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CONSERVATION LAWS AND ENERGY RELEASE RATES FOR THE NONLINEAR ELASTIC SHELLS

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1. Introduction

The conservation laws (or path-independent integrals) in shell theory have been consider by various authors [1], [2], [3]. In this paper, we examine similar type of integrals for nonlinear elastic shell theory in the context of thin shell theory obeying Love-Kirchhoff's hypothesis.

Path-independent integrals in shells have been considered by Bergez and Radenković [1] and Bergez [2]. However, they did not place any restrictions on the geometry of the shells and, based on the considerations on invariance, it is obvious that their integrals are not path independent in general.

Nicholson and Simmonds [3] have shown - in the context of shallow shell theory - that Sanders energy release rate integral is path–independent for all mid–surfaces geometries, Lo [4] examined path–independent integrals for cylindrical shells and shells of revolution. He concluded that path– independent integrals do not exist in general for shells except the ones which enjoy a high degree of symmetry.

Recently, a new method for the study of conservation laws has been proposed and used by Kienzler and Golebiewska-Herrmann [5] in the context of higher–order shell theories. In this paper our intention is to derive conservation law of J integral type using invariant characteristics of variational principles. These integrals are also related to energy release rates associated with translation. Finally, one of the integrals is applied as an example to illustrate the theory.

2. Equations of variational invariance

Let $\xi = \xi_\alpha$ $(\alpha = 1,2)$ be the Gaussian coordinates of the middle surface of a shell, $u_i(\xi)\,(i = 1,2,3)$ arbitrary vector fields, respectively and L is the Lagrangian density.

We can define a special form of Noether's theorem which is used here to derive the conservation laws [6]:

Theorem:. *If the fields u_i satisfy the corresponding Euler–Lagrange equations $E(L)_u = 0$, then the L remains infinitesimally invariant at u_i under the small transformations, if and only if u_i, also satisfies*

$$\int_C \left[L\alpha_a + \frac{\partial L}{\partial u_{i|a}} p_i \right] n_a dl = 0, \tag{2.1}$$

where the vector p_i is given by

$$p_i = \beta_i - u_{i|a}\alpha_a,$$

C is the smooth closed curve, bounding S and n_a is the unit normal (in S) to C.

As a special case we consider

$$\alpha_a \neq 0, \quad \beta_i = 0, \quad p_i = -u_{i|a}\alpha_a \tag{2.2}$$

which represent a family of coordinate translations and leads us to the conservation law which is of a special interest for us.

Then the conservation law (2.1) reads

$$\int_C \left[L\delta_{ab} - \frac{\partial L}{\partial u_{i|b}} u_{i|a} \right] n_b dl = 0 \tag{2.3}$$

This is the integral we are very familiar with, whose component along crack line is the Rice's J-integral [7].

3. Nonlinear shell theory

The starting point of the study is the elastic shell theory given by Budiansky and Sanders [8] and Koiter [9]. Only the elements of the theory are given here and details can be found in [9], [10].

The displacement vector $\mathbf{u}$ of a material point in the middle surface of the shell is given by

$$\mathbf{u} = u^a \mathbf{a}_a + w\mathbf{n} \tag{3.1}$$

where u_a, w are the surface and surface–normal components of the displacement vector and $\mathbf{a}_a = r_{,a}$, surface base vector.

The membrane strain measures ϑ_{ab} given in terms of these components (u_a, w) and rotation are

$$\vartheta_{ab} = \frac{1}{2}\left(u_{a|b} + u_{b|a}\right) - b_{ab}w, \tag{3.2a}$$

$$\phi_a = w_{,a} + b^l_a u_l, \tag{3.2b}$$

where b^l_a is the (mixed) curvature tensor of the middle surface.

The mid surface strain tensor γ_{ab} and the tensor of change of curvature k_{ab} can be expressed in the following forms [10]:

$$\gamma_{ab} = \vartheta_{ab} + \frac{1}{2}\phi_a\phi_b, \tag{3.3a}$$

$$k_{ab} = -w_{,ab} - b^l_{a|b}u_l - b^l_a u_{l|b} - b^l_b u_{l|a} + b^l_a b_{lb} w. \tag{3.3b}$$

For elastic thin shell, the stored energy function $W(\gamma, k)$ is quadratic in γ and k:

$$W(\gamma, k) = \frac{1}{2}H^{ablm}\gamma_{ab}\gamma_{lm} + \frac{1}{2}h^{ablm}k_{ab}k_{lm}. \tag{3.4}$$

So the constitutive equations can be given as

$$N^{ab} = \frac{\partial W}{\partial \gamma_{ab}} = H^{ablm}\gamma_{lm}, \quad M^{ab} = \frac{\partial W}{\partial k_{ab}} = h^{ablm}k_{lm} \tag{3.5}$$

in which N^{ab}, M^{ab} denote the membrane force tensor and the bending moment tensor, respectively.

In the absence of surface loads, they satisfy the following equilibrium equations

$$\left(N^{ab} - b^a_l M^{lb}\right)\big|_b - b^a_l\left(N^{lb}\phi_b + M^{ab}_{|b}\right) = 0 \quad \text{in S}, \tag{3.6a}$$

$$M^{ab}_{|ab} + b_{ab}\left(N^{ab} - b^a_l M^{lb}\right) + \left(N^{ab}\phi_b\right)\big|_a = 0 \quad \text{in S}. \tag{3.6b}$$

The above equations are exact and are derivable from the energy principles [10] applied to the deformed shell.

4. Conservation laws

If we identify the vector field u_i as $[u_a, w]$ in relations (2.3) we obtain expressions adapted for elastic shells.

Defining now a Lagrangian density (with negative sign) by the relation

$$-L\left(u_a, u_{a|b}, w, w_{|b}\right) = W(\gamma_{ab}, k_{ab}) \tag{4.1}$$

where the stored energy function is denoted by W, it may be verified that

$$\frac{\partial W}{\partial u_a} = N^{lb}b^a_l\phi_b - M^{lb}b^a_{l|b}, \quad \frac{\partial W}{\partial u_{a|b}} = N^{ab} - M^{lb}b^a_l - M^{al}b^b_l, \tag{4.2a}$$

$$\frac{\partial W}{\partial w} = -N^{ab}b_{ab} + M^{ab}b^l_a b_{lb}, \quad \frac{\partial W}{\partial w_{|b}} = N^{ab}\phi_a + M^{ab}_{|a}. \tag{4.2b}$$

The equilibrium equations can be directly derived from L as Euler–Lagrange equations

$$E(L)_u i = \frac{\partial L}{\partial u_i} - \left(\frac{\partial L}{\partial u_{i|b}}\right)_{|b} = 0, \quad u_i = \{u_a, w\}, \tag{4.3}$$

i.e. the relations (3.6) coincide with Euler–Lagrange eqns (4.3).

Using the above expression for L (4.1) we can rewrite (2.3) to derive the conservation law for elastic shell theory

$$\int_C \left[W\delta_{ab} - \frac{\partial W}{\partial u_{c|b}} u_{c|a} - \frac{\partial W}{\partial w_{|b}} w_{|a} \right] n_b \, \mathrm{d}l = 0 \tag{4.4}$$

The conservation law (4.4), using the above expression for W, $W_{,u_{c|b}}$, and $W_{,w_{|b}}$ is given by

$$J_l = \int_C \left[W n_l - T_c u_{c|l} - M_n w_{(n)|l} - Q w_{|l} \right] \mathrm{d}l \tag{4.5}$$

where

$$\mathbf{T} = T^a \mathbf{a}_a + Q\mathbf{n} \tag{4.6}$$

the boundary force,

$$T^a = \left(N^{ab} - b^a_l M^{lb} \right) n_b - b^a_b M^{lb} n_l, \tag{4.7a}$$

$$Q = \left(N^{ab}\phi_b + M^{ab}_{|b} \right) n_a + \frac{\partial}{\partial s}\left(M^{ab} t_a n_b \right), \tag{4.7b}$$

and

$$M_n = M^{ab} n_a n_b, \quad w_{(n)} = \frac{\partial w}{\partial \mathbf{n}}, \tag{4.7c}$$

is the bending moment, while $\mathbf{t}$, s, $\mathbf{n}$ is the boundary triad on C. Equations (4.5) represent conservation law for elastic shell theory which is believed to be new.

Proof of the path independence of the integral (4.5) is straightforward and the details are shown in [4].

5. Energy release rates

While the proof of the path independence of the integrals in (4.5) is straightforward, it does not afford any particular physical insight in the interpretation of these conservation laws. In this section we relate the integrals in (4.5) to energy release rates associated with translation, thus identifying them with the corresponding conservation laws in elasticity [11].

Denoting J to be the $\xi = \xi_1$ component of the integral (4.5), the path independent integral takes the form

$$J = J_\xi = \int_C \left(W n_\xi - T_a u_{a,\xi} - M_n w_{(n),\xi} - Q w_{,\xi} \right) \mathrm{d}l \tag{5.1}$$

This result can also be obtained by considering the total potential energy rate P of a shell with a crack, in the absence of body forces:

$$P(a) = \int_C W \, \mathrm{d}s - \int_{C_T} \left(T_a u_a + M_n w_{(n)} + Q w \right) \mathrm{d}l \tag{5.2}$$

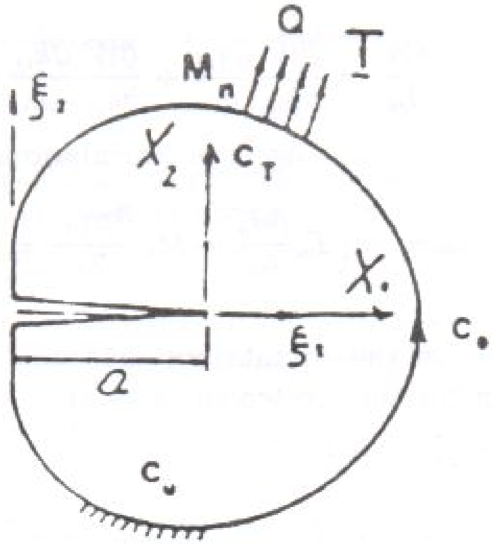

Fig. 1

where C_T denotes the contour of the shell on which the tractions T_a, M_n and Q are prescribed, (4.7). The tractions are assumed to be independent of crack length, a, and the crack surfaces are taken to be traction free, Fig. 1.

The stored energy function W is defined by (3.4), u_a, w are the corresponding surface and surface–normal displacement on an arc length $\mathrm{d}l$, and S is the middle surface boundary.

Differentiating (5.2) with respect to a, we obtains:

$$\frac{\mathrm{d}P}{\mathrm{d}a} = \int_S \frac{\mathrm{d}W}{\mathrm{d}a}\,\mathrm{d}s - \int_{C_0} \left(T_a \frac{\mathrm{d}u_a}{\mathrm{d}a} + M_n \frac{\mathrm{d}w_{(n)}}{\mathrm{d}a} + Q \frac{\mathrm{d}w}{\mathrm{d}a} \right) \mathrm{d}l \tag{5.3}$$

The contour of line integral can be extended along the boundary C_0 of the shell since $\mathbf{du}/\mathrm{d}a$, $\mathbf{dw}_{(n)}/\mathrm{d}a$, $\mathrm{d}w/\mathrm{d}a$ are on the boundary Cu, where $\mathbf{u}$, $\mathbf{w}_{(n)}$ and $\mathbf{w}$ are prescribed independently of a.

If the position of material point can be also defined relative to a Cartesian coordinate system $X_i = \xi_i - a\delta_{i1}$ attached to the crack tip, it follows that

$$\frac{\mathrm{d}}{\mathrm{d}a} = \frac{\partial}{\partial a} + \frac{\partial}{\partial x_i}\frac{\partial x_i}{\partial a} = \frac{\partial}{\partial a} - \frac{\partial}{\partial \xi_1} \tag{5.4}$$

Hence, (5.3) becomes

$$\frac{\mathrm{d}P}{\mathrm{d}a} = \int_S \left(\frac{\mathrm{d}W}{\mathrm{d}a} - \frac{\partial W}{\partial \xi_1} \right) \mathrm{d}s - \int_C \left[T_a \left(\frac{\partial u_a}{\partial a} - \frac{\partial u_a}{\partial \xi_1} \right) + M_n \left(\frac{\partial w_{(n)}}{\partial a} - \frac{\partial w_{(n)}}{\partial \xi_1} \right) + Q \left(\frac{\partial w}{\partial a} - \frac{\partial w}{\partial \xi_1} \right) \right] \mathrm{d}l \tag{5.5}$$

Furthermore

$$\frac{dW}{da} = \frac{\partial W}{\partial \gamma_{ij}}\frac{\partial \gamma_{ij}}{\partial a} + \frac{\partial W}{\partial k_{ij}}\frac{\partial k_{ij}}{\partial a} \tag{5.6}$$

Using (3.3) and (3.5), after some algebraic calculations, we obtain

$$\int_S \frac{dW}{da}\, ds = \int_{C_0} \left(T_a \frac{\partial u_a}{\partial a} + M_n \frac{\partial w_{(n)}}{\partial a} + Q\frac{\partial w}{\partial a} \right) dl \tag{5.7}$$

where we have made use of the equilibrium equations (3.6), the divergence theorem and an expression for surface tractions (4.7).

Therefore, equation (5.5) reduces to

$$-\frac{dP}{da} = \int_{C_0} \frac{\partial W}{\partial \xi_1}\, ds - \int_{C_0} \left(T_a \frac{\partial u_a}{\partial \xi_1} + M_n \frac{\partial w_{(n)}}{\partial \xi_1} + Q\frac{\partial w}{\partial \xi_1} \right) dl \tag{5.8}$$

which upon an application of the divergence theorem becomes:

$$-\frac{dP}{da} = \int_{C_0} \left\{ W\, d\xi_2 - \left[\left(T_a \frac{\partial u_a}{\partial \xi_1} + M_n \frac{\partial w_{(n)}}{\partial \xi_1} + Q\frac{\partial w}{\partial \xi_1} \right) \right] dl \right\} \tag{5.9}$$

The crack driving force, G, can then be calculated as the energy release rate in propagating the crack along an infinitesimal distance, i.e.,

$$G = -\frac{dP}{da} \tag{5.10}$$

This result is the same as that obtained by evaluating the path independent integral (5.1) on a contour enclosing a crack tip. Therefore, where (5.4) applicable, the value of J is identical to the crack driving force, G:

$$J = G$$

giving the physical meaning to J integral for the nonlinear elastic shells.

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DIE KONSERVATIONSGESETZE UND DIE GESCHWINDIGKEIT DER ENERGIEBEFREIUNG FUER NICHTLINEARE ELASTISCHE SCHALLEN

In dieser Arbeit wird die Verwendung der Noether's Theoreme sowie die Methode zur Gewinnung der Konservationsgesetze erortert.

Auf den Beispielen der nichtlinearen elastischen Schallen wird die Gruppe der koordinaten Translationen genommen, welcher der entsprechende Konservationsgesetz von J-Integral Typ entspricht, dessen Anwendung in der Bruchmechanik bekannt ist. Dadurch wird auch seine Verbingung mit der Geschwindigkeit der Energiebefreiung gezeigt.

ZAKONI KONZERVACIJE I BRZINA OSLOBADJANJA ENERGIJE ZA NELINEARNE ELASTICNE LJUSKE

U radu se razmatra primena teoreme E. Neter kao metoda za dobijanje zakona konzervacije. Iz osobine invarijantnosti varijacionog principa u odnosu na grupu infinitezimalnih transformacija izvodi se opsti oblik zakona konzervacije za vektorska polja.

Na primeru nelinearnih elasticnih ljuski, uzima se grupa koordinatnih translacija i njoj asocira odgovarajuci zakon konzervacije tipa J-integrala nezavisnog od putanje. Zatim se pokazuje da je brzina oslobodjanja energije pri razvoju prsline, za klasu problema koji zadovoljavaju uvedene pretpostavke jednaka vrednosti J integrala.

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