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CONSERVATION LAWS AND NOETHER'S THEOREM IN A PARAMETRIC FORMULATION OF MECHANICS

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1. Introduction

As it is shown, the energy conservation law $E = T + U$ =const. in general is valid only for the scleronomic systems, but for the rheonomic ones it is not valid. For these K. Jacobi [1] and P. Painlevé [2], [3] have obtained the corresponding energy law in the form $T_2 - T_0 + U = F(t) + h$, where $T = T_2 + T_1 + T_0$ is the kinetic energy and U the potential energy of the system.

However, recently V. Vujičić [4]–[7] has given a modification of analytical mechanics of rheonomic systems. Here a convenably chosen function of time $q_0 = \tau(t)$ is taken as an additional generalized coordinate, by which the nonstationary constrains and the Lagrangian of system are expressed. On this basis an extended system of Lagrange's equations is formulated, with an additional equation corresponding to the quoted quantity q_0. Hence, inter alias, the energy law is obtained in an integral form $\mathcal{E} = T + U + P$+const., where P is so-called rheonomic potential of system, arising from the nonstationarity of constraints.

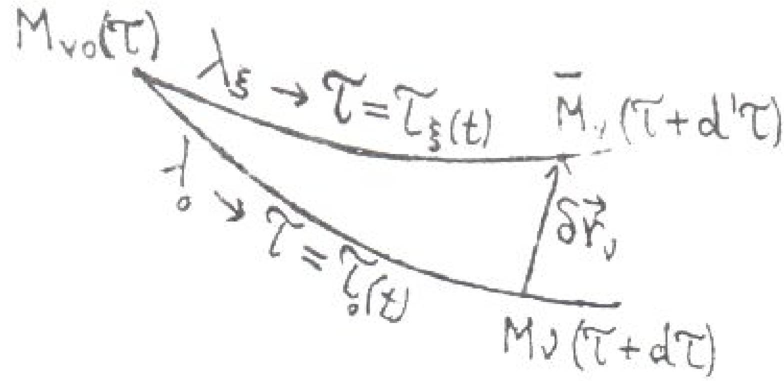
But, one other way of approach is given by the author himself [8]: a parametric formulation of the mechanics of rheonomic systems, which is founded on the family of varied paths and on the transition to a new parameter, depending on chosen path. In this paper we shall study the conservation laws and Noether's theorem in this parametric formulation, and compare these results with those obtained by Vujičić.

2. A Parametric Formulation of Mechanics

Let us consider a system of N particles, under the influence of arbitrary forces, with k holonomic nonstationary constraints

$$f_\mu(\vec{r}_\nu, t) = 0, \quad (\mu = 1, 2, \ldots, k;\ \nu = 1, 2, \ldots, N) \tag{2.1}$$

described by a set of generalized coordinates $(q_1, q_2, \ldots, q_n)$, where $n = 3N - k$. We can imagine a family of possible varied paths of system, drawn from the initial position of the system in the instantt (see the graph)



$$\vec{r}_\nu = \vec{r}_\nu(t,\lambda) \iff q_i = q_i(t,\lambda) \quad (\mu = 1,2,\ldots,k;\ \nu = 1,2,\ldots,N) \tag{2.2}$$

Here M_ν and $\bar{M}_\nu$ denote the position of ν–th particle in the same instant $t + dt$ on the actual and virtual path respectively, following by virtual displacements from $M_{\nu 0}$.

In the mechanics of rheonomic systems the time t has the double role: this is an independent variable (as in the mechanics in general), and in certain specific relations it has the character of a parameter (for example in the constraints). Instead of time as parameter let us introduce a new parameter τ, in dependence of the chosen path

$$\tau = \tau(t,\lambda) \iff t = t(\tau,\lambda) \tag{2.3}$$

and keep the time t as independent variable. In this way, to each varied path, corresponding to a value λ_ξ, corresponds certain function of time $\tau = \tau(t,\lambda_\xi) \equiv \tau_\xi(t)$, so that for the same instant $t + dt$ we have the different values of this parameter τ on the actual and varied path.

In all relations where the time t has the role of a parameter we can pass from time to so introduced parameter along a fixed path. So for $\lambda = \lambda_\xi$ we can express the constraints in the form

$$f_\mu\left[\vec{r}_\nu, t(\tau,\lambda_\xi)\right] \equiv f_\mu^*\left(\vec{r}_\nu, \tau\right) = 0 \tag{2.4}$$

and if we take τ as an additional generalized coordinate $q_0 = q_0(t)$, the form of Lagrangian is transformed into

$$L = L\left[q_i, \dot{q}_i, t(\tau,\lambda_\xi)\right] \equiv L^*\left(q_\alpha, \dot{q}_\alpha\right). \tag{2.5}$$

In this parametric formulation of mechanics the total work of ideal reaction forces on arbitrary virtual displacements is different from zero

$$\vec{R}_\nu^{\text{id}} \cdot \delta\vec{r}_\nu = R_0 \delta\tau, \quad R_0 = -\lambda_\mu \frac{\partial f_\mu}{\partial \tau}, \tag{2.6}$$

where the summation over the repeated indices is understood, by the aid of which d'Alembert–Lagrange's principle can be obtained in the form

$$\left(\vec{F}_\nu + \vec{R}^*_\nu - m_\nu \vec{a}_\nu\right) \cdot \delta\vec{r}_\nu = -R_0\,\delta\tau \tag{2.7}$$

From this principle, transformed in generalized coordinates, follows the corresponding extended system of Lagrange's equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} = Q_i + R^*_i, \quad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{q}_0} - \frac{\partial T}{\partial q_0} = Q_0 + R^*_0 + R_0, \tag{2.8}$$

here R_0 represents the generalized reaction force corresponding to q_0 and is given by (2.6). If we separate the generalized forces into the generally potential and nonpotential ones, and if $R_0 dq_0$ is a total differential, those equations can be transformed into

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha} - \frac{\partial \mathcal{L}}{\partial q_\alpha} = Q^*_\alpha + R^*_\alpha, \quad (\alpha = 0, 1, \cdots, n) \tag{2.9}$$

where

$$\mathcal{L}(q_\alpha, \dot{q}_\alpha) = L - P = T - V - P \tag{2.10}$$

and P is given by

$$P \overset{\mathrm{def}}{=} -\int R_0\,\mathrm{d}q_0 \iff R_0 = -\frac{\mathrm{d}P}{\mathrm{d}q_0}. \tag{2.11}$$

The corresponding canonical formalism can be formulated in analogous way as in habitual case. Namely, if we introduce the generalized momenta by

$$p_\alpha = \frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha} = a_{\alpha\beta}\dot{q}_\beta - b_\beta, \quad (\alpha = 0, 1, \ldots, n) \tag{2.12}$$

Lagrange's equations (2.9) can be substituted by an equivalent system of differential equations of the first order

$$\dot{p}_\alpha = -\frac{\partial \mathcal{H}}{\partial q_\alpha} + Q^*_\alpha + R^*_\alpha, \quad \dot{q}_\alpha = \frac{\partial \mathcal{H}}{\partial p_\alpha}, \quad (\alpha = 0, 1, \ldots, n) \tag{2.13}$$

where

$$\mathcal{H}(q_\alpha, p_\alpha) = p_\alpha \dot{q}_\alpha - \mathcal{L}. \tag{2.14}$$

Those are the corresponding Hamilton's equations and this Hamiltonian represents the extended energy of system in the sense

$$\mathcal{H}(q_\alpha, p_\alpha) = T + U + P. \tag{2.15}$$

3. Energy Laws and Differential Equations of Motion

a) In this aim, let us depart from the fundamental equation of motion

$$m_\nu \frac{\mathrm{d}\vec{v}_\nu}{\mathrm{d}t} = \vec{F}_\nu + \vec{R}^{\mathrm{id}}_\nu + \vec{R}^*_\nu, \quad (\nu = 1, 2, \ldots, N) \tag{3.1}$$

where the reaction forces are decomposed into the ideal $\vec{R}_\nu^{\rm id}$ and nonideal ones $\vec{R}_\nu^*$. As it is known, from this equation, by multiplication by $d\vec{r}_\nu = \vec{v}_\nu dt$ and summation over ν, one obtains the kinetic energy theorem of the system [9]

$$m_\nu \vec{v}_\nu \cdot d\vec{v}_\nu = dT = \vec{F}_\nu \cdot d\vec{r}_\nu + \vec{R}_\nu^{\rm id} \cdot d\vec{r}_\nu + \vec{R}_\nu^* \cdot d\vec{r}_\nu \tag{3.2}$$

If we still divide the active forces into generally potential with $V = \vec{b}_\nu \cdot \vec{v}_\nu + U$ and nonpotential ones $\vec{F}^*$, by application of Euler's theorem for homogeneous functions and by utilization of condition for virtual displacements, this relation passes into

$$\frac{dE}{dt} = \frac{d}{dt}(T+U) = \frac{\partial V}{\partial t} - \lambda_\mu \frac{\partial f_\mu}{\partial t} + \left(\vec{F}_\nu^* + \vec{R}_\nu^*\right) \cdot \vec{v}_\nu \tag{3.3}$$

Here the second term on the right hand–side expresses the influence of the non-stationarity of constrains to energy relations, what is specific for the considered problem.

b) If we start from Lagrange's equations, with Lagrangian including the general potential $V = b_i \dot{q}_i + U$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = Q_i^* + R_i^*, \quad (i = 1, 2, \ldots, n) \tag{3.4}$$

where Q_i^* and R_i^* are the nonpotential active forces respectively nonideal reaction ones, in the similar manner (multiplying by $dq_i = \dot{q}_i dt$) follows

$$\frac{d\mathcal{E}}{dt} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L\right) = -\frac{\partial L}{\partial t} + (Q_i^* + R_i^*)\,\dot{q}_i \tag{3.5}$$

But, this form of energy law differs from (3.3) and therefore the corresponding conservation laws are also different. Under the condition that the expression on the right hand–side is equal to zero, in the first case we have $E = T + U =$const., and in the second one $\mathcal{E} = T_2 - T_0 + U =$const. (Jacobi–Painlevé energy integral).

c) However, if one departs from extended system of Lagrange's equations (2.9), one obtains an another energy law, as it is shown by V.Vujičić [5]. His proof can be generalized to general case of potential forces, by multiplication of these equations by $dq_\alpha = \dot{q}_\alpha dt$, from where one yields

$$\frac{d\mathcal{E}^{\rm ext}}{dt} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha}\dot{q}_\alpha - \mathcal{L}\right) = (Q_\alpha^* + R_\alpha^*)\,\dot{q}_\alpha \tag{3.6}$$

If $(Q_\alpha^* + R_\alpha^*)\,\dot{q}_\alpha = 0$, i.e. if the effect of all nonpotential forces is equal to zero, we get an energy integral

$$\mathcal{E}^{\rm ext} \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha}\dot{q}_\alpha - \mathcal{L} = \text{const.} \tag{3.7}$$

and the sense of this quantity can be perceived by application of Euler's theorem to homogeneous functions T and $V_1 = b_\beta \dot{q}_\alpha$

$$\mathcal{E}^{\rm ext} = T + U + P. \tag{3.8}$$

This is in accordance with the result of Vujičić and the same demonstrates that this result remains valid also in the most general case.

But, so formulated energy integral differs from habitual energy conservation law and does not represent the first integral in the usual sense. Namely, the quantities R_0 and $P = -\int R_0 \, \mathrm{d}q$, in the general case can be founded only after finding the solutions of first n Lagrange's equations $q_i = q_i(t)$ $(n = 1, 2, \ldots, n)$. This quantity (3.8) represents an integral (or constant) of motion in such sense that his total derivative with respect to time is equal to zero, no matter whether this quantity can be found or not without finding these solutions $q_i(t)$.

The distinction of this energy law from the one obtained from n Lagrange's equations arises from the fact that in this parametric formulation of mechanics is included also the contribution of the nonstationarity of constraints (through the term R_0). Therefore this form of energy law is essentially equivalent to the first one obtained from the kinetic energy theorem, but expressed in respect to one extended system of generalized coordinates.

4. General Criterion for the Integrals of Motion

For any function of canonical variables $F(q_\alpha, p_\alpha)$ the total time derivative, with the aid of Hamilton's equations (2.13) can be written in the concise form

$$\frac{\mathrm{d}F}{\mathrm{d}t} = [F, \mathcal{H}]_\alpha + (Q^*_\alpha + R^*_\alpha)\,\dot{q}_\alpha, \tag{4.1}$$

where the extended Poisson's bracket is introduced by

$$[F, G]_\alpha = \frac{\partial F}{\partial q_\alpha}\frac{\partial G}{\partial p_\alpha} - \frac{\partial F}{\partial p_\alpha}\frac{\partial G}{\partial q_\alpha}. \tag{4.2}$$

On this ground one can formulate a general criterion for the integrals of motion in the above quoted sense: if the following condition is satisfied

$$\frac{\mathrm{d}F}{\mathrm{d}t} = [F, \mathcal{H}]_\alpha + (Q^*_\alpha + R^*_\alpha)\,\dot{q}_\alpha = 0 \iff F(q_\alpha, p_\alpha) = \text{const.}, \tag{4.3}$$

this is the necessary and sufficient condition for a quantity F to be an integral of motion. Because of the same structure of this Poisson's bracket, from here follows also the corresponding Poisson's theorem: if F_1 and F_2 are two integrals of motion, their Poisson's bracket is also an integral of motion.

By applying this criterion to the quantity (2.14)

$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = (Q^*_\alpha + R^*_\alpha)\,\dot{q}_\alpha = 0, \tag{4.4}$$

one can conclude that under condition $(Q^*_\alpha + R^*_\alpha)\,\dot{q}_\alpha = 0$ this function represents an integral of motion

$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = 0 \iff \mathcal{H}(q_\alpha, p_\alpha) = T + U + P = \text{const.} \tag{4.5}$$

This result coincides with that which was obtained from $n+1$ Lagrange's equations, but in this parametric formulation the habitual Hamiltonian $H(q_i, p_i, t)$ is not an integral of motion.

As a second example let us take the case of cyclic coordinates, when $q_1, q_2, \cdots, q_m$ are missing in Lagrangian and are linear functions of time $q_k = a_k t + b_k$, $(k = 0, 1, \ldots, m)$. Then the Hamiltonian can be decomposed in two corresponding parts and if one introduces the "truncated" Hamiltonian

$$I = \mathcal{H} - p_k a_k = \frac{\partial \mathcal{L}}{\partial \dot{q}_\mu} \dot{q}_\mu - \mathcal{L}, \quad (\mu = m+1, \ldots, n) \tag{4.6}$$

since $p_k = \partial \mathcal{L} / \partial \dot{q}_k$ =const., this quantity satisfies the criterion (4.3), i.e. it represents an integral of motion

$$\frac{\mathrm{d}I}{\mathrm{d}t} = [\mathcal{H} - p_k a_k, \mathcal{H}] = -[\mathrm{Const.}, \mathcal{H}] = 0, \tag{4.7}$$

in accordance with the result obtained by Vujičić [10].

5. Conservation Laws from D'Alembert–Lagrange Principle

We can demonstrate, analogously to the method of B. Vujanović [11], that one can obtain here the conservation laws from transformed d'Alembert–Lagrange's principle. In considered formulation of mechanics this principle has the form (2.7), and we can transform it in generalized coordinates, decomposing the generalized forces into potential and nonpotential ones. In this manner, after identical transformations one obtains

$$\delta L + (Q^*_\alpha + R^*_\alpha)\, \delta q_\alpha + R_0 \delta q_0 = \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{q}_\alpha} \delta q_\alpha \right) \tag{5.1}$$

and this is the corresponding central Lagrange's equation. Let us still pass from synchronous to total variations putting $\delta q_\alpha = \Delta q_\alpha - \dot{q}_\alpha \Delta t$, bearing in mind (2.11), and add to the both sides the term $\mathrm{d}\Lambda/\mathrm{d}t$, where Λ can be any function of q_α and $\dot{q}_\alpha$

$$\begin{aligned} \Delta \mathcal{L} + \mathcal{L} \frac{\mathrm{d}}{\mathrm{d}t}(\Delta t) + (Q^*_\alpha + R^*_\alpha)(\Delta q_\alpha - \dot{q}_\alpha \Delta t) + \frac{\mathrm{d}\Lambda}{\mathrm{d}t} = \\ = \frac{\mathrm{d}}{\mathrm{d}t} \left[\frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha} (\Delta q_\alpha - \dot{q}_\alpha \Delta t) + \mathcal{L} \Delta t + \Lambda \right] \end{aligned} \tag{5.2}$$

From here we can deduce the following conclusion, concerning the conservation laws: if

$$\Delta \mathcal{L} + \mathcal{L} \frac{\mathrm{d}}{\mathrm{d}t}(\Delta t) + (Q^*_\alpha + R^*_\alpha)(\Delta q_\alpha - \dot{q}_\alpha \Delta t) + \frac{\mathrm{d}\Lambda}{\mathrm{d}t} = 0, \tag{5.3}$$

then exists an integral of motion of the form

$$I \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha} (\Delta q_\alpha - \dot{q}_\alpha \Delta t) + \mathcal{L} \Delta t + \Lambda = \mathrm{const.} \tag{5.4}$$

Consequently, to each transformation of generalized coordinates and time with suitable chosen gauge function Λ, which satisfies the condition (5.3), corresponds an integral of motion (5.4). This statement represents an indirect generalization of Noether's theorem to nonconservative systems in this parametric formulation of mechanics, as a result analogous to quoted one obtained by Vujanović.

6. Total Variation of Action

However, this generalization can be realized also directly, by analogy with the one in the habitual formulation of mechanics [12], departing from the total variation of action and taking $q_0 = \tau(t)$ as an additional generalized coordinate. This variation is given by

$$\Delta W = \int_{\bar{t}_0}^{\bar{t}_1} \mathcal{L}\left(\bar{q}_\alpha, \dot{\bar{q}}_\alpha, \bar{t}\right) \mathrm{d}\bar{t} - \int_{t_0}^{t_1} \mathcal{L}\left(q_\alpha, \dot{q}_\alpha, t\right) \mathrm{d}t \tag{6.1}$$

and if we effectuate the transformations similar to ones in the usual analytical mechanics ([13], pp. 142–146), passing from $\bar{q}_\alpha(\bar{t})$ and $\bar{t}$ to $q_\alpha(t)$ and t, as well as extending the summation over the index α from 0 to n, one obtains in the first approximation

$$\Delta W = \int_{t_0}^{t_1} \left\{ \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha}\left(\Delta q_\alpha - \dot{q}_\alpha \Delta t\right) + \mathcal{L}\Delta t\right] + \right. \\ \left. + \left(\frac{\partial \mathcal{L}}{\partial q_\alpha} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha}\right)\left(\Delta q_\alpha - \dot{q}_\alpha \Delta t\right) \right\} \mathrm{d}t \tag{6.2}$$

Let us choose the transformations of generalized coordinates and time in the form of a r–parametric transformation group with r infinitesimal parameters

$$\Delta q_\alpha = \varepsilon_a \xi^a_\alpha \left(q_\beta, \dot{q}_\beta, t\right), \quad \Delta t = \varepsilon_a \xi^a_{(0)}\left(q_\beta, \dot{q}_\beta, t\right) \quad (a = 1, 2, \ldots, r) \tag{6.3}$$

By inserting these expressions in (6.2), the total variation of action can be presented through the functions ξ^a_α and $\xi^a_{(0)}$

$$\Delta W = \int_{t_0}^{t_1} \varepsilon_\alpha \left\{ \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha}\bar{\xi}^a_\alpha + \mathcal{L}\xi^a_{(0)}\right] - \left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha} - \frac{\partial \mathcal{L}}{\partial q_\alpha}\right)\bar{\xi}^a_\alpha \right\} \mathrm{d}t, \tag{6.4}$$

where instead of ξ^a_α we introduced the quantities

$$\bar{\xi}^a_\alpha = \xi^a_\alpha - \dot{q}_\alpha \xi^a_{(0)} \quad (a = 1, 2, \ldots, r) \tag{6.5}$$

7. Generalization of Emmy Noether's theorem

If one applies the formula (6.2) to any nonconservative system, the variational derivative according to Lagrange's equations (2.9) can be substituted by

$Q^*_\alpha + R^*_\alpha$

$$\Delta W = \int_{t_0}^{t_1} \left\{ \frac{\mathrm{d}}{\mathrm{d}t} \left[\frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha} (\Delta q_\alpha - \dot{q}_\alpha \Delta t) + \mathcal{L} \Delta t \right] - (Q^*_\alpha + R^*_\alpha)(\Delta q_\alpha - \dot{q}_\alpha \Delta t) \right\} \mathrm{d}t \quad (7.1)$$

By adding and subtracting the term $\mathrm{d}\Lambda/\mathrm{d}t$, where so introduced gauge function Λ can be any function of q_α and $\dot{q}_\alpha$, this relation passes to

$$\Delta W = \int_{t_0}^{t_1} \left\{ \frac{\mathrm{d}}{\mathrm{d}t} \left[\frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha} (\Delta q_\alpha - \dot{q}_\alpha \Delta t) + \mathcal{L} \Delta t + \Lambda \right] - \right.$$
$$\left. - (Q^*_\alpha + R^*_\alpha)(\Delta q_\alpha - \dot{q}_\alpha \Delta t) - \frac{\mathrm{d}\Lambda}{\mathrm{d}t} \right\} \mathrm{d}t \quad (7.2)$$

On the other hand, the total variation of action can also be transformed by application of the standard operations

$$\Delta W = \int_{t_0}^{t_1} \Delta(\mathcal{L}\mathrm{d}t) = \int_{t_0}^{t_1} \left[\Delta \mathcal{L} + \mathcal{L} \frac{\mathrm{d}}{\mathrm{d}t}(\Delta t) \right] \mathrm{d}t$$

and by inserting this expression in (7.2), one obtains

$$\int_{t_0}^{t_1} \left\{ \frac{\mathrm{d}}{\mathrm{d}t} \left[\frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha} (\Delta q_\alpha - \dot{q}_\alpha \Delta t) + \mathcal{L} \Delta t + \Lambda \right] - \right.$$
$$\left. - \left[\Delta \mathcal{L} + \mathcal{L} \frac{\mathrm{d}}{\mathrm{d}t}(\Delta t) + (Q^*_\alpha + R^*_\alpha)(\Delta q_\alpha - \dot{q}_\alpha \Delta t) + \frac{\mathrm{d}\Lambda}{\mathrm{d}t} \right] \right\} \mathrm{d}t = 0 \quad (7.3)$$

Therefore, if the following condition is satisfied

$$\Delta \mathcal{L} + \mathcal{L} \frac{\mathrm{d}}{\mathrm{d}t}(\Delta t) + (Q^*_\alpha + R^*_\alpha)(\Delta q_\alpha - \dot{q}_\alpha \Delta t) + \frac{\mathrm{d}\Lambda}{\mathrm{d}t} = 0, \quad (7.4)$$

since the time interval (t_0, t_1) is arbitrary, the integrand must be equal to zero, whereby it results

$$I \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha} (\Delta q_\alpha - \dot{q}_\alpha \Delta t) + \mathcal{L} \Delta t + \Lambda = \text{const.} \quad (7.5)$$

By substituting here Δq_α and Δt by expressions (6.3) and putting $\Lambda = \varepsilon_a \Lambda^a$, because of independence of parameters ε_a this conservation law will be decomposed into r independent ones

$$I^a \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha} \bar{\xi}^a_\alpha + \mathcal{L} \xi^a_{(0)} + \Lambda^a = \text{const.} \quad (a = 1, 2, \ldots, r) \quad (7.6)$$

Consequently, to each r–parametric transformation of generalized coordinates and time, which satisfies the condition (7.4), correspond r mutually independent integrals of motion (7.6). This statement represents the direct generalization of Emmy Noether's theorem to any nonconservative systems in this parametric formulation of mechanics. In the special case when $(Q^*_\alpha + R^*_\alpha)\,\dot{q}_\alpha = 0$ the total variation of action is reduced to

$$\Delta W = \Delta \int_{t_0}^{t_1} \mathcal{L}\,\mathrm{d}t = -\int_{t_0}^{t_1} \frac{\mathrm{d}\Lambda}{\mathrm{d}t}\,\mathrm{d}t$$

$$\text{if} \quad \frac{\mathrm{d}\Lambda}{\mathrm{d}t} = 0: \quad \Delta W = 0. \tag{7.7}$$

and we obtain as a special case the habitual formulation of Noether's theorem, namely the well known condition under which this theorem is valuable, with the same integral of motion (7.6).

8. Generalized Killing's Equations

Finally, let us still find the equations which must be satisfied by functions ξ^a_α and $\xi^a_{(0)}$, analogously to the procedure given by B. Vujanović [11] in the habitual formulation of mechanics. In this aim, depart from the condition (7.4) in the developed form

$$\frac{\partial \mathcal{L}}{\partial q_\alpha}\Delta q_\alpha + \frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha}\Delta \dot{q}_\alpha + \mathcal{L}\frac{\mathrm{d}}{\mathrm{d}t}(\Delta t) + \tag{8.1}$$
$$+ (Q^*_\alpha + R^*_\alpha)(\Delta q_\alpha - \dot{q}_\alpha \Delta t) + \frac{\mathrm{d}\Lambda}{\mathrm{d}t} = 0$$

and apply the relation between the total variation of time derivative and vice versa in the inverse sequence ([14], pp. 11)

$$\Delta \dot{q}_\alpha = \Delta \frac{\mathrm{d}q_\alpha}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(\Delta q_\alpha) - \dot{q}_\alpha \frac{\mathrm{d}}{\mathrm{d}t}(\Delta t)$$

So, if we substitute Δq_α and Δt by expressions (6.3) and put $\Lambda = \varepsilon_a \Lambda^a$, one obtains a linear combination of the parameters ε_a, and because of their independence each coefficient of ε_a must be equal to zero

$$\frac{\partial \mathcal{L}}{\partial q_\alpha}\xi^a_\alpha + \frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha}\left(\frac{\mathrm{d}\xi^a_\alpha}{\mathrm{d}t} - \dot{q}_\alpha \frac{\mathrm{d}\xi^a_{(0)}}{\mathrm{d}t}\right) + \mathcal{L}\frac{\mathrm{d}\xi^a_{(0)}}{\mathrm{d}t} +$$
$$+ (Q^*_\alpha + R^*_\alpha)\left(\xi^a_\alpha - \dot{q}_\alpha \xi^a_{(0)}\right) + \frac{\mathrm{d}\Lambda^a}{\mathrm{d}t} = 0, \quad (a = 1, 2, \ldots, r) \tag{8.2}$$

If we develop all time derivatives, bearing in mind that the functions ξ^a_α, $\xi^a_{(0)}$ and Λ^a depend on q_α and $\dot{q}_\alpha$, after grouping the similar terms these equations can be written in the form $A^a + \ddot{q}_\beta B^a_\beta = 0$. Since the functions A^a and B^a_β do not depend on $\ddot{q}_\alpha$, these equations will be satisfied only if $A^a = 0$ and $B^a_\beta = 0$, i.e.

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha}\left(\frac{\partial \xi^a_\alpha}{\partial \dot{q}_\beta} - \dot{q}_\alpha \frac{\partial \xi^a_{(0)}}{\partial \dot{q}_\beta}\right) + \mathcal{L}\frac{\partial \xi^a_{(0)}}{\partial \dot{q}_\beta} + \frac{\partial \Lambda^a}{\partial t} = 0 \tag{8.3}$$

$$(\alpha, \beta = 0, 1, \dots, n;\ a = 1, 2, \dots, r)$$

as well as

$$\frac{\partial \mathcal{L}}{\partial q_\alpha}\xi^a_\alpha + \frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha}\left(\frac{\partial \xi^a_\alpha}{\partial q_\beta}\dot{q}_\beta - \frac{\partial \xi^a_{(0)}}{\partial q_\beta}\dot{q}_\alpha \dot{q}_\beta\right) + \mathcal{L}\frac{\partial \xi^a_{(0)}}{\partial q_\beta}\dot{q}_\beta +$$

$$+(Q^*_\alpha + R^*_\alpha)\left(\xi^a_\alpha - \dot{q}_\alpha \xi^a_{(0)}\right) + \frac{\partial \Lambda^a}{\partial q_\beta}\dot{q}_\beta = 0, \quad (a = 1, 2, \dots, r) \tag{8.4}$$

These equations represent the corresponding generalized Killing's equations for the functions ξ^a_α and $\xi^a_{(0)}$, and their sense consists in the following. If exists at least one particular solution of this system of $r(n+2)$ equations, the condition for the existence of integrals of motion will be satisfied, and to each such transformation of generalized coordinates and time correspond r mutually independent integrals of motion. In this manner, the finding of integrals of motion is reduced to discover the particular solutions of this system of generalized Killing's equations, corresponding to the fundamental Vujanović's idea for finding the conservation laws of nonconservative systems in this parametric formulation of mechanics.

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LES LOIX DE CONSERVATION ET LE THEOREME DE NOETHER DANS UNE FORMULATION PARAMETRIQUE DE MECANIQUE

Dans cet article on a étudié les loix de conservation dans une formulation paramétrique de mécanique des systèms rhéonomes, qui est basée sur une famille des trajectoires variées, tirées à partir de leur position initiale et sur la transition à un paramètre nouvel, qui dépend de la trajectoire choisie. D'abord sont analysées les formes diverses de la loi d'énergie, obtenues à l'aide des équations differentielles du mouvement, dans habituelle et cette formulation de mécanique. En outre, ici on a formulé un criterium général pour les intégrales du mouvement, exprimées au moyen des crochets de Poisson étendus.

Dans la seconde part de cet article on a obtenu le théoreme de Noether généralisé dans cette formulation paramétriue de mécanique, à partir tant du principe correspondant de d'Alembert et Lagrange, ainsi que de la variation totale d'action. En applicant cela aux cets systèmes rhéonomes, la loi d'énergie est trouvée dans la forme $\mathcal{E} = T+U+P=$ const., oú P est le potential rhéonome, de même qu'à l'aide du système étendu des équations de Lagrange. Ces résultats sont en accordance avec tels obtenus par V. Vujičić dans sa modification de la mécanique des systèmes rhéonomes.

ZAKONI ODRŽANJA I NOETHER-INA TEOREMA U JEDNOJ PARAMETARSKOJ FORMULACIJI MEHANIKE

U ovom radu proučavani su zakoni održanja u parametarskoj formulaciji mehanike reonomnih sistema, koja se zasniva na familiji variranih trajektorija, povučenih iz njihovog početnog položaja i na prelazu na jedan novi parametar, koji zavisi od izabrane trajektorije. Prvo su analizirani razni oblici zakona energije dobijeni pomoću diferencijalnih jednačina kretanja, u uobičajenoj i ovoj formulaciji mehanike. Sem toga, ovde je formulisan i jedan opšti kriterijum za integrale kretanja, izražen pomoću proširenih Poisson-ovih zagrada.

U drugom delu ovog rada dobijena je generalisana Noether-ina teorema u ovoj parametarskoj formulaciji mehanike, polazeći kako od odgovarajućeg d'Alembert-Lagrange-evog principa tako i od totalne varijacije dejstva. Primenjujući to na ove reonomne sisteme, nađen je zakon energije u obliku $\mathcal{E} = T+U+P=$const., gde je P reonomni potencijal, kao i pomoću proširenog sistema Lagrange-evih jednačina. Ovi rezultati su u saglasnosti sa rezultatima koje je dobio V. Vujičić u svojoj modifikaciji mehanike reonomnih sistema.

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