omega\Delta t = \pi/2)$.

The comparison of the results in columns 1 and 2 with those in column 3 (theoretical), shows that the results derived here by the presented method of solution are of excellent accuracy. The errors are a small portion of a percentage, within the accuracy of computation (by a pocket calculator). Somewhat higher errors have the results with damping, but still much less than a percentage.

Another illustrative example is the response of the system to a harmonic (sinusoidal) loading. The results of the analysis of this example are given in table 2.

In columns 1 and 2 is given the transient response, computed by eq. 2.20–21, with $\xi = 0$. The increment solution given in columns 3, 4 is the same as given in table 1. The response at the end of the time interval given in columns 5, 6 is computed as a sum of the values in columns 1, 2 and columns 3, 4. The theoretical response is given in columns 7, 8..

The analysis of the results presented in the table shows that the accuracy of the applied incremental method is very good. The difference between the computed and the theoretical results is very small and can not be represented graphically. The analysis is carried out for one cycle of vibration of the system and 2 cycles of the harmonic loading. After that the response is repeated. At the end of the cycle of vibration the response has to be equal zero. The error of the computed response is negligible.

Table 2. Response of the system to harmonic loading; $T = 0.4$ sec. $\Delta t = 0.05$ sec. $\xi = 0$, $\overline{\omega} = 2\omega$, multipl. factor 10^2.

Time Sec.	u^0	$\dot{u}^0$	Δu	$\Delta \dot{u}$	u	$\dot{u}$	Theory u	Theory $\dot{u}$
	1	2	3	4	5	6	7	8
0.05	0	0	0.0557	2.993	0.0557	2.993	0.0560	3.001
0.10	0.1741	1.4980	0.0955	2.738	0.2696	4.236	0.2702	4.244
0.15	0.3813	0.0008	-0.0557	-2.993	0.3256	-2.992	0.3262	-3.001
0.20	0.0955	-5.7321	-0.0955	-2.738	0	-8.470	0	-8.488
0.25	-0.3821	-6.002	0.0557	2.993	-0.3264	-3.009	-0.3262	-3.001
0.30	-0.3662	1.498	0.0955	2.738	-0.2707	4.236	-0.2702	4.244
0.35	-0.0007	6.002	-0.0557	-2.993	-0.0558	3.009	-0.0560	3.001
0.40	0.0960	2.747	-0.0955	-2.738	0.0005	0.009	0	0

The results presented in the both tables show that both: displacement and velocity are of similar accuracy. The accuracy of the acceleration computed from the equation of equilibrium (2.22), also should be of the same order accuracy.

4. Conclusions

A method of solution of the differential equation of vibration of a single degree of freedom system has been presented. In a way the method represents extension of the finite element method to the solution of this problem. However, the time finite element is developed directly from the differential equation of the problem, by approximation of the high order derivatives with a product of lower order derivatives of the interpolation function. Two methods of solution were presented: incremental and direct method of derivation of the displacement and velocity. The acceleration is computed by satisfaction of the equilibrium equation. All of them are of the same order accuracy.

The method was tested by computation of the response of a system to impulsive and harmonic loading, for which there is a theoretical solution. The errors of the computed response represent small portion of a percentage. The method gives practically unconditionally stable solution. Even in the case of quite long step of integration $\Delta t < T/5$, the results are good.

The method offers application of long steps of integration and saving of the computational time. The advantage of this method would be the reliability of the results and the saving of computational time.

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НУМЕРИЧЕСКОЕ РЕШЕНИЕ ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ ВИБРАЦИИ

Расматривается нумерическое решение дифференциального уравнения вибрации системы с одним степенем свободы. Решение постепенно, шаг – по шаг интегрирование. Высшие диференциаллы уравнения апроксимируются произведением низшых диференциалов. Таким образом получается так как временсий конечны элемент. Нумерические результаты показывют очен хорошую точност примененного потхода. Решение стабильное даже в случай применения долгова шага интегрирования $\Delta t = T/5$.

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