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AGING CREEP OF COMPOSITE BEAMS

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Introduction

The paper deals with coaction in composite beams consisting of a finite number of aging linear viscoelastic materials. This is a generalization of the problem existing in the theory of composite and prestressed beam structures.

The considerations are limited to the change of strains and stresses in the cross section and the generalized displacements of the beam when the axial force and the bending moment follow the prescibed laws. The results are stated in the form of three theorems.

Lazić's and Bažant's theorems, known in extant literature, represent the special cases of those theorems set in the present paper.

1. Assumptions and the basic equations

A composite beam structure of variable cross section is considered and suppositions of engineering beam-bending theory are retained. The cross section consists of f different aging linear viscoelastic materials symmetrically arranged with respect to the loading plane.

The uniaxial aging creep law can be symbolically written in the form:

$$\varepsilon - \varepsilon_j^0 = \frac{1}{E_{j0}} \tilde{P}_j' \sigma_j, \quad j = 1, 2, \dots, f, \tag{1.1}$$

i.e.

$$\sigma_j = E_{j0} \tilde{Q}_j' (\varepsilon - \varepsilon_j^0), \quad j = 1, 2, \dots, f, \tag{1.2}$$

(see Appendix and Notations).

The linear integral operations, the creep operator $\tilde{P}_j'$ and the relaxation operator $\tilde{Q}_j'$, are inverse satisfying the known relation:

$$\tilde{P}_j' \tilde{Q}_j' = \tilde{Q}_j' \tilde{P}_j' = \tilde{1}', \quad j = 1, 2, \dots, f. \tag{1.3}$$

In a special case when the k[th] material is linear elastic the following is valid:

$$\tilde{P}_k' \equiv \tilde{1}', \quad \tilde{Q}_k' \equiv \tilde{1}'. \tag{1.4}$$

Bernoulli's assumption of plane cross section:

$$\varepsilon = \eta + y\kappa, \tag{1.5}$$

and the equilibrium equations:

$$\sum_{j=1}^{f} \int_{A_j} \sigma_j \, \mathrm{d}A = N, \quad \sum_{j=1}^{f} \int_{A_j} y\, \sigma_j \, \mathrm{d}A = M, \tag{1.6}$$

yield the basic equation [4] given in the operator form, too:

$$\begin{aligned} E_u\, A_i\, \tilde{R}_{11}'\, \eta + E_u\, S_i\, \tilde{R}_{12}'\, \kappa &= N + N^0, \\ E_u\, S_i\, \tilde{R}_{21}'\, \eta + E_u\, J_i\, \tilde{R}_{22}'\, \kappa &= M + M^0, \end{aligned} \tag{1.7}$$

where the operators $\tilde{R}_{hs}'$ are defined as follows:

$$\tilde{R}_{11}' = \frac{1}{A_i} \sum_{j=1}^{f} A_{jr} \tilde{Q}_j', \quad \tilde{R}_{22}' = \frac{1}{J_i} \sum_{j=1}^{f} J_{jr} \tilde{Q}_j', \tag{1.8}$$

$$\tilde{R}_{12}' = \tilde{R}_{21}' = \frac{1}{S_i} \sum_{j=1}^{f} S_{jr} \tilde{Q}_j',$$

and:

$$N^0 = E_u \sum_{j=1}^{f} A_{jr} \tilde{Q}_j' \varepsilon_j^0, \quad M^0 = E_u \sum_{j=1}^{f} S_{jr} \tilde{Q}_j' \varepsilon_j^0. \tag{1.9}$$

The system of two inhomogeneous integral equations Eq.(1.7) has the solution:

$$\begin{aligned} E_u \eta &= \frac{1}{A_i} \tilde{F}_{11}' (N + N^0) + \frac{1}{S_i} \tilde{F}_{12}' (M + M^0), \\ E_u \kappa &= \frac{1}{S_i} \tilde{F}_{21}' (N + N^0) + \frac{1}{J_i} \tilde{F}_{22}' (M + M^0). \end{aligned} \tag{1.10}$$

Further, we will deal with materials the relaxation operators of which $\tilde{Q}_j'$ have the form:

$$\tilde{Q}_j' = \varrho_j' \tilde{1}' + \varrho_j \tilde{R}', \quad \varrho_j' = 1 - \varrho_j, \quad 0 \le \varrho_j \le 1, \quad j = 1, 2, \ldots, f, \tag{1.11}$$

ϱ_j being time-independent quantities and $\tilde{R}'$ the relaxation operator corresponding to an arbitrarily chosen aging linear viscoelastic material. It can be one of f materials coacting in the composite cross section.

Let $\tilde{F}'$ be the creep operator on the same material. Then the following is valid:

$$\tilde{F}'\tilde{R}' = \tilde{R}'\tilde{F}' = \tilde{1}'. \tag{1.12}$$

The basic equations Eq. (1.7) and their solution Eq. (1.10) have the same form as the corresponding equations of the mathematical theory of composite and prestressed beam structures [2]. Only the elements of the matrix $\|\gamma_{hs}\|_{2,2}$, appearing in the operators $\tilde{R}'_{hs}$ $(h,\ s = 1,2)$, are defined by different expressions.

The operator matrix $\left\|\tilde{R}'_{hs}\right\|_{2,2}$ of Eq.(1.7) has the individual forms:

$$\tilde{R}'_h = \gamma'_h\tilde{1}' + \gamma_h\tilde{R}', \quad \gamma'_h = 1 - \gamma_h, \quad h = 1,2. \tag{1.13}$$

The quantities γ_h represent the individual values of the matrix $\|\gamma_{hs}\|_{2,2}$ associated to the composite cross section. Its elements depend on the cross section geometrical properties and, through the coefficients ϱ_j, on the rheological properties of coacting materials (see Notations).

The individual forms of the operator matrix $\left\|\tilde{F}'_{hs}\right\|_{2,2}$ of Eq. (1.10) are denoted by $\tilde{F}'_h$ (h=1,2). Operators $\tilde{F}'_h$ and $\tilde{R}'_h$ satisfy the expression corresponding to the inverse operators:

$$\tilde{F}'_h\tilde{R}'_h = \tilde{R}'_h\tilde{F}'_h = \tilde{1}', \quad h = 1,2. \tag{1.14}$$

The operators $\tilde{F}'_{hs}$, appearing in Eq. (1.10), are directly expressible in terms of the individual forms of the operator matrix $\left\|\tilde{F}'_{hs}\right\|_{2,2}$:

$$\tilde{F}'_{11} = \frac{1}{\Delta\gamma}\left(\delta\gamma_2\tilde{F}'_1 + \delta\gamma_1\tilde{F}'_2\right), \quad \tilde{F}'_{22} = \frac{1}{\Delta\gamma}\left(\delta\gamma_1\tilde{F}'_1 + \delta\gamma_2\tilde{F}'_2\right),$$
$$\tilde{F}'_{12} = \tilde{F}'_{21} = \frac{\gamma_{12}}{\Delta\gamma}\left(\tilde{F}'_1 - \tilde{F}'_2\right), \tag{1.15}$$

(see Notations).

Cross sections consisting of q $(q < f)$ aging linaer viscoelastic materials and $f - q$ linear elastic materials are examined [4]. It is shown that a composite cross section can be considered as homogeneous made of a hypotetical aging linear viscoelastic material, too, the proprties of which depend on the rheological properties of coacting materials and on the cross section geometry. Two creep functions F^*_h or two relaxation functions R^*_h (h=1,2), (see Apendix), determine the rheological properties of the hypothetical material. They are called the cross section creep and relaxation functions, respectively. The same conclusion can be applied in the case we consider here.

For linear integral operators the laws of algebra of ordinary numbers are valid including, in our case, the commutativity law. Keeping in mind the foregoing we can derive the following operator expressions.

Starting from Eq. (1.14), where Eq. (1.13) is applied, we arrive at:

$$\tilde{F}_h^{'}\tilde{R}^{'} = \frac{1}{\gamma_h}\left(\tilde{1}^{'} - \gamma_h^{'}\tilde{F}_h^{'}\right), \quad h = 1, 2, \tag{1.16}$$

i.e.

$$\tilde{F}_h^{'}R^* = \frac{1}{\gamma_h}\left(1^* - \gamma_h^{'}F_h^*\right), \quad h = 1, 2, \tag{1.17}$$

(see Appendix). Multiplying Eq. (1.16) by the operator $\tilde{F}^{'}$, and substituting Eq. (1.12), after integration we get:

$$\tilde{F}_h^{'}F^* = \frac{1}{\gamma_h^{'}}\left(F^* - \gamma_h F_h^*\right), \quad h = 1, 2. \tag{1.18}$$

When Eq. (1.14) for $h = 1$ is multiplied by the operator $\tilde{F}_2^{'}$, using Eq. (1.16) for $h = 2$, we derive:

$$\tilde{F}_2^{'}R_1^* = \frac{1}{\gamma_2}\left(\gamma_1 1^* - \Delta\gamma F_2^*\right). \tag{1.19}$$

Finally, by the similar procedure we obtain:

$$\tilde{F}_1^{'}R_2^* = \frac{1}{\gamma_1}\left(\gamma_2 1^* + \Delta\gamma F_1^*\right). \tag{1.20}$$

The operator relations will be applied to proove the desired theorems.

2. Theorems

We suppose that the axial force $N + N^0$ and the bending moment $M + M^0$, Eqs. (1.7) and (1.9), depend linearly on the cross section relaxation functions R_h^* (h=1,2) and on the relaxation and creep functions R^* and F^* of the introduced, arbitrarily chosen, material:

$$\begin{aligned} N + N^0 &= n_0 1^* + n_1 R_1^* + n_2 R_2^* + n_3 R^* + n_4 F^*, \\ M + M^0 &= m_0 1^* + m_1 R_1^* + m_2 R_2^* + m_3 R^* + m_4 F^*, \end{aligned} \tag{2.1}$$

n_i and m_i (i=0,1,2,3,4) being time-independent quantities. Here is included the assumed law, referring to the stress-independent strain ε_j^0 Eq. (1.1), in the form:

$$\varepsilon_j^0 = \varepsilon_{0j}^0 1^* + \varepsilon_{1j}^0 F^*, \quad j = 1, 2, \ldots, f, \tag{2.2}$$

ε_{0j}^0 and ε_{1j}^0 being constants.

Theorem 1. *If the axial force and the bending moment follow the law given by Eq. (2.1) then the strain ε at an arbitrary point of the cross section depends linearly on the cross section creep functions F_h^* (h=1,2) and on the creep function F^*:*

$$E_u\varepsilon = \overline{\varepsilon}_0 1^* + \overline{\varepsilon}_1 F_1^* + \overline{\varepsilon}_2 F_2^* + \overline{\varepsilon}_3 F^*, \tag{2.3}$$

$\overline{\varepsilon}_i$ $(i = 0, 1, 2, 3,)$ being time-independent quantities.

The proof of this theorem is developed using Eqs. (1.10), (2.1), (1.15), (1.14), (1.17) - (1.20) and (1.5). After some transformations we get:

$$E_u\alpha = \overline{\alpha}_0 1^* + \overline{\alpha}_1 F_1^* + \overline{\alpha}_2 F_2^* + \alpha_3 F^*, \tag{2.4}$$

for $\alpha = \eta$, $\overline{\alpha}_i = \overline{\eta}_i$ and $\alpha = \kappa$, $\overline{\alpha}_i = \overline{\kappa}_i$ $(i = 0, 1, 2, 3)$. Introducing:

$$\overline{\varepsilon}_i = \overline{\eta}_i + y\,\overline{\kappa}_i, \quad i = 0, 1, 2, 3, \tag{2.5}$$

Eq. (2.3) is obtained. The values $\overline{\eta}_i$ and $\overline{\kappa}_i$ are defined by the following expressions:

$$\begin{aligned}
\overline{\eta}_0 &= \gamma_{22}\frac{\mathcal{N}_0}{A_i} - \gamma_{12}\frac{\mathcal{M}_0}{S_i}, \quad \overline{\eta}_1 = \delta\gamma_2\frac{\mathcal{N}_1}{A_i} + \gamma_{12}\frac{\mathcal{M}_1}{S_i},\\
\overline{\eta}_2 &= \delta\gamma_1\frac{\mathcal{N}_2}{A_i} - \gamma_{12}\frac{\mathcal{M}_2}{S_i}, \quad \overline{\eta}_3 = \gamma_{22}'\frac{\mathcal{N}_3}{A_i} + \gamma_{12}\frac{\mathcal{M}_3}{S_i},\\
\overline{\kappa}_0 &= \gamma_{11}\frac{\mathcal{M}_0}{J_i} - \gamma_{12}\frac{\mathcal{N}_0}{S_i}, \quad \overline{\kappa}_1 = \delta\gamma_1\frac{\mathcal{M}_1}{J_i} + \gamma_{12}\frac{\mathcal{N}_1}{S_i},\\
\overline{\kappa}_2 &= \delta\gamma_2\frac{\mathcal{M}_2}{J_i} - \gamma_{12}\frac{\mathcal{N}_2}{S_i}, \quad \overline{\kappa}_3 = \gamma_{11}'\frac{\mathcal{M}_3}{J_i} + \gamma_{12}\frac{\mathcal{N}_3}{S_i},
\end{aligned} \tag{2.6}$$

where:

$$\begin{aligned}
\mathcal{A}_0 &= \frac{1}{\gamma_1\gamma_2}\left(\gamma_1 a_1 + \gamma_2 a_2 + a_3\right),\\
\mathcal{A}_1 &= \frac{1}{\gamma_1\gamma_1'}\frac{1}{\Delta\gamma}\left(\gamma_1\gamma_1' a_0 + \gamma_1'\Delta\gamma\, a_2 - \gamma_1'^2 a_3 - \gamma_1^2 a_4\right),\\
\mathcal{A}_2 &= \frac{1}{\gamma_2\gamma_2'}\frac{1}{\Delta\gamma}\left(\gamma_2\gamma_2' a_0 - \gamma_2'\Delta\gamma\, a_1 - \gamma_2'^2 a_3 - \gamma_2^2 a_4\right),\\
\mathcal{A}_3 &= \frac{1}{\gamma_1'\gamma_2'} a_4.
\end{aligned} \tag{2.7}$$

For $\mathcal{A}_i = \mathcal{N}_i$, $a_j = n_j$ and $\mathcal{A}_i = \mathcal{M}_i$, $a_j = m_j$ $(i = 0, 1, 2, 3)$, $(j = 0, 1, 2, 3, 4)$ the expressions for time-independent quantities $\mathcal{N}_i$ and $\mathcal{M}_i$, appearing in Eq. (2.6), are defined.

Theorem 2. *If the axial force and the bending moment follow the law given by Eq. (2.1) then the stress σ_j at an arbitrary point of the part j of the cross section* *depends linearly on the cross section creep functions F_h^* $(h = 1,2)$ and on the creep and relaxation functios F^* and R^*:*

$$\sigma_j = \overline{\sigma}_{0j} 1^* + \overline{\sigma}_{1j} F_1^* + \overline{\sigma}_{2j} F_2^* + \overline{\sigma}_{3j} F^* + \overline{\sigma}_{4j} R*, \quad j = 1, 2, \ldots, f, \tag{2.8}$$

$\overline{\sigma}_{ij}$ $(i = 0, 1, 2, 3, 4)$ being time-independent quantities.

The proof of this theorems is derived using Eqs. (1.2), (1.11), (2.2), (2.3), (2.5)–(2.7), (1.17) and (1.12). After some transformations we get Eq. (2.8) where:

$$\begin{aligned} \overline{\sigma}_{0j} &= \frac{\nu_j}{\gamma_1\gamma_2} \left\{ \varrho_j' \gamma_1 \gamma_2 \left(\overline{\varepsilon}_0 + E_u \varepsilon_{0j}^0 \ \right) + \varrho_j \left[\gamma_2 \overline{\varepsilon}_1 + \gamma_1 \overline{\varepsilon}_2 + \gamma_1\gamma_2 \left(\overline{\varepsilon}_3 + E_u \varepsilon_{1j}^0\right)\right]\right\}, \\ \overline{\sigma}_{1j} &= \frac{\nu_j}{\gamma_1} (\gamma_1 - \varrho_j) \overline{\varepsilon}_1, \qquad \overline{\sigma}_{2j} = \frac{\nu_j}{\gamma_2} (\gamma_2 - \varrho_j) \overline{\varepsilon}_2, \\ \overline{\sigma}_{3j} &= \nu_j \varrho_j' \left(\overline{\varepsilon}_3 + E_u \varepsilon_{1j}^0\right), \qquad \overline{\sigma}_{4j} = \nu_j \varrho_j \left(\overline{\varepsilon}_0 + E_u \varepsilon_{0j}^0\right), \quad j = 1, 2, \ldots, f. \end{aligned} \tag{2.9}$$

Theorem 3. *If in a composite beam structure the axial force and the bending moment depend linearly on the relaxation and creep functions R^* and F^* then the reduced generalized displacement of the beam Δ^* depends linearly on the creep functions of a finite number of beam cross sections $F_h^{*(a)}$ $(h = 1, 2)$ and on the creep function F^*:*

$$\Delta^* = \sum_{(a)} \left(\overline{\Delta}_0^{(a)} 1^* + \overline{\Delta}_1^{(a)} F_1^{*(a)} + \overline{\Delta}_2^{(a)} F_2^{*(a)} + \overline{\Delta}_3^{(a)} F^* \right), \tag{2.10}$$

$\overline{\Delta}_i^{(a)}$ $(i = 0, 1, 2, 3)$ being time-independent eqantities.

We proove this theorem starting from the expression for a generalized displacement of the beam based on the principle of virtual forces:

$$\Delta^* = E_u J_u \int_L \left[\hat{M}(\zeta, z)\, \kappa(\zeta, t, t_0) + \hat{N}(\zeta, z)\, \eta(\zeta, t, t_0)\right] \mathrm{d}\zeta, \tag{2.11}$$

where $\hat{M}$ and $\hat{N}$ are the bending moment and the axial force, respectively, at point ζ due the corresponding unit generalized force $\hat{P} = 1^*$ at point z.

We adopt the beam model with constant cross section in a finite number of intervals. To the interval (a) correspond the length $L^{(a)}$, the values $\gamma_h^{(a)}$, the cross section creep functions $F_h^{*(a)}$ $(h = 1, 2)$ and so on. Then we can write:

$$\Delta^* = E_u J_u \sum_{(a)} \int_{L^{(a)}} \left[\hat{M}(\zeta, z)\, \kappa^{(a)}(\zeta, t, t_0) + \hat{N}(\zeta, z)\, \eta^{(a)}(\zeta, t, t_0)\right] \mathrm{d}\zeta. \tag{2.12}$$

The axial force and the bending moment change according to Eq. (2.1) where:

$$n_1 = n_2 = m_1 = m_2 = 0. \tag{2.13}$$

Then Eqs. (2.4), (2.6) and (2.7) retain the same form, only in Eq. (2.7) the quantities a_1 and a_2 no longer exist.

When we substitute Eq. (2.4), referring to the interval (a), into Eq. (2.12), we arrive at Eq. (2.10) where:

$$\overline{\Delta}_i^{(a)} = J_u \int\limits_{L^{(a)}} \left[\hat{M}(\zeta, z)\,\overline{\kappa}_i^{(a)}(\zeta) + \hat{N}(\zeta, z)\,\overline{\eta}_i^{(a)}(\zeta)\right] \mathrm{d}\zeta. \tag{2.14}$$

Time-independent values $\overline{\kappa}_i^{(a)}$ and $\overline{\eta}_i^{(a)}$ are defined from Eqs. (2.4), (2.6) and (2.7) where all quantities, except a_i $(i = 0, 3, 4)$, receive the superscript (a).

3. Special cases

From the derived theorems we can develop, as special cases, the known theorems and expressions of the mathematical theory of composite and prestressed beam structures. In this theory it is supposed that concrete (c) is an aging linear viscoelastic material, prestressing steel (p) has the relaxation property, the relaxation function of which depends linearly on the concrete relaxation function, while two other kinds of steel, steel parts (n) and reinforcing steel (m) are linear elastic materials [2]. Then the relaxation functions Q_j^* for $f = 4$ Eq. (1.11) are defined as follows:

$$\begin{gathered} Q_1^* = R_c^* = R^*, \quad \varrho_1 = 1; \quad Q_2^* = R_p^* = \varrho_p' \, 1^* + \varrho_p \, R^*, \quad \varrho_2 = \varrho_p; \\ Q_3^* = R_n^* \equiv 1^*, \quad \varrho_3 = 0; \quad Q_4^* = R_m^* \equiv 1^*, \quad \varrho_4 = 0; \end{gathered} \tag{3.1}$$

where Eq. (1.4) is used (see Appendix).

In the mentioned mathematical theory two Lazić's theorems are derived [2] assuming that:

$$\varepsilon_j^0 \equiv 0, \quad \text{i.e.} \quad \varepsilon_{0j}^0 = \varepsilon_{1j}^0 = 0, \quad j = 1, 2, \ldots, f, \tag{3.2}$$

see Eq. (2.2).

1. If the axial force N and the bending moment M depend linearly on the cross section relaxation functions R_h^* $(h = 1, 2)$ then the functions η and κ depend linearly on the cross section creep functions F_h^* $(h = 1, 2)$.

In that case in Eq. (2.1) four coefficients do not exist:

$$n_3 = n_4 = m_3 = m_4 = 0. \tag{3.3}$$

From Eqs. (2.7) and (2.6) it follows:

$$\mathcal{N}_3 = 0, \quad \mathcal{M}_3 = 0; \quad \overline{\eta}_3 = 0, \quad \overline{\kappa}_3 = 0. \tag{3.4}$$

Then Eq. (2.4) represents the statement of the theorem.

2. If the axial force N and the bending moment M depend linearly on the concrete relaxation function R^* then the stress σ_j at an arbitrary point of the part j of the cross section depends linearly on the cross section creep functions F_h^* $(h = 1, 2)$ and on the concrete relaxation function R^*.

In that case in Eq. (2.1) we introduce:

$$n_1 = n_2 = n_4 = m_1 = m_2 = m_4 = 0. \tag{3.5}$$

Applying Eqs. (2.7) and (2.6) we show that Eq. (3.4) is valid again. Now Eq. (2.5) produces:

$$\bar{\varepsilon}_3 = 0, \tag{3.6}$$

and from Eqs. (2.9) and (3.2) it ensues:

$$\overline{\sigma}_{3j} = 0, \quad j = 1, 2, \ldots, f. \tag{3.7}$$

Then Eq. (2.8) represents the statement of the theorem,

In Ref. [3] it is shown that a generalized displacement of the composite and prestressed beam structure depends linearly on the cross section creep function of a finite number of beam cross section $F_h^{*(a)}$ $(h = 1, 2)$ if the axial force $N + N^0$ and bending moment $M + M^0$ depend linearly on the concrete relaxation function R^*.

From Eqs. (3.5), (2.6), (2.7), (2.4) and (2.14) we get:

$$\overline{\Delta_3^{(a)}} = 0, \tag{3.8}$$

for all values of the superscript (a). Then Eq. (2.10) is in accordance with the expression for generalized displacement given in Ref. [3].

Bažant's theorem [1] refers to concrete as an aging linear viscoelastic material and it reads: if the strain depends linearly on the concrete creep function F^* then the stress depends linearly on the concrete relaxation function R^*. Ref. [2] shows that Bažant's theorem represents the special case of Lazić's theorem, quoted in item 2, so that it certainly represents the special case of theorem 2 developed in the present paper.

Conclusions

In a general case, when in the cross section of a composite beam f aging linear viscoelastic materials coact, the basic equations and their solution have the same form and the same conclusions may be derived as in the mathematical theory of composite and prestressed beam structures. Only the elements of the matrix, representing the reduced cross section geometry, are defined by the different expressions.

The theorems determine the change of the deformations and stresses in a composite cross section and generalized displacements of the composite beams when the axial force and the bending moment follow the prescribed laws. Lazić's theorems, referring to composite and prestressed beam structures, as well as Bažant's theorem, concerning an aging linear viscoelastic material, known in existing literature, are derived from the general theorems as their special cases.

Appendix – Some mathematical explanations

The linear integral operator $\tilde{G}$ is associated to a function of two time arguments t and τ, $G = G(t, \tau)$ which satisfies the condition $G(t, \tau) \equiv 0$ for $t < \tau$. The linear integral operator $\tilde{G}$ is defined for any function $U = U(t, \tau)$, $\tau \geq t_0$, by the following expression:

$$I = I(t, \tau) = \int_{\tau}^{t} G(t, \theta)\, U(\theta, \tau)\, \mathrm{d}\theta = \tilde{G}U. \tag{A.1}$$

In the functions I and U the second argument becomes a parameter if $\tau = t_0$.

The following notations are used:

$$G' = G'(t, \tau) = \frac{\partial G(t, \tau)}{\partial \tau}, \quad 1' = 1'(t, \tau) = \delta(\tau - t),$$

$$1^* = 1^*(t, \tau) = H(t - \tau) = \begin{cases} 1 & \text{for} \quad t > \tau \\ 0 & \text{for} \quad t \leq \tau, \end{cases}$$

where $\delta(\tau - t) = \delta(t - \tau)$ is the Dirac function and $H(t - \tau)$ is the Heaviside step function.

The operator $\tilde{1}'$, associated to the Dirac function, is the unity operator. From the definition of the Dirac function it follows:

$$\tilde{1}'U = U, \quad \tilde{1}'\tilde{G} = \tilde{G}\tilde{1}' = \tilde{G}. \tag{A.2}$$

If G' is substituted for G ($G' \neq 1'$) and 1^* for U in Eq. (A.1) we get:

$$G^* = \tilde{G}'\, 1^* = G^*(t, \tau) = \int_{\tau}^{t} \frac{\partial G(t, \theta)}{\partial \theta} H(\theta - \tau)\, \mathrm{d}\theta = G(t, t) - G(t, \tau). \tag{A.3}$$

The function G^* is called the integral of the function G'. In such a way to the operators $\tilde{R}'$, $\tilde{F}'$, $\tilde{R}'_h$, $\tilde{F}'_h$, $(h = 1, 2)$ and so on correspond the functions R^*, F^*, R^*_h, F^*_h, $(h = 1, 2)$,... respectively.

In particular, the integral of the Dirac function is the Heaviside step function:

$$\tilde{1}'\, 1^* = 1^*. \tag{A.4}$$

Each of the operator relations can be integrated. That is symbolically represented as the right side multiplication by the Heaviside step function. For example:

$$\tilde{G}'_1 \tilde{G}'_2 = \alpha \tilde{1}' + \beta \tilde{G}'_3, \tag{A.5}$$

(α and β are constants) upon integration yields:

$$\tilde{G}_1'\tilde{G}_2^* = \alpha 1^* + \beta G_3^*. \tag{A.6}$$

The reader can operator relations, appearing in the present paper, substitute by the corresponding one where the integrations are carried out.

Notations

t, τ, θ	= time mesured from a prescribed instant,
t_0	=time of the first load application,
y	=ordinate of an arbitrary point of cross section referring to the centroid of the transformed cross section area
z, ζ	=coordinate along the beam axis
$\varepsilon = \varepsilon(z, y, t, t_o)$	= strain of an arbitrary point
$\varepsilon_j^o = \varepsilon_j^o(t, t_o)$	= prescribed stress-independent inelastic strain
$\eta = \eta(z, t, t_o)$	= normal strain of the beam axis
$\kappa = \kappa(z, t, t_o)$	= change in curvature of the beam axis
$\sigma_j = \sigma_j(z, y, t, t_o)$	= stress in part j of the composite cross section
$P_j^* = P_j^*(t, t_o)$, $F^* = F^*(t, t_o)$	= nondimensional creep function
$Q_j^* = Q_j^*(t, t_o)$, $R^* = R^*(t, t_o)$	= nondimensional relaxation function
ϱ_j, ϱ_j^i	= defined by Eq. (1.11)
$N = N(z, t, t_o)$	= axial force
$M = M(z, t, t_o)$	= bending moment
E_{jo}, E_u	= Young's moduli: at t_o and arbitraily chosen
ν_j	= reducing factor: $\dfrac{E_{jo}}{E_u}$
$A_j = A_j(z)$, $I_j = I_j(z)$	= area and centroidal moment of inertia of part j of composite cross section
$A_{jr} = A_{jr}(z)$, $I_{jr} = I_{jr}(z)$	$= A_{jr} = \nu_j A_j;\quad I_{jr} = \nu_j I_j$
$A_i = A_i(z)$	= transformed cross section area:
y_j	= ordinate of the centroid A_j referring to the centroid A_i
$S_{jr} = S_{jr}(z)$	= first moment of A_{jr} : $S_{jr} = y_j\, A_{jr};\ \sum_{j=1}^{f} S_{jr} = 0$
$J_i = J_i(z)$	= moment of inertia of transformed cross section: $J_i = \sum_{j=1}^{f} J_{jr} = \sum_{j=1}^{f}\left(I_{jr} + y_j^2\, A_{jr}\right)$
J_u	= moment of inertia arbitrarily chosen
$S_i = S_i(z)$	$= \sqrt{A_i\, J_i}$
$\gamma_{hs} = \gamma_{hs}(z)$	= elements of the matrix $\Vert\gamma_{hs}\Vert_{2,2}$

$$\gamma_{11} = \frac{1}{A_i} \sum_{j=1}^{f} \varrho_j A_{jr},$$

$$\gamma_{22} = \frac{1}{J_i} \sum_{j=1}^{f} \varrho_j J_{jr},$$

$$\gamma_{12} = \gamma_{21} = \frac{1}{S_i} \sum_{j=1}^{f} \varrho_j S_{jr}$$

$\gamma_h = \gamma_h(z)$ = individual values of the matrix $\|\gamma_{hs}\|_{2,2}$

$\gamma_h' = \gamma_h'(z)$ = $1 - \gamma_h$

$\Delta\gamma = \Delta\gamma(z)$ = $\gamma_1 - \gamma_2$

$\delta\gamma_h = \delta\gamma_h(z)$ = coefficients: $\delta\gamma_1 = \gamma_1 - \gamma_{11}$, $\delta\gamma_2 = \gamma_{11} - \gamma_2$

$R_h^* = R_h^*(\gamma_h, t, t_o)$ = nondimensional cross section relaxation function

$F_h^* = F_h^*(\gamma_h, t, t_o)$ = nondimensional cross section creep function

$\Delta^* = \Delta^*(z, t, t_{,o})$ = reduced generalized displacement defined by Eq. (2.11).

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LE FLUAGE VIEILLISSANT DANS LES CTRUCTURES MIXTES

On a dérivé trois théorèmes concernant la co-action dans la structure mixte composée par les différents materiaux viscoélastiques linéaires vieillissants. Les cas particuliers sont les théorèmes connus dans la litterature: les théorèmes de la théorie mathématique des structures mixtes et précontraintes et la théoreème concernant le matériau viscoélastique linéaire vieillisant.

PUZANJE I STARENJE KOD SPREGNUTIH GREDA

Izvedene su tri teoreme koje se odnose na sadejstvo spregnute grede saćinjene od različitih linearnih viskoelastičnih materijala s osobinom starenja. Kao njihovi specijalni slučajevi dobijaju se, u literaturi poznate, teoreme matematičke teorije spregnutih i prethodno napregnutih linijskih nosača i teorema koja važi za linearan viskoelastičan materijal s osobinom starenja.

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