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NONLOCAL EQUATIONS OF THE GREASING LAYER OF SUSPENSION

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1. Introduction

It is well known that the clasical continuum mechanics is based on the principle of local action as one of the basic principles. The domain of applicability of such continuum theories is determined by physical reasons which allows that the material properties at a given point of the body in consideration depend only of the infinitesimal neighborhood about that point. In other words, in the classical continuum mechanics only short range interactions among the particles are considered.

But, there exists a whole range of physical sitiuations in which the dimensions of the system in considerations are compatible whith a characteristic inner lenght of the material. This means that such system can be studied by the theory in which the fundamental equations depend on the properties of the whole body, i.e the constitutive equations involve integrals of the state variables as functions of space and time.

A whole range of effects appearing in experiment has been noticed which the classical continuum mechanics cannot explain. Papers [9]–[15] point out a necessity of describing such effects from the stand point of the nonlocal micropolar theory. The examples of such effects can be met in mechanics and liquid crystal physics, polymers, as well as in turbulence in fluids. The agreements of the theoretical results with experimental studies is the reason for the increase of the volume of literature on the subject of nonlocal continuum mechanics.

The previous remarks motivate an extension of the nonlocal theory to the consideration of the suspension, which is the main idea in the present paper.

Considering the suspension as a two–phase mixture the following fields are defined: velocities, microrotation velocity, pressure and concentration. The equations of balance are formulated for each phase separately, and for the mixture as a whole. The nonlocal effects are included through the constitutive equations.

The aim of the present paper is to include the nonlocal effects into the micropolar continuum theory of suspension flow. The theory has been applied to the case of the motion of the thin layer of suspension between two approximately parallel surfaces with radius curve sufficiently large in comparison with average depth of the layer δ.

2. The balance equation

Generaly, the problem of suspension is very difficult. Because of that we limit our consideration to the case when a suspension is considered as isotropic mixture of two components of chemically nonreactive constituents. We proceed further refering the reader to [1] for the balance laws stated below:

2.1 The balance of mass of the α-th constituent and the mixture as a whole

$$\frac{d\varrho_{(\alpha)}}{dt} + \varrho_{(\alpha)} \nabla \mathbf{v}_{(\alpha)} = \hat{\varrho}_{(\alpha)}, \tag{2.1}$$

$$\frac{d\varrho}{dt} + \varrho \nabla \mathbf{v} = \hat{\varrho}, \tag{2.2}$$

where $\varrho_{(\alpha)}$, $\mathbf{v}_{(\alpha)}$, ϱ and $\mathbf{v}$ are the mass density and velocity of the α-th constituent and the mixture as a whole, and where

$$\varrho = \sum_{\alpha=1}^{n} \varrho_{(\alpha)}, \qquad \varrho \mathbf{v} = \sum_{\alpha=1}^{n} \varrho_{(\alpha)} \mathbf{v}_{(\alpha)} . \tag{2.3}$$

Let us introduce the flux of the diffused mass of the α-th constituent:

$$\mathbf{J}_{(\alpha)} = \varrho_{(\alpha)}(\mathbf{v}_{(\alpha)} - \mathbf{v}) = \varrho_{(\alpha)} \mathbf{u}_{(\alpha)}, \tag{2.4}$$

where $\mathbf{u}_{(\alpha)}$ is the diffusion rate of the α-th constituent. If the concentration of mass of the α-the constituent is

$$c_{(\alpha)} = \frac{\varrho_{(\alpha)}}{\varrho}, \tag{2.5}$$

then, by applying (2.1), it is possible to establish the relationship between the concentration of mass and the flux of the diffused mass of the α-th constituent

$$\varrho \frac{dc_{(\alpha)}}{dt} = -\nabla \cdot \mathbf{J}_{(\alpha)} . \tag{2.6}$$

2.2 The balance of momentum of the mixture

$$\varrho \frac{d\mathbf{v}}{dt} = \nabla \cdot \mathbf{t} + \varrho \mathbf{f} + \varrho \hat{\mathbf{f}} - \hat{\varrho} \mathbf{v}, \tag{2.7}$$

where $\mathbf{t}$ and $\mathbf{f}$ are the nonsymmetric stress tensor and the body force.

2.3 The balance of moment of momentum

$$\varrho I \frac{d\vec{\nu}}{dt} + \hat{\varrho} I \vec{\nu} + \sum_{a} \nabla \left(\varrho_{(\alpha)} I_{(\alpha)} \vec{\nu}_{(\alpha)} \vec{u}_{(\alpha)} \right) =$$

$$\vec{t} \times \vec{I} + \nabla \cdot \vec{m} + \sum_{a} \varrho_{(\alpha)} \left(\vec{r} \times \vec{f}_{(\alpha)} + \vec{m}_{\alpha} \right), \tag{2.8}$$

where ν is the microrotation vector, and $\mathbf{m}$ is the couple stress tensor, respectively.

Now, the phenomenological equation can be derived in the following form [3]:

$$\begin{aligned} \varrho D(\nabla c_p + k_p \nabla p) = \mathbf{J}_p + a_1 a_2 \mathbf{J}_p \times (\nu - \Omega) + \\ + a_1 a_3 (\nabla \times \nu) + a_1 a_4 [(\nabla \times \nu) \times (\nu - \Omega)]. \end{aligned} \tag{2.9}$$

where D is the diffusion coefficient of the disperse phase, $k_p = a_1 \gamma / (\varrho D)$ and $\Omega = \frac{1}{2} \nabla \times \mathbf{v}$, whereas a_1, a_2, a_3 and a_4 are the scalars which characterize the isotropic features of the medium.

In nonlocal theory nonlocal residuals of various fields, denoted by carring a carat "^" are introduced to localize the global laws.

3. Differential equations of motion

In the case of incompressible suspensions, without body forces, couples and temperature influences, the balance equations can be written in the following forms:

$$v_{i,i} = 0, \tag{3.1}$$

$$\varrho \frac{dv_i}{dt} = t_{ij,i}, \tag{3.2}$$

$$\varrho \frac{d(I\nu_i)}{dt} = m_{ij,j} + \varepsilon_{ijk} t_{kj}, \tag{3.3}$$

$$\varrho \frac{dc_p}{dt} = -J_{pi,i}, \tag{3.4}$$

where the nonsymmetric stress tensor and the couple stress tensor are [10], [16]:

$$\begin{aligned} t_{kl} = (-\pi + \lambda v_{r,r}) \delta_{kl} + \mu (v_{k,l} + v_{l,k}) + k (v_{l,k} - \varepsilon_{klr} \nu_r) + \\ + \int_v [\sigma (|\mathbf{x}\prime - \mathbf{x}|) \delta_{kj} + \lambda\prime (|x\prime - x|) v_{r,r}(|\mathbf{x}\prime|) \delta_{kl} + \mu\prime (|x\prime - x|) \cdot 2d_{kl} + \\ + k\prime (v_{l,k}(|\mathbf{x}\prime|) - \varepsilon_{klr} \nu_r(|\mathbf{x}\prime|))] dv(|\mathbf{x}\prime|) \end{aligned} \tag{3.5}$$

$$\begin{aligned} m_{kl} = \\ = \alpha \nu_{r,r} \delta_{kl} + \beta \nu_{k,l} + \gamma \nu_{l,k} + \\ \int_v (\alpha\prime (|\mathbf{x}\prime - \mathbf{x}|) \delta_{kl} \nu_{r,r}(|\mathbf{x}\prime|) + \beta\prime (|\mathbf{x}\prime - \mathbf{x}|) \nu_{k,l}(|\mathbf{x}\prime|) + \gamma\prime (|\mathbf{x}\prime - \mathbf{x}|) \nu_{l,k}(|\mathbf{x}\prime|)) \, dv(|\mathbf{x}\prime|) \end{aligned} \tag{3.6}$$

where k, α, β and γ are the coefficients of the micropolar continuum viscosity, and k', α', β' and γ' modulus nonlocal viscosity. By substituting (3.5) in (3.2) and (3.5), as well as (3.6) in (3.3), in the case of a stationary flow, the equation of motion in the vector form will be obtained:

$$(\lambda v_{i,i})_{,l} + [\mu (v_{k,l} + v_{l,k}) + k (v_{l,k} - \varepsilon_{klr}\nu_r)]_{,k} - \pi_l +$$
$$\int_v \{\sigma\delta_{kl} + \lambda' v_{i,i}\delta_{kl} + [2\mu' d_{kl} + k'(v_{l,k} - \varepsilon_{klr}\nu_r)]\delta_{kl}\}\, dv = 0 \tag{3.7}$$

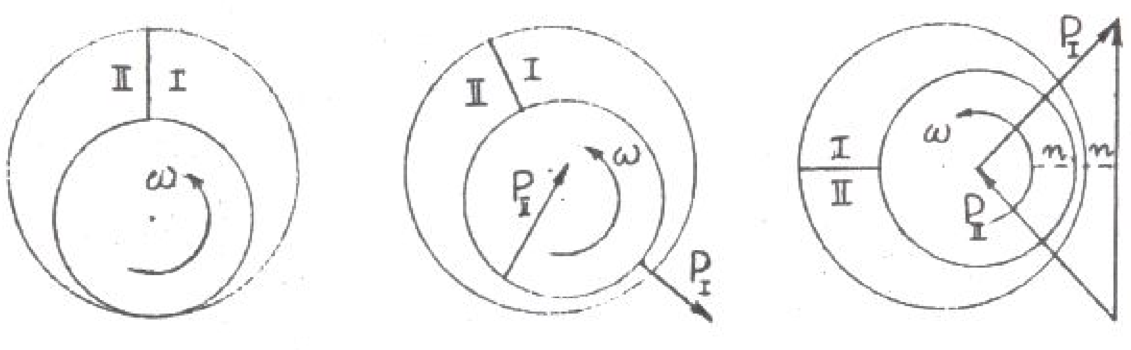
$$(\alpha\nu_i)_{,l} + (\beta\nu_{k,l} + \gamma\nu_{l,k})_{,k} + k(\varepsilon_{lmn}v_{n,m} - 2\nu_i) +$$
$$\int_v [\alpha'\nu_{i,i}\delta_{kl} + (\beta'\nu_{k,l} + \gamma'\nu_{l,k})\delta_{kl} + k'(\varepsilon_{lmn}v_{n,m} - 2\nu_i)]\, dv = 0 \tag{3.8}$$

To the above equations the continuity equation is added:

$$v_{i,i} = 0. \tag{3.9}$$

4. Approximate equations of the greasing layer of suspension

It is well known that machine members often rub. In order to prevent intensive friction, the engine oil put between them. This effect is called lubrication. It is most frequently used with shafts revolving in bearings.



a) b) c)

Fig. 1

Prior to revolving, the shaft will touch the bearing surface at the lowest point (Fig. 1a). While revolving, the shaaft will move radially to the right. The consequences of such movement are the resulting forces of pressure P_I and P_{II} of the region I and II, which are not balanced with the vertical loading (Fig. 1b). The movement of the shaft towards the right will last until the direction of the resulting pressure upon the shaft becomes parallel to the vector direction of the external loading. Such balance will be reached when the line of the smallest clearance n–n between the shaft and the bearing becomes horizontal (Fig. 1c).

The engine oil sticks to the hard surface of the shaft and the bearing; thus, the problem reduces to the study of the flow of a viscous suspension (the oil and the metal dust) between two approximately parallel planes at the point of the smallest clearance.

Hence, let us consider the stationary motion of a viscous suspension with a nonsymmetric stress tensor in a thin layer between approximately parallel surfaces with the radius curve sufficiently great in comparison with the mean thickness of the layer δ (Fig. 2). In the Fig. 2, U_1, U_2 and V_2 are shown the corresponding projections of the vector of the point velocity in the first and in the second surfaces.

The component V_2 results from the assumption of very small oscillations in the rotation of the shaft at the point of the smallest clearance.

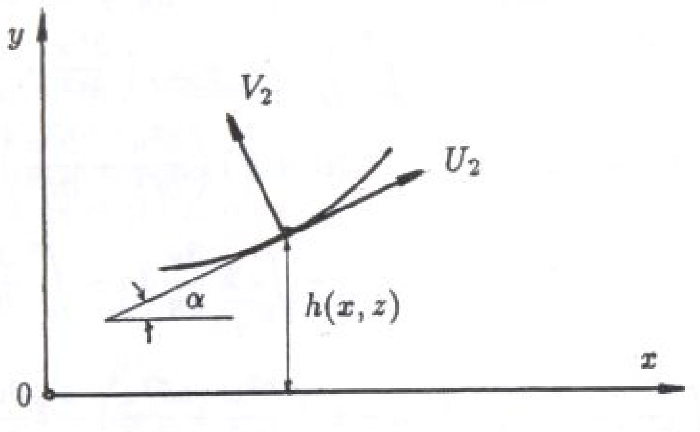


Fig. 2

In order that the mean thickness of the layer be small during motion, it is necessary to assume that the transverse component of velocity V_2 is small enough as compared with velocity U_1 ($V_2 \ll U_1$). Therefore the ratio of the two velocities has been designated by ε, where $\varepsilon \ll 1$:

$$\frac{V_2}{U_1} = \varepsilon. \tag{4.1}$$

Let us denote the mean value of the curve radius of the given surfaces with L_0. Since it has been assumed that the thickness of the layer δ is small enough in comparison with the radius of curvature, the ratio of those values could also be designated with ε, i.e.,

$$\frac{\delta}{L_0} = \varepsilon. \tag{4.2}$$

According to the way of motion of the suspension described above, the velocity vector, the microrotation velocity vector, pressure, diffused mass flux and the angular velocity will have the following components:

$$
\begin{aligned}
\vec{v} &= \vec{v}\,[u(x,y),\, v(x,y),\, 0]\,,\\
\vec{\nu} &= \vec{\nu}[0,\, 0,\, \nu(x,y)],\\
p &= p(x,y)\\
\vec{J_p} &= \vec{J_p}\,[J_{px}(x,y),\, J_{py}(x,y),\, 0]\,,\\
\vec{\Omega} &= \vec{\Omega}\,[0,\, 0,\, \Omega(x,y)]\,.
\end{aligned}
\tag{4.3}
$$

By using (4.3), the equations (3.9), (3.7) and (3.8) can be written in the following form:

$$
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{4.4}
$$

$$
\begin{aligned}
(\lambda+\mu)\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y}\right) + (\mu+k)\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + k\frac{\partial \nu}{\partial y} - \frac{\partial p}{\partial x} +\\
\int_0^x \int_0^y \left[(\lambda'+\mu')\left(\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 v}{\partial \xi \partial \eta}\right) + \right.\\
\left. + (\mu'+k')\left(\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2}\right)\right] \mathrm{d}\eta \mathrm{d}\xi +\\
+ \int_0^y k'\frac{\partial \nu}{\partial \eta}\,\mathrm{d}\eta - \int_0^x \frac{\partial p}{\partial \xi}\mathrm{d}\xi = 0,
\end{aligned}
\tag{4.5}
$$

$$
\begin{aligned}
(\lambda+\mu)\left(\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial y^2}\right) + (\mu+k)\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) - k\frac{\partial \nu}{\partial x} - \frac{\partial p}{\partial y} +\\
+ \int_0^x \int_0^y \left[(\lambda'+\mu')\left(\frac{\partial^2 u}{\partial \eta \partial \xi} + \frac{\partial^2 v}{\partial \eta^2}\right) + (\mu'+k')\right.\\
\left.\left(\frac{\partial^2 v}{\partial \xi^2} + \frac{\partial^2 v}{\partial \eta^2}\right)\right] \mathrm{d}\eta \mathrm{d}\xi - \int_0^x k'\frac{\partial \nu}{\partial \xi}\,\mathrm{d}\xi - \int_0^y \frac{\partial p}{\partial \eta}\mathrm{d}\eta = 0,
\end{aligned}
\tag{4.6}
$$

$$
\begin{aligned}
\gamma\left(\frac{\partial^2 \nu}{\partial x^2} + \frac{\partial^2 \nu}{\partial y^2}\right) + k\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) - 2k\nu +\\
\int_0^x \int_0^y \left[\gamma'\left(\frac{\partial^2 \nu}{\partial \xi^2} + \frac{\partial^2 \nu}{\partial \eta^2}\right)\right] \mathrm{d}\eta \mathrm{d}\xi +\\
+ \int_0^x k'\frac{\partial v}{\partial \xi}\,\mathrm{d}\xi - \int_0^y k'\frac{\partial u}{\partial \eta}\mathrm{d}\eta - \int_0^x 2k'\nu\,\mathrm{d}\xi - \int_0^y 2k'\nu\,\mathrm{d}\eta = 0,
\end{aligned}
\tag{4.7}
$$

Let us introduce dimensionless coordinates, taking into account the fact that the order of magnitude of the coordinate and velocity in the direction of the normal upon the first surface is small in comparison with the order of magnitude of the coordinates and the velocity in the x direction:

$$x = L_o x^*, \quad y = \delta y^*, \tag{4.8}$$

$$u = U_1 u^*, \quad v = V_2 v^*, \quad \nu = \Omega \nu^*, \quad J_{px} = \varrho U_1 J^*_{px}, \quad J_{py} = \varrho V_2 J^*_{py}. \tag{4.9}$$

Substituting these coordinates into the equation (4.4) one obtains:

$$\frac{\partial u^*}{\partial x^*} + \frac{V_2 L_0}{U_1 \delta} \frac{\partial v^*}{\partial y^*} = 0. \tag{4.10}$$

Since all terms of the this equation should be of the same order of magnitude, it follows:

$$\frac{V_2 L_0}{U_1 \delta} = 1. \tag{4.11}$$

Reynolds's number, dimensionless pressure and the microrotation velocity have the form:

$$R_l = \frac{U_1 L_o}{\mu + k}, \quad p = \frac{\varrho U_1^2}{\varepsilon^2 R_l} p^*, \quad \nu = \frac{U_1}{\delta} \nu^*. \tag{4.12}$$

If the dimensionless variables (4.8),(4.9) and (4.12) are introduced into the equations (4.4)–(4.7) and if all the values not containing the multiplier $1/\varepsilon^2$ are neglected as much smaller than the ones containing this multiplier, the approximate equations of the lubricating suspension film are reduced to the form:

$$\begin{gathered}(\mu + k) \frac{\partial^2 u}{\partial y^2} + k \frac{\partial \nu}{\partial y} - \frac{1}{\varrho} \frac{\partial p}{\partial x} + \\ \int_0^y \int_0^y (\mu' + k') \frac{\partial^2 u}{\partial \eta^2} \, d\eta d\eta + \int_0^y k' \frac{\partial \nu}{\partial \eta} \, d\eta - \frac{1}{\varrho} \int_0^x \frac{\partial p}{\partial \xi} \, d\xi = 0,\end{gathered} \tag{4.13}$$

$$\frac{\partial p}{\partial y} + \int_0^y \frac{\partial p}{\partial \eta} \, d\eta = 0, \tag{4.14}$$

$$\begin{gathered}\gamma \frac{\partial^2 \nu}{\partial y^2} - k \left(2\nu + \frac{\partial u}{\partial y}\right) + \\ \int_0^y \int_0^y \gamma' \frac{\partial^2 \nu}{\partial \eta^2} \, d\eta d\eta - \int_0^y k' \frac{\partial u}{\partial \eta} \, d\eta - \int_0^y 2k' \nu \, d\eta = 0.\end{gathered} \tag{4.15}$$

In the case when $k = 0$ and $\gamma = 0$ and neglecting the nonlocal effects (classical viscous fluid), equations (4.13)–(4.15) become Reynolds's differential equations for lubrication layer [18].

For the solution of the differential equations (4.13) and (4.15) it is necessary to define the boundary conditions for the velocity and the velocity of microrotation.

According to the assumption introduced, the points upon the first surface move at the velocity of U_1 only in the direction of the x–axis (Fig. 3). In adition to that, let us assume that the curve radius of the first surface is also large enough to consider it as flat.

The boundary conditions for the velocity and the velocity of microrotation upon the first surface read:

$$\text{for} \quad y = 0: \quad u = U_1, \quad v = 0, \quad w = 0, \quad \nu = 0. \tag{4.16}$$

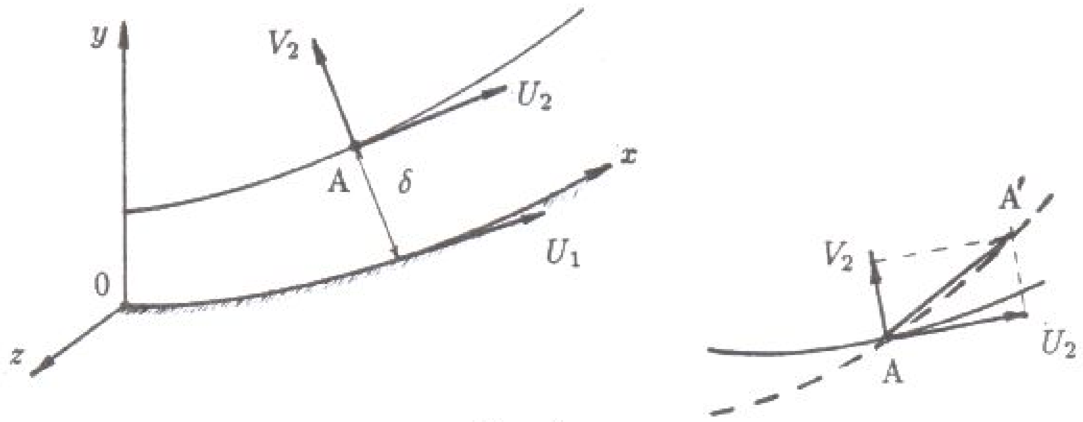


Fig. 3

The points of the other surface move at the tangential velocity U_2 and the normal velocity of V_2. If α is used to designate the angle of taper of the second surface in relation to the first, the velocity components for the variavle layer thickness h:

$$\text{for} \quad y = h(x,z): \quad \begin{aligned} u =& U_2 \cos\alpha - V_2 \sin\alpha, \\ v =& U_2 \sin\alpha + V_2 \cos\alpha, \\ w =& 0. \end{aligned} \tag{4.17}$$

Since, according to the assumption, the angle α small, it follows that

$$\sin\alpha \cong \tan\alpha = \frac{\partial h}{\partial x}, \quad \cos\alpha \cong 1$$

Then the velocity components take the form:

$$\begin{aligned} u =& U_2 - V_2 \frac{\partial h}{\partial x}, \\ v =& U_2 \frac{\partial h}{\partial x} + V_2, \\ w =& 0. \end{aligned} \tag{4.18}$$

Regarding the fact that the velocity V_2 is small, the product $V_2 \partial h / \partial x$ can be neglected as a small value of the second order, so that the boundary conditions on the second surface for the velocity and microrotation velocity finally read:

$$\text{for} \quad y = h(x,z): \quad \begin{aligned} u &= U_2, \\ v &= U_2 \frac{\partial h}{\partial x} + V_2, \\ w &= 0, \\ \nu &= 0. \end{aligned} \tag{4.19}$$

In this way, we have obtained a complete equation system. The given system of intego–differential equations is very complicated for solution. Due to the fact that this problem has not been solved even in the case of the local theory with a nonsymetric stress tensor, let us solve it neglecting the integral terms.

The aim of further investigations will be the solution of this complex system of equations, and, therefore, the results obtained in the present paper could serve as their testing.

First of all, let us demonstrate that, in this case, without neglecting the integral terms, the pressure p is not a function of y. By applying the Laplace's transformation to the relation (4.14), we obtain:

$$\begin{gathered} \mathcal{L}\left(\frac{\partial p}{\partial y}\right) + \mathcal{L}\left[\int_0^y \frac{\partial p}{\partial \eta}\,\mathrm{d}\eta\right] = 0, \\ P(s) + \frac{1}{s}P(s) = 0 \Rightarrow P(s) = \mathcal{L}\left(\frac{\partial p}{\partial y}\right) = 0 \\ \mathcal{L}\left(\frac{\partial p}{\partial y}\right) = s\mathcal{L}(p) - p(x,0) = 0 \\ \mathcal{L}^{-1}(\mathcal{L}(p)) = \mathcal{L}^{-1}(\frac{1}{s}p(x,0)) = p(x,0) \end{gathered} \tag{4.20}$$

By differentiating (4.15) along y, and by using (4.13), we obtain:

$$\frac{\partial^3 \nu}{\partial y^3} - c^2 \frac{\partial \nu}{\partial y} = P(x), \tag{4.21}$$

where

$$c^2 \equiv \frac{k}{\gamma}\frac{2\mu + k}{\mu + k}, \quad P(x) \equiv \frac{k}{\gamma\varrho(\mu + k)}\frac{\mathrm{d}p}{\mathrm{d}x}$$

From this, by integration , we obtain:

$$\nu = A_1(x)\cdot e^{cy} + A_2(x)\cdot e^{-cy} - \frac{P(x)}{c^2}y - \frac{C_1(x)}{c^2} \tag{4.22}$$

By substituting the microrotation velocity ν in (4.15), we obtain:

$$\frac{\partial u}{\partial y} = \left(\frac{\gamma c^2}{k} - 2\right)A_1\cdot e^{cy} + \left(\frac{\gamma c^2}{k} - 2\right)A_2\cdot e^{-cy} + \frac{2}{c^2}P(x)\cdot y + \frac{2}{c^2}C_1(x) \tag{4.23}$$

From there, we get:

$$u = \frac{b}{c}A_1(x)\cdot e^{cy} - \frac{b}{c}A_2(x)\cdot e^{-cy} + \frac{1}{c^2}P(x)\cdot y^2 + \frac{2}{c^2}C_1(x)\cdot y + C_2(x), \tag{4.24}$$

where

$$b = \frac{\gamma c^2}{k} - 2.$$

By using the continuity equation (4.4) as well as relation (4.24), we obtain:

$$v = -\frac{d}{c}\left(A_1' e^{cy} - A_2' e^{-cy}\right) - \frac{P' y^3}{3c^2} + \frac{C_1' y^2}{c^2} - C_2' y - C_3(x) \tag{4.25}$$

where $d = b/c$ and $(') \equiv d/dx$. The funtions A_1, A_2, C_1, C_2, and C_3 be able to derive from boundary conditions (4.16) and (4.18) (see Appendix).

The concentration distribution will be obtained from Eq. (2.9), by using $(4.3)_4$ and (4.14), so that it is reduced to:

$$\varrho D\frac{\partial C_p}{\partial x} + \varrho D k_p \frac{\partial p}{\partial x} = J_{px} + \left[a_1 a_2 J_{py} - a_1 a_4 \frac{\partial \nu}{\partial x}\right]\left[\nu - \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\right], \tag{4.26}$$

$$\varrho D\frac{\partial C_p}{\partial y} = J_{py} - \left[a_1 a_2 J_{px} - a_1 a_4 \frac{\partial \nu}{\partial y}\right]\left[\nu - \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\right]. \tag{4.27}$$

If the dimensionless variables (4.6), (4.7) and (4.8) are introduced into the above equations, and if, as previously, only the members of the highest order are taken into account, one can obtain:

$$\varrho D k_p \frac{\partial p}{\partial x} = J_{px}, \tag{4.28}$$

$$\varrho D\frac{\partial C_p}{\partial y} = \left[-a_1 a_2 J_{px} - a_1 a_4 \frac{\partial \nu}{\partial y}\right](\nu - \Omega), \tag{4.29}$$

where $\Omega = \frac{1}{2}\frac{\partial u}{\partial y}$.

By using the expressions (4.28) and (4.29), one can obtain the concentration distribution law for the motion of a thin suspension layer:

$$\varrho D\frac{\partial C_p}{\partial y} = \left[-a_1 a_2 \varrho D k_p \frac{\partial p}{\partial x} - a_1 a_4 \frac{\partial \nu}{\partial y}\right](\nu - \Omega). \tag{4.30}$$

The determination of the influence of nonlocality in some problems will be the subject of a further investigation.

Note. In this paper a set of differential equations are derived for fields of velocities, microrotation velocity, pressure and concentration, (4.13)–(4.15) and (4.30), for nonlocal theory.

If we ignore the nonlocal effects, the present set of equations reduces to the system given in [8].

Note also, that Equation (4.30) is formally the same, as the corresponding one from [8] and [9]. The essential difference is that the pressure p and the microrotation velocity ν differ in the local [4], [8] and the present nonlocal theory.

Appendix.

From the boundary conditions (4.16) and (4.18), and the relations (4.22), (4.24) and (4.25) we obtain:

$$u|_{y=0} = d(A_1 + A_2) + C_2 = U_1, \tag{A-1}$$

$$v|_{y=0} = -\frac{d}{c}\left(A_1' - A_2'\right) - C_3 = 0, \tag{A-2}$$

$$u|_{y=h(x,z_0)} = d\left(A_1\cdot e^{ch} + A_2\cdot e^{-ch}\right) + \frac{Ph^2}{c^2} + \frac{2C_1}{c^2}h + C_2 = U_2, \tag{A-3}$$

$$v|_{y=h(x,z_0)} = -\frac{d}{c}\left(A_1'\cdot e^{ch} - A_2'\cdot e^{-ch}\right) - \frac{P'h^3}{3c^2} - \frac{C_1'h^2}{c^2} - C_2'h - C_3 = V_2 + U_2 h', \tag{A-4}$$

$$\nu|_{y=0} = A_1 + A_2 - \frac{C_1}{c^2} = 0 \tag{A-5}$$

$$\nu|_{y=h(x,z_0)} = A_1\cdot e^{ch} + A_2\cdot e^{-ch} - \frac{Ph}{c^2} - \frac{C_1}{c^2} = 0. \tag{A-6}$$

Now, we have the system of six equations for six unknown functions A_1, A_2, C_1, C_2, C_3 and P.

The combinations of relations (A-1), (A-5) and (A-3), (A-6) yields:

$$dC_1 + c^2C_2 = c^2U_1, \tag{A-7}$$

$$2(h+d)C_1 + c^2C_2 = U_2c^2 - dPh + Ph^2. \tag{A-8}$$

Hence, from (A-7) and (A-8), we get:

$$C_1 = \frac{c^2(U_2 - U_1) + Ph(h-d)}{2h}, \tag{A-9}$$

$$C_2 = U_1 - \frac{d}{c^2}\frac{c^2(U_2 - U_1) + Ph(h-d)}{2h} \tag{A-10}$$

From (A-5), (A-6) and (A-10) we are able to calculate unknowns A_1 and A_2:

$$A_1 = \frac{Ph}{2c^2\sinh(ch)} + \frac{(U_2 - U_1) + Ph(h-d)}{2h}\left(1 + \frac{1 - e^{ch}}{2\sinh(ch)}\right), \tag{A-11}$$

$$A_2 = \frac{2Ph(h-d) + c^2\left(1 - e^{ch}\right)\left[(U_2 - U_1) + Ph(h-d)\right]}{4c^2h\sinh(ch)}. \tag{A-12}$$

Now, from (A-2), we determine C_3:

$$C_3 = \frac{d}{c}\left(A_2' - A_1'\right) = \frac{d}{c}\left(A_2 - A_1\right)'. \tag{A-13}$$

But, from (A-11) and (A-12), we have:

$$A_2 - A_1 = \frac{U_1 - U_2}{2h} - \frac{P(h-d)}{h},$$

and from that, we obtain for C_3:

$$C_3 = \frac{d}{2c}\left[\frac{\left(U_1' - U_2'\right)(h-d) - h'\left(U_1 - U_2\right)}{h^2} - P'(h-d) - Ph'\right]. \tag{A-14}$$

At the end, we obtain, from (A-4), the differential equation for P:

$$P' + \frac{f_9}{f_8}P = \frac{f_{10}}{f_8},$$

and solution for P:

$$P = \exp\left(-\int \frac{f_9}{f_8}\,\mathrm{d}x\right)\left[\int \frac{f_{10}}{f_8}\exp\left(\int \frac{f_9}{f_8}\,\mathrm{d}x\right)\mathrm{d}x + C\right], \tag{A-15}$$

where are

$$f_1 \equiv \frac{h}{2c^2\sinh(ch)} + \frac{h-d}{2}\left(1 + \frac{1-e^{ch}}{2\sinh(ch)}\right),$$

$$f_2 \equiv \frac{U_2 - U_1}{2h}\left(1 + \frac{1-e^{ch}}{2\sinh(ch)}\right), \quad f_3 \equiv \frac{(h-d)\left[2 + c^2\left(1-e^{ch}\right)\right]}{4c^2\sinh(ch)},$$

$$f_4 \equiv \frac{\left(1-e^{ch}\right)\left(U_2 - U_1\right)}{4h\sinh(ch)}, \quad f_5 \equiv \frac{c^2\left(U_2 - U_1\right)}{2h}, \quad f_6 \equiv U_1 - \frac{d\left(U_2 - U_1\right)}{2h},$$

$$f_7 \equiv \frac{d}{2c}\frac{\left(U_1' - U_2'\right)(h-d) - h'\left(U_1 - U_2\right)}{h^2},$$

$$f_8 \equiv -\frac{d}{c}e^{ch}f_1 + \frac{d}{c}e^{-ch}f_3 - \frac{5h^3}{6c^2} + \frac{dh^2}{2c^2} + \frac{dh}{2c},$$

$$f_9 \equiv -\frac{d}{c}e^{ch}f_1' + \frac{d}{c}e^{-ch}f_3' - \frac{h^2h'}{2c^2} + \frac{dh}{2c^2}h' + \frac{dh'}{2c},$$

$$f_{10} \equiv \frac{d}{c}e^{ch}f_2' - \frac{d}{c}e^{-ch}f_4' + \frac{h^2}{c^2}f_5' + hf_6' + f_7 + V_2 + U_2h'.$$

From the boundary condition for P, or equivalently for pressure p (see 4.2), we can determine C. Than substituting P in (A-9), (A-10), (A-11), (A-12) and (A-14), we have the solutions for A_1, A_2, C_1, C_2 and C_3.

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НЕЛОКАЛЬНЫЕ УРАВНЕНИЯ СМАЗОЧНОГО СЛОЯ СУСПЕНЗИИ

В работе применяется микрополярная теория континуума, в которой несимметричный тензор напряжения пользуется для описании состояния напряжения. Теория примененна для случая движения суспензии в тонком слое между приблизительно параллельными поверхностями и со радиусамы кривизны достаточно великимы по сравнению со средней толщиной слоя δ.

Полученные проближенные дифференциальные уравнения движения смазочного слоя суспензий, а также и приближенные уравнения распределения концентрации тонкого слоя суспензии.

Потом полученная система дифференциальных уравнений решается, при чем не учитывамы интегральние членны.

NELOKALNE JEDNAČINE PODMAZUJUĆEG SLOJA SUSPENZIJE

U radu se primenjuje mikropolarna teorija kontinuuma u kojoj se nesimetrični tenzor napona koristi za opisivanje naponskog stanja. Teorija je primenjena za slučaj kretanja suspenzija u tankom sloju, između približno paralelnih površina sa poluprečnicima krivine dovoljno velikih u poređenju sa srednjom debljinom sloja δ.

Izvode se približne diferencijalne jednačine raspodele koncentracije tankog sloja suspenzije. Zatim se dati sistem jednačina rešava, ako se zanemare integralni članovi.

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