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A TWO–FIELD FINITE ELEMENT MODEL RELATED TO THE REISSNER'S PRINCIPLE

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1. Introduction

The state of art of the above problem is described by Zienkiewicz [1], p.333 as follows: "*It is ... possible to derive elements that exhibit complete continuity of the appropriate components along interfaces ... Extension to the full stress problem is difficult and as yet such elements have not been successfully used*".

The main formal problem with such type of continuity folows from the common opinion that "*the disconnection of stress variables at corner nodes can only be accomplished for all variables*" ([1], p 332.).

In the present, and in some previous papers, [2], [3], [16], [17], this problem is completely resolved by considering only the nodes at the boundary surfaces (and interfaces) as a points where the stress variables are to be prescribed or disconnected, what appropriate.

To handle stress components adequately and conveniently, we accommodate the boundary (interface) nodal coordinate surfaces to be coincident or at least tangent to the local boundary surfaces and/or interfaces. Then it is possible, in addition to the displacement constraints, to treat also the stress constraints as essential boundary conditions. As it can be concluded on the basis of the existing (iterative) computational evidence of Cantin et al. [15], (Loubignac method [23]) where the stress boundary conditions were satisfied in an iterative manner, such an approach can significantly improve the results, especially at the vicinity of a boundary, or when the number of elements is small. Hence, this approach is a very promising way towards improvement of the performances of mixed finite elements.

Although either the authors of [11] and [13] were aware about this possibility, as it can be concluded by careful reading of these papers, none of them used it practically, and in both papers the mechanical boundary conditions are treated as the natural ones. It looks that theoretical and practical aspects of the direct treatment of stress constraints as essential boundary conditions, at least in the most general case, remained an unsolved problem until the paper [2] appeared. In that paper the coordinate independent (tensorial) interpolations/approximations [4] are used allowing easy adjustment of the local coordinate systems to be tangent to the boundary or interface.

Albeit important for the correct task definition, the problems discussed in [2] are of, more or less, "technical" nature, and, once formally solved, induce the appearance of some fundamental difficulties.

It has been indicated in [16] that mixed FEM problems in elasticity, especially if the polynomials of the same order are used for both the displacement and stress shape functions (an obvious choice), and if the boundary traction conditions are fulfilled as the essential ones, usually do not satisfy Brezzi conditions, and hence cannot be solved. The problem can be superseded formally, by the use of higher order interpolation functions for the stresses. Hierarchic interpolation (Appendix B) is proposed, because in this case the addition or elimination of the particular basis functions does not influence the remaining functions.

In the present paper solvability of a problem is studied in detail. Special care has been devoted to some quantitative properties (sizes) of the finite element subspaces under consideration. Superconvergent numerical results justified the proposed approach.

2. Field equations

Let us consider a complete system of the field equations in the linear elasticity, where

$$\mathbf{div}\,\mathbf{T} + \mathbf{f} - \varrho\mathbf{a} = 0 \quad \text{in}\,\mathcal{B} \tag{2.01}$$

$$\mathbf{e} - \tfrac{1}{2}\left(\nabla\mathbf{u} + \nabla\mathbf{u}^T\right) = 0 \quad \text{in}\,\mathcal{B} \tag{2.02}$$

$$\mathbf{T} - \mathbb{C}{:}\mathbf{e} = 0 \quad \text{in}\,\mathcal{B} \tag{2.03}$$

$$\mathbf{T}\cdot\mathbf{n} - \mathbf{p} = 0 \quad \text{on}\,\partial\mathcal{B}_t \tag{2.04}$$

$$\mathbf{u} - \mathbf{w} = 0 \quad \text{on}\,\partial\mathcal{B}_u \tag{2.05}$$

are respectively the equations of motion, strain-displacement and stress-strain relationships, boundary traction conditions and geometric boundary conditions. In these expressions, $\mathbf{T}$ is the stress tensor, ϱ the mass density, $\mathbf{f}$ the vector of the body forces, $\mathbf{a}$ the acceleration vector, $\mathbf{e}$ the strain tensor, $\mathbf{u}$ the displacement vector, $\mathbb{C}$ the elasticity tensor, $\mathbf{p}$ the vector of the boundary tractions, $\mathbf{w}$ the vector of the prescribed displacements. Finally, $\mathcal{B}$ is an open, bounded domain of the elastic body, $\mathbf{n}$ is the unit normal vector to the boundary $\partial\mathcal{B}$; $\partial\mathcal{B}_t$ and $\partial\mathcal{B}_u$ are the portions of $\partial\mathcal{B}$ where the stresses or the displacements are prescribed, respectively.

3. Boundary traction conditions

As it has been already mentioned, if one wants to treat also the stress constraints as essential boundary conditions, it is necessary to introduce special coordinate systems, having coordinate surfaces tangent the boundary surfaces of a body. In this special case it is possible to determine some of the stress tensor components from the boundary tractions at the point of consideration. Per instance, if $y^{(r)}$=const. is the equation of a boundary surface, then [2]

$$t^{t(r)} = p^t n^{(r)}. \tag{3.01}$$

Hence, if the boundary tractions p^t are known, one can easily determine the corresponding stresses $t^{t(r)}$ at the boundary surface $y^{(r)}$=const.

In practice, it is convenient to introduce local right-hand Cartesian systems, having axes y^1 and y^2 in a plane tangential to the boundary surface at the boundary node under consideration, say L. The remaining axis y^3 can be taken in the direction of the outer normal to the boundary surface. Then the above equation becomes:

$$t^{Lt(3)} = p^{Lt}. \tag{3.02}$$

In the plane stress case the above expression reduces to

$$t^{Lt(2)} = p^{Lt}, \tag{3.03}$$

or, more detailed

$$t^{L1(2)} = p^{L1}, \quad t^{L2(2)} = p^{L2}. \tag{3.04}$$

4. Weak form of the field equations

Let us suppose that both boundary conditions (2.04) and (2.05) are essential, and hence exactly satisfied by the trial functions of a problem. Then we need to consider only the weak forms of the equations (2.01-03).

4.01. The equations of equilibrium

By the use of the Galerkin procedure, one can seek the weak solution of the static counterpart (**a**=0) of (2.01) from the scalar product

$$\int_{\mathcal{B}} \mathbf{v} \cdot \operatorname{\mathbf{div}} \mathbf{T}\, dV = -\int_{\mathcal{B}} \mathbf{v} \cdot \mathbf{f}\, dV. \tag{4.01}$$

where $\mathbf{v}$ is taken from the space $\mathbf{L}^2$ of all square integrable vectorfields.

4.02 The strain-displacement and the stress-strain relationships

We will consider the standard case, i.e. invertible constitutive equations (2.03), when one can write

$$\mathbf{e} = \mathbb{A}{:}\mathbf{T}, \tag{4.02}$$

where $\mathbb{A}$ is the *elastic compliance* tensor. From the comparison of (2.02) and (4.02) it follows that

$$\mathbb{A}{:}\mathbf{T} = \frac{1}{2}\left(\nabla\mathbf{u} + \nabla\mathbf{u}^T\right). \tag{4.03}$$

If the test functions $\mathbf{S}$ are taken from the space $\mathbf{T}^2$ of all square integrable *symmetric* tensorfields, the weak solution of (4.03) can be determined by the help of a relatively simple expression

$$\int_{\mathcal{B}} \mathbf{S}:(\mathbb{A}:\mathbf{T} - \nabla\mathbf{u})\,\mathrm{d}V = 0. \tag{4.04}$$

4.03 Weak formulation of a mixed problem

By the simple summation of (4.01) and (4.04) one obtains a new and far-reaching expression (4.05), which allows *asymmetric* weak formulation of a mixed problem, associated with the mixed (Reissner's) variational principle:

Find $\mathbf{T}\in\mathbf{H}(\mathbf{div})$ satisfying $\mathbf{Tn}|_{\partial\mathcal{B}_t}=\mathbf{p}$ and $\mathbf{u}\in\mathbf{H}^1$ such that $\mathbf{u}|_{\partial\mathcal{B}_u}=\mathbf{w}$ and

$$\int_{\mathcal{B}} (\mathbf{S}:\mathbb{A}:\mathbf{T} - \mathbf{S}:\nabla\mathbf{u} + \mathbf{v}\cdot\mathbf{div}\,\mathbf{T})\,\mathrm{d}V = -\int_{\mathcal{B}} \mathbf{v}\cdot\mathbf{f}\,\mathrm{d}V \tag{4.05}$$

for all $\mathbf{S}\in\mathbf{T}^2$ and all $\mathbf{v}\in\mathbf{L}^2$.

In this expression $\mathbf{H}(\mathbf{div})$ is the space of all symmetric tensorfields which are square integrable and have *square integrable divergence* [5], while $\mathbf{H}^1$ is the space of all vectorfields which are square integrable and have square integrable gradient.

However, it is presently a common sense that asymmetric formulations are impractical from the computational point of view. By the use of the divergence theorem over the second integral in the above expression, we obtain the well-known [5] symmetric weak formulation for the mixed problem:

Find $\mathbf{T}\in\mathbf{H}(\mathbf{div})$ satisfying $\mathbf{Tn}|_{\partial\mathcal{B}_t}=\mathbf{p}$ and $\mathbf{u}\in\mathbf{L}^2$ such that

$$\int_{\mathcal{B}} (\mathbf{S}:\mathbb{A}:\mathbf{T} + \mathbf{divS}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{divT})\,\mathrm{d}V = -\int_{\mathcal{B}} \mathbf{v}\cdot\mathbf{f}\,\mathrm{d}V + \int_{\partial\mathcal{B}_u} (\mathbf{Sn})\cdot\mathbf{w}\,\mathrm{d}A \tag{4.06}$$

for all $\mathbf{S}\in\mathbf{H}(\mathbf{div})$ satisfying $\mathbf{Sn}|_{\partial\mathcal{B}_t}=0$ and all $\mathbf{v}\in\mathbf{L}^2$.

Although symmetric, this form is not very popular amongst the finite element practitioners, at least for the two reasons. First, both the stresses $\mathbf{T}$ and their variations $\mathbf{S}$ are taken from $\mathbf{H}(\mathbf{div})$, the space unfamiliar for an average finite element user, and second, the use of the discontinuous displacements from the space $\mathbf{L}^2$ and natural displacement boundary conditions is awkward for the same people. Nevertheless, this interesting formulation is frequently elaborated by mathematicians [5], [6].

At variance with the preceding case, application of the divergence theorem over the third term of (4.05) gives a more popular (especially amongst the engineering oriented scientists [1], [10]) form of a mixed problem:

Find $\mathbf{T}\in\mathbf{T}^2$ and $\mathbf{u}\in\mathbf{H}^1$ such that $\mathbf{u}|_{\partial\mathcal{B}_u}=\mathbf{w}$ and

$$\int_{\mathcal{B}} (\mathbf{S}:\mathbb{A}:\mathbf{T} - \mathbf{S}:\nabla\mathbf{u} - \nabla\mathbf{v}:\mathbf{T})\,\mathrm{d}V = -\int_{\mathcal{B}} \mathbf{v}\cdot\mathbf{f}\,\mathrm{d}V - \int_{\partial\mathcal{B}_t} \mathbf{v}\cdot\mathbf{p}\,\mathrm{d}A \tag{4.07}$$

for all $\mathbf{S} \in \mathbf{T}^2$ and $\mathbf{v} \in \mathbf{H}^1$ such that $\mathbf{v}|_{\partial\mathcal{B}_u} = 0$.

If the constitutive equations are locally satisfied (see (2.03) and (4.02)) the above equation straightforwardly reduces to the primal (displacement) problem: Find $\mathbf{u} \in \mathbf{H}^1$ such that $\mathbf{u}|_{\partial\mathcal{B}_u} = \mathbf{w}$ and

$$\int_{\mathcal{B}} \mathrm{e}(\mathbf{v}){:}\mathbb{C}{:}\mathrm{e}(\mathbf{u})\,\mathrm{d}V = \int_{\mathcal{B}} \mathbf{v}\cdot\mathbf{f}\,\mathrm{d}V + \int_{\partial\mathcal{B}_t} \mathbf{v}\cdot\mathbf{p}\,\mathrm{d}A, \tag{4.08}$$

for all $\mathbf{v} \in \mathbf{H}^1$ such that $\mathbf{v}|_{\partial\mathcal{B}_u} = 0$.

Note also that a finite element model based on (4.07) for the linear displacement – linear discontinuous stress triangle is numerically equivalent to the classical linear displacement (constant stress) triangle based on (4.08). Some authors even regard (4.08) and (4.07) to be equivalent, or simply neglect (4.07).

However, if the bilinear displacement – five term stress quadrilateral is considered, from (4.07) follows one of the most successful quadrilaterals – a Pian–Sumihara [7] mixed-hybrid element.

5. Finite element approximations of the field equations

In this section and in the rest of the paper we will consider only the problem defined by (4.07).

5.1. Classical mixed approach

We let $\mathcal{C}_h$ be the partitioning of $\bar{\mathcal{B}}$ ($\mathcal{B}$ closed) into elements $\mathcal{E}$ and define the finite element subspaces for the displacement vector, the stress tensor and the appropriate weight functions (variations) respectively as

$$\mathbf{U}_h = \left\{\mathbf{u} \in \mathbf{H}^1(\mathcal{B}) \mid \mathbf{u}|_{\partial\mathcal{B}_u} = \mathbf{w}, \quad \mathbf{u}|_{\mathcal{E}} = U^K(\mathcal{E})\mathbf{u}_K, \quad \forall\mathcal{E} \in \mathcal{C}_h\right\}, \tag{5.01}$$

$$\mathbf{T}_h = \left\{\mathbf{T} \in \mathbf{T}^2(\mathcal{B}) \mid \mathbf{T}_{\mathcal{E}} = T_L(\mathcal{E})\mathbf{T}^L, \quad \forall\mathcal{E} \in \mathcal{C}_h\right\}, \tag{5.02}$$

$$\mathbf{V}_h = \left\{\mathbf{v} \in \mathbf{H}^1(\mathcal{B}) \mid \mathbf{v}|_{\partial\mathcal{B}_u} = 0, \quad \mathbf{v}|_{\mathcal{E}} = V^M(\mathcal{E})\mathbf{v}_M, \quad \forall\mathcal{E} \in \mathcal{C}_h\right\}, \tag{5.03}$$

$$\mathbf{S}_h = \left\{\mathbf{S} \in \mathbf{T}^2(\mathcal{B}) \mid \mathbf{S}|_{\mathcal{E}} = S_N(\mathcal{E})\mathbf{S}^N, \forall\mathcal{E} \in \mathcal{C}_h\right\}. \tag{5.04}$$

In these expressions, $\mathbf{u}_K$ and $\mathbf{T}^L$ are the nodal values of the vector $\mathbf{u}$ and tensor $\mathbf{T}$, respectively. Accordingly, U^K and T_L are the corresponding values of the interpolation functions, connecting the displacements and stresses at an arbitrary poiht in $\mathcal{E}$ (the body of an element), and the nodal values of these quantities. The complete analogy holds for the diŝplacement and stress variations $\mathbf{v}$ and $\mathbf{S}$ respectively. The test subspaces $\mathbf{V}_h \subset \mathbf{U}_h$ and $\mathbf{S}_h = \mathbf{T}_h$ in this particular case. Note that $\mathbf{v}|_{\partial\mathcal{B}_u} = 0$.

Because the displacement spaces are the same as in the classical displacement approach, and the stress space can be discontinuous at the element boundaries, it is a straightforward task to construct the elements of the above type.

5.2. Continuous stress and displacement mixed approach

The stresses and stress variations here are taken from a smaller space, the same as in (4.06), or even more restricted $\mathbf{T} \in \mathbf{T}^1 \subset \mathbf{H}(\mathbf{div}) \subset \mathbf{T}^2$, i.e.

$$\mathbf{T}_h = \left\{ \mathbf{T} \in \mathbf{T}^1(\mathcal{B}) \mid \mathbf{T}|_{\mathcal{E}} = T_L(\mathcal{E})\mathbf{T}^L, \quad \forall \mathcal{E} \in \mathcal{C}_h \right\}, \tag{5.05}$$

$$\mathbf{S}_h = \left\{ \mathbf{S} \in \mathbf{T}^1(\mathcal{B}) \mid \mathbf{S}|_{\mathcal{E}} = S_N(\mathcal{E})\mathbf{S}^N, \quad \forall \mathcal{E} \in \mathcal{C}_h \right\}. \tag{5.06}$$

Technically, the boundary traction conditions are treated as the natural ones (identically as in the displacement method), while at the interfaces the stress components are disconnected (again as in the displacement method). From the point of view of known finite element techniques, the part of a mesh limited by the boundaries and interfaces of a body under consideration (Regular Part of Structure, RPS in [13] and [14]), can be understood as a large composite mixed-hybrid [7], [10] element, having continuous stress and displacement fields in the interior, and requiring only the displacement continuity at the boundaries.

The advantage of the described approach is in a fact that for the displacements and stresses one can use interpolation functions of the same type, and that for an average finite element user the stress continuity, where appropriate, is an obvious and attractive property of a model. This approach has been succesfully used by Mirza and Olson [11] for linear triangles, and by the present authors [13] for bilinear quadrilaterals, and the numerical results indicated superconvergent properties of a model. Eigenvalue analysis performed by Olson [12] for the triangular and rectangular elements of this type is also very useful for the further development of a model.

5.3. Satisfaction of the boundary traction conditions

The purpose of the present, and of some previous papers, [2], [3], [16], [17], is to discuss, at variance with the preceding case, the situation when stresses satisfy also the boundary traction conditions. Although this approach is not necessary from the point of view of the variational principle (4.07), where the stress boundary conditions are satisfied as the natural ones, we will see that, in the finite element equations (5.09), one can separate known and unknown nodal stress values, and hence also satisfy the stress boundary conditions as the essential ones,. i.e. equilibrate the boundary tractions (components of the stress vector).

Hence (only) at the boundaries and interfaces we allow that our $\mathbf{T}\in\mathbf{H}^1$ behaves as $\mathbf{T}\in\mathbf{H}(\mathbf{div})$, instead of $\mathbf{T}\in\mathbf{T}^2$ from the preceding case. Finally, the stress and its variation subspaces will be

$$\mathbf{T}_h = \left\{ \mathbf{T} \in \mathbf{T}^1(\mathcal{B}) \mid \mathbf{Tn}|_{\partial\mathcal{B}_t} = \mathbf{p}, \quad \mathbf{T}|_{\mathcal{E}} = T_L(\mathcal{E})\mathbf{T}^L, \quad \forall \mathcal{E} \in \mathcal{C}_h \right\}, \tag{5.07}$$

$$\mathbf{S}_h = \left\{ \mathbf{S} \in \mathbf{T}^1(\mathcal{B}) \mid \mathbf{Sn}|_{\partial\mathcal{B}_t} = 0, \quad \mathbf{T}|_{\mathcal{E}} = S_N(\mathcal{E})\mathbf{S}^N, \quad \forall \mathcal{E} \in \mathcal{C}_h \right\}. \tag{5.08}$$

As it has been already mentioned, in the paper of Cantin et al. [15], where the stress boundary conditions were satisfied in an iterative manner, it has been shown that such an approach can significantly improve the results.

5.4. Compact matrix form of the finite element equations

As it has been shown in [2], the finite element equations based on (4.07) can be written in a form

$$\begin{bmatrix} A_{vv} & -D_{vv} \\ -D_{vv}^T & 0 \end{bmatrix} \begin{bmatrix} t_v \\ u_v \end{bmatrix} = \begin{bmatrix} -A_{vp} & D_{vp} \\ D_{pv}^T & 0 \end{bmatrix} \begin{bmatrix} t_p \\ u_p \end{bmatrix} - \begin{bmatrix} 0 \\ F_p + P_p \end{bmatrix} \tag{5.09}$$

In this expression unknown (variable) stresses t_v and displacements u_v, and the known (prescribed) ones t_p and u_p, are separated. The members of the matrices A and D, and vectors (column matrices) F and P (the discretized body and surface forces) are, respectively:

$$A_{NuvLst} = \sum_e \int_{\mathcal{E}} S_N\, g^a_{(N)u}\, g^b_{(N)v}\, A_{abcd}\, g^c_{(L)s}\, g^d_{(L)t} T_L \,\mathrm{d}V, \tag{5.10}$$

$$D^{Kq}_{Nuv} = \sum_e \int_{\mathcal{E}} S_N\, U^K_a\, g^a_{(N)u} \,\mathrm{d}V\, g^{(K)q}_{(N)v}, \tag{5.11}$$

$$F^{Mq} = \sum_e \int_{\mathcal{E}} g^{(M)q}_a\, V^M f^a \,\mathrm{d}V, \tag{5.12}$$

$$P^{Mq} = \sum_e \int_{\mathcal{E}} g^{(M)q}_a\, V^M p^a \,\mathrm{d}V. \tag{5.13}$$

In these expressions $g^{(K)q}_{(N)v}$ and $g^{(M)q}_a$ are the Euclidean shifters. Computation of these quantities is described in the Appendix A. Furthermore, $U^K_a = \partial U^K / \partial \xi^a$, and ξ^b $(a, b, c, d = 1, 2, 3)$ are local (element) coordinates, usually convected (parametric, isoparametric). A_{abcd} are the components of the elastic compliance tensor $\mathbb{A}$.

A full flexibility of the present, coordinate independent formulation allows that nodal coordinates $x^{(K)n}$ and $y^{(L)s}$ (see Appendix A) can be chosen arbitrarily and independently at each node, and hence enables the exact enforcement of the (essential) boundary conditions. In addition, some or all of the stress components, which are already prescribed, can be kept variable, which allows a lot of experimentation with the boundary conditions. Note that in the last case, when all stress components are retained variable, the spaces (5.07) and (5.08) reduce to (5.05) and (5.06).

6. Solvability and stability of a system

6.1. Fundamentals

Let us consider a system (5.09), and rewrite this expression shortly as

$$\begin{bmatrix} A & -D \\ -D^T & 0 \end{bmatrix} \begin{bmatrix} t \\ u \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}. \tag{6.01}$$

In this expression, D is an $n_t \times n_u$ matrix. D^t maps an n_t dimensional vector t into a n_u dimensional vector g.

Let us apply now the QR decomposition over the matrix B:

$$D = Q \begin{bmatrix} R \\ 0 \end{bmatrix}, \tag{6.02}$$

where Q is an *orthogonal* matrix, and R is an *upper (right) triangular* matrix. Q can be partitioned [6] into Y, whose columns span the *range space* of D, and Z, whose columns span the *null space* of D:

$$D = [Y \quad Z] \begin{bmatrix} R \\ 0 \end{bmatrix}. \tag{6.03}$$

Consequently, one can write that

$$D = YR. \tag{6.04}$$

Any n_t dimensional vector can be expressed as a linear combination of

$$n_t = n_Y + n_Z, \tag{6.05}$$

linearly independent orthogonal vectors–columns of Y and Z and hence

$$t = Yp + Zq = [Y \quad Z] \begin{bmatrix} p \\ q \end{bmatrix}. \tag{6.06}$$

After some algebra the equation (6.01) can be rewritten as

$$\begin{bmatrix} -R & Y^tAZ & Y^tAY \\ 0 & Z^tAZ & Z^tAY \\ 0 & 0 & -R^t \end{bmatrix} \begin{bmatrix} u \\ q \\ p \end{bmatrix} = \begin{bmatrix} Y^tf \\ Z^tf \\ g \end{bmatrix}. \tag{6.07}$$

6.2. On the effective solution of a system

It is evident that, at least in principle, one can determine p and q from (6.07). Hence, it is possible to determine the stresses t from (6.06), before than the displacements u, which can be found *a'posteriory* also from (6.07).

Consequently, we can call such an approach by the traditional name "*the force method*", where p and q are, respectively, "*statically determinate*" and "*statically indeterminate*" generalized forces.

Although (6.07) can be used for the effective solution of (6.01) or (5.09), especially when A is a singular matrix, the examples considered in this paper have regular A matrices, and hence can be solved by the symmetric solver described in [14]. Note also that the system matrix in (6.01), although not necessarily positive definite, is nonsingular and symmetric, and hence also can be solved by the symmetric Gaussian elimination.

6.3. Solvability of a system

According to [9] the conditions for solvability of (6.01) and hence of (5.09) are now clear: we need that

$$R^t, \quad \text{as a mapping:} \quad Y \rightarrow \mathbb{R}^{n_u}, \quad \text{is invertible.} \tag{6.08}$$

On the other hand

$$Z^t A Z, \quad \text{as a mapping:} \quad Z \rightarrow Z, \quad \text{is invertible.} \tag{6.09}$$

Note that from (6.07) it follows that R^t is a lower triangular and hence square matrix. Consequently

$$n_Y = n_u. \tag{6.10}$$

However, if one wants for R^t to be invertible, it is necessary also that

$$\operatorname{rank} D = n_u. \tag{6.11}$$

Note also that from (6.09) it follows that

$$\operatorname{rank} A \geq n_Z. \tag{6.12}$$

Now we can express the conditions (6.08) and (6.09) respectively in terms of the matrices D and A, and of the null space Z:

$$Du = 0 \quad \Rightarrow \quad u = 0, \tag{6.13}$$

$$y^t A y \geq 0 \quad \forall y \in Z. \tag{6.14}$$

Due to (6.03) and (6.05)

$$n_t = n_Z + n_Y = n_Z + n_u \geq n_u, \tag{6.15}$$

which is (obviously) a necessary condition for the solvability of (6.01). However, it is not sufficient. Already, solvability can be defined by the following proposition:

Let A be an $n_t \times n_t$ square matrix and D an $n_t \times n_u$ matrix. Furthermore, let Z be defined as in (6.03). *The linear system* (6.01) *is uniquely solvable for every $f \in \mathbb{R}^{n_t}$ and for every $g \in \mathbb{R}^{n_u}$ if and only if conditions* (6.13,14) *are satisfied.*

However, from the practical point of view, the rank conditions (6.11-12) also are a useful check for solvability of (6.01).

6.4. Stability of a system

The stability and solvability are closely related. I.e., the system (6.01) is not solvable if its matrix

$$M = \begin{bmatrix} A & -D \\ -D^T & 0 \end{bmatrix}, \tag{6.16}$$

is *singular* (not invertible) i.e. if the determinant of M is *zero*. Analogously, the solution of (6.01) is *unstable*, if the determinant of (6.16) is *small*, i.e. if the matrix M is *ill-conditioned.*

Anyhow, the stability deserves to be more precisely determined. According to Arnold [5], the stability property of (6.01) refers to the continuity of mapping from the data f and g to the solution u and t.

The stability constant

The stability constant S is a smallest constant such that

$$\frac{||\delta t|| + ||\delta u||}{||t|| + ||u||} \leq S \, \frac{||\delta f|| + ||\delta g||}{||f|| + ||g||}, \tag{6.17}$$

for all t, u, δt, δu and f, g, δf, δg with $At - Du = f$; $Dt = g$; $A\delta t - D\delta u = \delta f$; $D\delta t = \delta g$. Roughly speaking, the relative error of a solution is bounded by the relative error of the right side multiplied by the *stability constant* $S \geq 1$. In the problem under consideration, the stability constant can be defined as a product of norms of M and of its inverse [9].

Brezzi's theorem

The Brezzi's theorem will be explained following Arnold [5], without proof which can be found elsewhere [9]. Let us denote first the integrals from (4.07) as

$$a(\mathbf{S}, \mathbf{T}) = \int_{\mathcal{B}} \mathbf{S}{:}\mathbb{A}{:}\mathbf{T} \, dV, \quad d(\mathbf{S}, \mathbf{u}) = \int_{\mathcal{B}} \mathbf{S}{:}\nabla \mathbf{u} \, dV. \tag{6.18}$$

In the next step we define the quantity

$$\beta_h = \inf_{S \in S_h} \; \sup_{\mathbf{u} \in \mathbf{U}_h} \frac{d(\mathbf{S}, \mathbf{u})}{||\mathbf{S}|| \; ||\mathbf{u}||}, \tag{6.19}$$

and suppose that there exists a positive constant α_h such that

$$a(\mathbf{T}, \mathbf{T}) = \alpha_h ||\mathbf{T}||^2 \quad \forall \mathbf{T} \in \mathbf{Z}_h, \tag{6.20}$$

where

$$\mathbf{Z}_h = \{\mathbf{T} \in \mathbf{T}_h(\mathcal{B}) \mid d(\mathbf{T}, \mathbf{u}) = 0 \quad \forall \mathbf{u} \in \mathbf{U}_h(\mathcal{B}) \quad \forall \mathcal{E} \in \mathcal{C}_h\}. \tag{6.21}$$

Then the stability constant S may be bounded in the terms of reciprocals of the constants α_h and β_h. Thus if for a sequence of problems of type (6.01) the α_h remain bounded uniformly above zero (this is the *first Brezzi condition*) and the β_h do likewise (*second Brezzi condition*), then the resulting method is stable.

According to Arnold [5], the first Brezzi's condition simply asserts the invertibility of the operator $Z^t AZ$ in (6.07). Similarly, the second Brezzi's condition just asserts the invertibility of R.

6.5. Some special properties of the system under consideration

Before of all, let us note that norms of the particular importance for the present case are:

$$||\mathbf{T}||^2_{\mathbf{H(div)}} = |\mathbf{T}|^2_{\mathbf{o}} + |\mathbf{div T}|^2_{\mathbf{o}}, \quad ||\mathbf{T}||^2_{\mathbf{T}^2} = |\mathbf{T}|^2_{\mathbf{o}}, \tag{6.22}$$

where the appropriate seminorms [6] are

$$|\mathbf{T}|^2_{\mathbf{o}} = \sum \int |\mathbf{T}|^2 \, \mathrm{d}V, \quad |\mathbf{div T}|^2_{\mathbf{o}} = \sum \int |\mathbf{div T}|^2 \, \mathrm{d}V. \tag{6.23}$$

If the weak form (4.06) is used, it follows that ([5], p. 294) "*Since it is possible to find* $\mathbf{T}$ *which is bounded by* 1 *everywhere but for which the divergence of* $\mathbf{T}$ *is arbitrarily large, there cannot exist a constant* α *such that* $a(\mathbf{T}, \mathbf{T}) \geq \alpha_h ||\mathbf{T}||^2_{\mathbf{H(div)}}$ *for all* $\mathbf{T}$ *in* $\mathbf{H}(\mathbf{div})$. *So the "a" form is indeed not coercive*".

By contrast, in the present paper a weak form (4.07) and hence a norm $||\mathbf{T}||_{\mathbf{T}^2}$ is used. Moreover, in the analysis of the regular mixed ($\mathbf{T}$, $\mathbf{u}$) formulation of elasticity problem, A is positive definite and (6.19) is satisfied for all $\mathbf{T} \in \mathbf{T}_h(\mathcal{B})$. Consequently, in the present problem "a" *is* coercive.

Remarks on the solvability

In the accordance with Oden [10] p. 134, if the second Brezzi's condition is to hold, so also must a condition on the rank of D, (6.11). Then an β_h exists such that (6.19) holds whenever

$$Dv = 0 \quad \text{implies that} \quad v = 0. \tag{6.24}$$

This means that we should have

$$\dim \mathbf{T}_h \geq \dim \nabla \mathbf{U}_h, \tag{6.25}$$

where

$$\nabla \mathbf{U}_h = \{2\mathbf{e}_h = \nabla \mathbf{u}_h + \nabla \mathbf{u}^t_h, \quad \mathbf{u}_h \in \mathbf{U}_h\} \tag{6.26}$$

For the problem considered, $\nabla \mathbf{U}_h$ is evidently a strain space.

Let us discuss now the dimensions of the spaces under consideration. In the absence of the boundary conditions, and if the same mesh is used for both the displacements and stresses, taking also into account the symmetry of the stress tensor, the dimensions of the displacement, strain and stress spaces will be respectively:

$$n_u = \dim \mathbf{U}_h = n\, N_u; \quad n_t = \dim \mathbf{T}_h = \tfrac{1}{2} n(n+1) N_t;$$
$$n_e = \dim \nabla \mathbf{U}_h = \tfrac{1}{2} n(n+1) N_u. \tag{6.27}$$

Evidently, $\mathbf{T}_h$ and $\nabla \mathbf{U}_h$ are the spaces of the second order symmetric tensors. Each of these tensors has $\frac{1}{2}n(n+1)$ components, where n is a number of the spatial dimensions of the problem under consideration. Furthermore, N_t is the number of nodes of a stress mesh, while N_u is the number of nodes of a displacement mesh. From (6.25) and (6.27) it follows directly that, in the absence of the stress boundary conditions, the rank condition (6.11) will be satisfied if

$$N_t \geq N_u. \tag{6.28}$$

This relationship justifies the success of the scheme defined by (4.07), (5.05) and (5.06) and described in [11]–[14], where $N_t \equiv N_u$.

Let us note also that, because at each node K_t of a stress mesh we have (due to the symmetry of a stress tensor) $\frac{1}{2}n(n+1)$ stress degrees of freedom, and at each node K_u of a displacement mesh n displacement degrees of freedom, the relationship (6.15) can be rewritten as

$$\frac{1}{2}n(n+1)N_t \geq nN_u. \tag{6.29}$$

If (6.28) holds, (6.29) will be satisfied for any and every value of n. However, the reverse is not true. Consequently, (6.28) is a much stronger condition than the popular rule (6.15) and hence more helpful in giving ideas how to construct and modify the trial space to maintain solvability.

Furthermore, if (some or all) of the stress boundary conditions are enforced, the condition (6.28) can be replaced by somewhat conservative heuristic rule

$$N_t - N_t^* \geq N_u. \tag{6.30}$$

In this expression, N_t^* is the number of nodes having at least one of the stress components prescribed.

It is evident that (6.30) cannot be satisfied for $N_t = N_u$. Hence, if the stress space (5.07) is used instead of (5.05), it is necessary to enrich the elements by the additional nodes (or eventually to enrich nodes by the additional degrees of freedom). More details will be given in the discussion on the numerical examples.

Remarks on stability

According to Arnold [5] p. 224, if "a" is coercive and if we choose a sequence of $\mathbf{T}_h$ and $\mathbf{H}_h$ for which β_h stays bounded away from zero, the corresponding method is stable. Moreover, as the space $\mathbf{T}_h$ increases, for fixed $\mathbf{U}_h$, the constant β_h increases. In other words, for problems for which the "a" form is coercive, enrichment of the space $\mathbf{T}_h$ increases stability. Practically any such scheme can be rendered stable by the sufficient enrichment of the $\mathbf{T}_h$ space, the only limitation being the cost associated to the extra (stress) degrees of freedom. Hence, if the stress space of the present problem is enriched in the accordance with (6.30) due to the solvability reasons, one also can expect that the computational scheme will be stable.

7. Plane stress problem

Numerical experiments will be conducted for the plane stress problem. Let us for a moment neglect the boundary conditions. Then at each node there are two displacement degrees of freedom, and three stress degrees of freedom. Hence, if the same mesh is used for both the stresses and displacements, from (6.27) it follows that

$$n_t = 1.5\, n_u. \tag{7.01}$$

However, in the absence of the boundary (and any other) constraints on the stresses and displacements it is required, due to (6.25) to have

$$n_t \leq 1.5\, n_u, \tag{7.02}$$

and, in the accordance with (7.01) the use of the shape functions of the same order for the stresses and displacements is barely sufficient to maintain solvability and stability of a system. In other words, the matrix D of the size $n_t \times n_u$ has only n_u linearly independent rows, minimum necessary to satisfy (6.11). This result is closely related with the fact that the strain (and stress) components in elasticity are not completely independent due to the compatibility conditions.

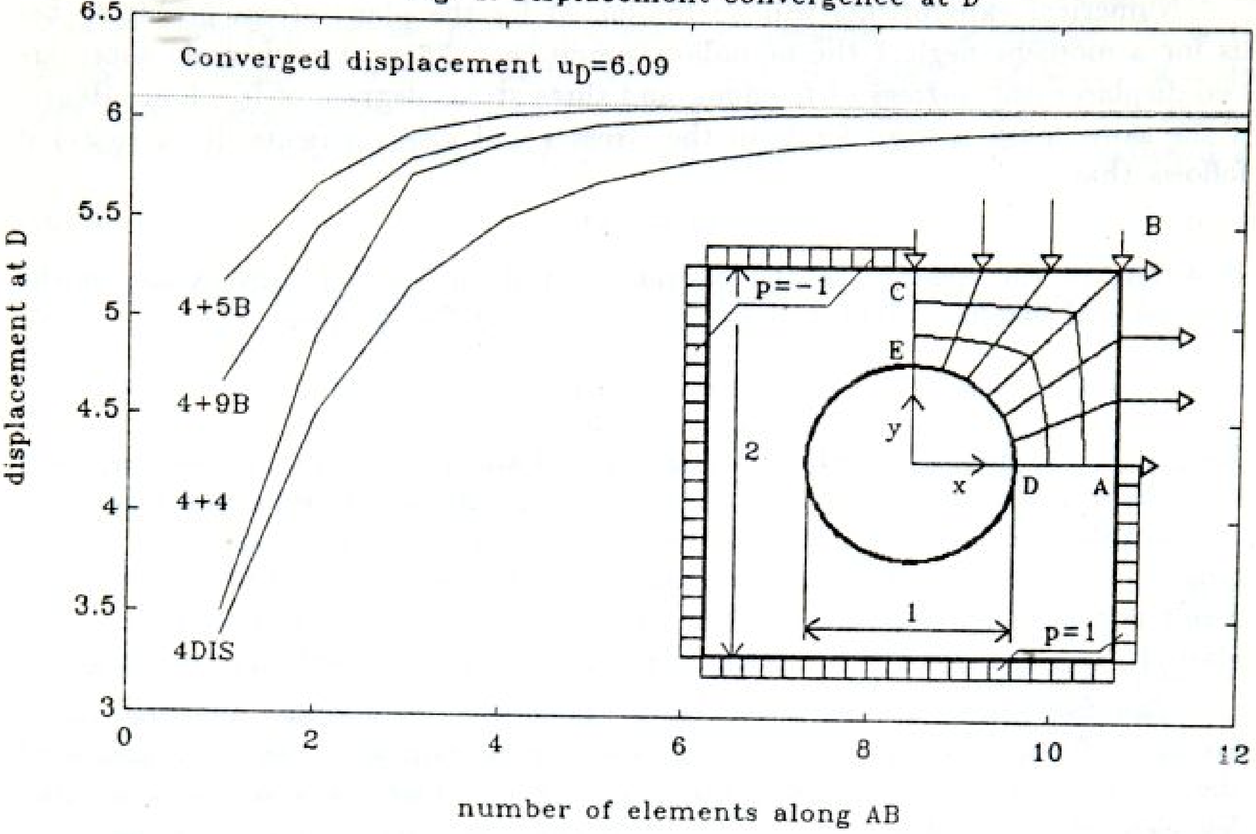
The fulfillment of the displacement boundary conditions, reducing n_u is favorable from the point of view of (7.02), in contrary to the satisfaction of the stress boundary conditions, which reduce n_t. Only in some very special, few-element cases (where the influence of the boundary conditions is decisive) (7.02) can be reduced to the algebraic minimum (6.15).

8. Illustrative examples

The particular example (Fig. 1.), has been used because its complex boundary and state of stress places a difficult requirements on the element behavior. The problem under consideration is a square plate of the unit semispan, with a central circular hole of the unit diameter. The plate is loaded along its sides by the unit load, tensile in $x = z^1$, and compressive in $y = z^2$ direction. Modulus of elasticity and Poisson's coefficient are taken to be $E = 1$ and $\nu = 0.3$. Local coordinate systems for the determination of the stresses at nodes situated along ED on Fig. 1. can be cylindrical or local Cartesian, having one axis tangential and other orthogonal to the interior contour circle at the boundary node.

For the comparison of the present results with the standard finite element ones, the first (lowest) curve on a Fig. 1. corresponds to a bilinear isoparametric four node element, labeled by 4DIS. The next curve, labeled by 4+4, corresponds to a standard mixed FE procedure with bilinear and continuous at the element interfaces both the displacements and stresses [13], [14]. Note that for very coarse meshes model 4+4 is not a much superior over 4DIS. Anyhow, for the realistic mesh densities the superiority is clear. Per instance, the displacement u_D for 5×5 mesh 4+4 is 5.99, while for 10×10 mesh 4DIS is 5.98. At this point let us notify that the converged value of u_D (dotted line on the Fig. 1.) is 6.09.

Fig. 1. Displacement convergence at D



However, relatively unsatisfactory behavior of the element for coarse meshes induced an idea of the improvement of a model by the introduction of the stress boundary conditions as the essential ones, described in detail by the present authors in [2].

Unfortunately, the satisfaction of the stress boundary conditions reduces n_t because some of the stresses become *predicted*, and move to the right side of (5.09). In the sequel (7.02) is violated, and (6.01) cannot be solved. The "brute force" remedy of a problem is to increase n_t by the use of the complete quadratic (nine node) stress distribution in the interior of an element. The results are promising (see curve 4+9B, where B denotes fulfillment of the stress boundary conditions). The improvement is noteworthy especially for very coarse meshes. A more detailed study of 4+9B model, not presented here, indicated the possible presence of the membrane locking due to the self-equilibrated stress modes in a model, and the possible improvement if only a central, "bubble" mode of the second-order stress distribution is retained. For easier manipulation with the stress modes, hierarchic (Appendix B and [1] p. 140) stress distribution is accepted in the present study. The results of this approach, shown at the curve 4+5B are more than satisfactory.

In addition to the qualitative discussion based on a Fig. 1, some quantitative measures (estimates) of the convergence rates (Fig. 2) can be very helpful.

Classical bilinear displacement (4 DIS) model obeys almost optimal convergence rate for the displacement value, i.e. error in the convergence is close to the approximation error in the Euclidean or in maximum norm, $O(h^2)$, or, the convergence rate is close to 2. However, if the model 4+4 is considered (4DIS + bilinear continuous stresses), one can see that the superconvergence properties are exhibited, i.e. rate approaches 3. Finally, if the model 4+5B is used, an average rate of convergence is close to 4. Mean convergence rates shown in Fig. 2 are determined by the least squares fit of the numerical results.

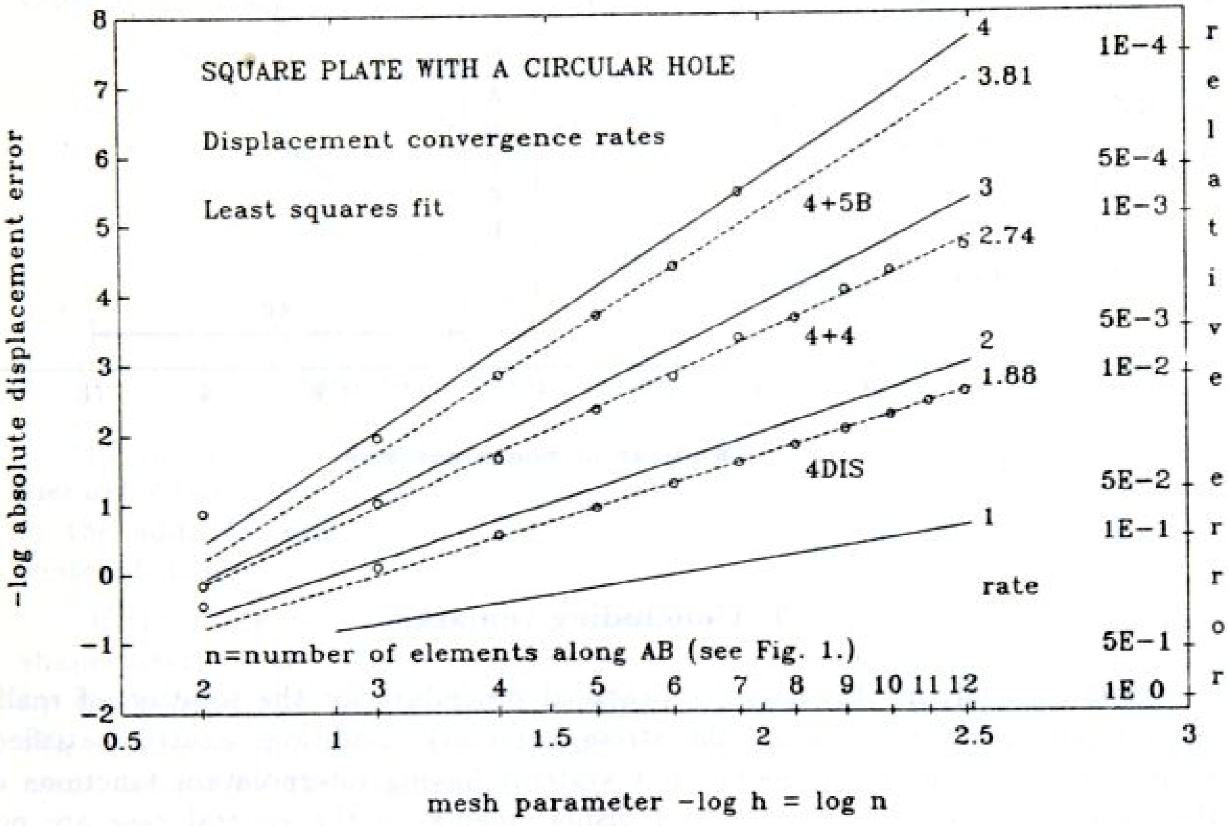
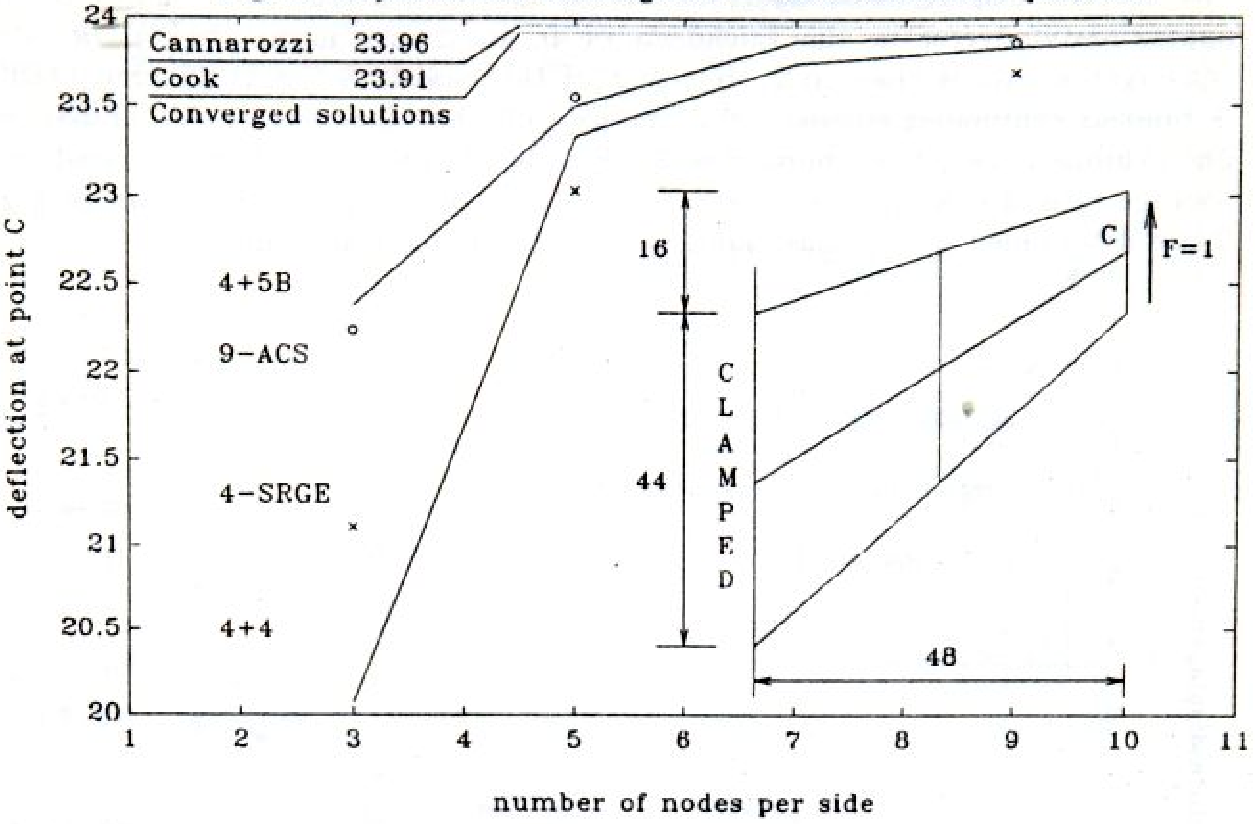


Fig.2.

An important measure of the quality of a model is also the comparison (Fig.3) of the present results for the Cook's skew cantilever benchmark problem (E=1, $\nu = 1/3$, t=1) with the best available contemporary elements, four node quadrilateral 4-SRGE of Simo *et al* [18] and nine node quadrilateral 9-ACS of Jang and Pinsky [19], (or equivalent of Ma and Kanok-Nukulchai [20]). The results are compared with converged solution. The tip displacement most probably closest to the theoretical value is the converged solution computed by Cannarozzi [21], $\nu = 23.96$, which is slightly at variance to the original result $\nu = 23.91$ of Cook [22].

Fig. 3. Displacement convergence – Cook membrane problem



9. Concluding remarks

Before of all, in this paper a practical procedure for the solution of really sized mixed problems, having the stress boundary conditions exactly satisfied, is described. It has been shown that systems having interpolation functions of the same order for the stresses and displacements, in the general case are not solvable. For the solvability and stability of a solution it is necessary to increase the number of the stress degrees of freedom, but not enormously. The reasonable compromise for the general use is to retain central, bubble nodes, similarly to the usual remedies in the Stokes problem [5]. Anyhow, the most important properties of the present approach are its remarkably high convergence rate and accuracy.

Appendix A. Calculation of the Euclidean shifters

Here we quote some of the results already developed in [2], and necessary in any kind of correct computations connected with the main results in this paper. In the following expressions z^i $(i, j, k, l = 1, 2, 3)$ are the global Cartesian coordinates, $x^{(K)n}$ $(m, n, p, q = 1, 2, 3)$ are local (nodal) coordinates, to be used for the determination of the nodal displacements. Coordinates for the nodal stresses (not necessarily) common for the adjacent elements will be denoted by $y^{(L)s}$ $(r, s, t, u, v = 1, 2, 3)$.

Note also that the components of the contravariant fundamental metric tensors, $g^{(K)mn}$ with respect to the $x^{(K)n}$ and g^{ab} with respect to ξ^b, should be computed as the members of the matrices $\left[g_{(K)mn}\right]^{-1}$ and $\left[g_{ab}\right]^{-1}$, inverse to the matrices of the covariant metric tensors

$$g_{(K)mn} = \delta_{kl} \frac{\partial z^k}{\partial x^{(K)m}} \frac{\partial z^l}{\partial x^{(K)n}}, \quad g_{ab} = \delta_{kl} \frac{\partial z^k}{\partial \xi^a} \frac{\partial z^l}{\partial \xi^b}, \tag{A.01}$$

respectively. Definitively,

$$g_{(L)s}^{(K)m} = \delta_{kl} g^{(K)mn} \frac{\partial z^k}{\partial x^{(K)n}} \frac{\partial z^l}{\partial x^{(L)s}}, \tag{A.02}$$

$$g_{(L)s}^{a} = \delta_{kl} g^{ab} \frac{\partial z^k}{\partial \xi^k} \frac{\partial z^l}{\partial y^{(L)s}}, \tag{A.03}$$

$$g_{b}^{(K)q} = \delta_{kl} g^{(K)qp} \frac{\partial z^k}{\partial \xi^b} \frac{\partial z^l}{\partial x^{(M)p}}. \tag{A.04}$$

Appendix B. Hierarchic shape functions

In the process of construction and modification of a trial space the use hierarchic shape functions is extremely useful, because these allow great flexibility in the addition and removal of the additional (midside, midsurface and body centered nodes).

If the finite elements we are considering are n-cubes, and the fundamental shape functions are n-linear, one can cover also the additional degrees of freedom by the n-quadratic family. Let us define now hierarchic interpolation functions W_K of the second order over the unit n-cube canonical (master, reference) element defined by the coordinates

$$-1 \le \xi^\alpha \le 1, \quad \alpha = 1, 2, \ldots, n \tag{B.01}$$

$$W_K^\alpha = \prod_{\alpha=1}^{n} W_K^\alpha; \quad W_K^\alpha = \left(1 - \frac{1}{2}|\xi_K^\alpha|\right)(1 + \xi_K^\alpha \xi^\alpha)\left[1 - (1 - |\xi_K^\alpha|)(\xi^\alpha)^2\right], \tag{B.02}$$

where ξ_K^α are the nodal coordinates. Note that we can distinguish $1+n$ types of nodes. The nodes of the 0-th type are those for which all nodal coordinates take values $\xi_K^\alpha = \pm 1$. The nodes of the k-th type are those for which $k = 1, 2, \ldots, n$ of the nodal coordinates take values $\xi_K^\alpha = 0$ in turn. In the accordance with the definition of members of D matrices, the first derivatives of W_K

$$\frac{\partial W_K}{\partial \xi^\gamma} = \frac{\partial W_K^\gamma}{\partial \xi^\gamma} \prod_{\substack{\alpha=1 \\ \alpha \ne \gamma}}^{n} W_K^\alpha; \quad \frac{\partial W_K^\gamma}{\partial \xi^\gamma} = \left(1 - \frac{1}{2}|\xi_K^\gamma|\right)\xi_K^\gamma - 2(1 - |\xi_K^\gamma|)\xi^\gamma, \tag{B.03}$$

are also of the computational interest.

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МОДЕЛЬ ДВУХ ПОЛЕЙ СВЯЗАННАЯ С ПРИНЦИПОМ РЕЙССНЕРА В МЕТОДЕ КОНЕЧНЫХ ЭЛЕМЕНТОВ

В этой работе показанно что смешанные проблемы МКЭ в теории упругости, если полиномы одного же порядка пользуются для интерполяции перемщений и напряжений, и если граничние условия по напряжениямх выпольняются как эссенциальные, как правило не удовлетворяют условиям Бреци (Brezzi). Проблему возможно разрешить формально, если брать интерполяционных функций высшего порядка для напряжений. Анализируются детали разрешимости данной проблемы. Специальное внимание посвящается некоторым количественным характеристикам (размерам) рассматрываемых подпространств конечных элементов. Сверхсходящиеся численные решения оправдывают предлагаемую расчетную схему.

MODEL DVA POLJA U METODI KONAČNIH ELEMENATA POVEZAN SA RAJSNEROVIM PRINCIPOM

U radu je pokazano da mešoviti problemi MKE u elastičnosti, ako se polinomi istog reda koriste za interpolaciju pomeranja i napona, i ukoliko su granični uslovi po naponima zadovoljeni kao esencijalni, po pravilu ne zadovoljavaju Brecijeve (Brezzi) uslove. Problem može da se reši formalno, ukoliko za napone koristimo interpolacione funkcije višega reda. Detaljno se analizira rešivost problema. Posebna pažnja je posvećena nekim kvantitativnim karakteristikama (razmerama) razmatranih potprostora konačnih elemenata. Superkonvergentna numerička rešenja opravdavaju predloženi prilaz.

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