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INFLUENCE OF ROTATORY INERTIA AND SHEAR ON THE DYNAMIC INSTABILITY OF BEAMS SUBJECTED TO RANDOM EXCITATIONS

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1. Introduction

Dynamic stability and instability of continous systems, subjected to random excitations has been studied for the past thirty years. One of the first analyses of the problem of the stability properties of beam was done by Samuels and Eringen [8]. In that paper the criterion for determining the mean-square stability in case of a single variationed being the white-noise process, while keeping the others constant, was derived.

An other significant study was published by Caughey and Gray [3], where a Liapunov's type of approach, to obtain sufficient conditions assertaining the almost sure stability of a simply supported beam subjected to stationary ergodic loads, was used.

The applicability of Liapunov's functional method applied to continuous systems as beam and plates, subjected to random parametric excitations was expanded by Plaut and Infante [7]. This developed method is very convenient for obtaining sufficient conditions for the almost sure asymptotic stability.

Kozin [4] introduced the "best" Liapunov's functional, suitable for studying of the almost sure asymptotic stability of beam and plates axially loaded by zero-mean stationary ergodic forces, whose samples are continuous with probability one.

The instability conditions in terms of spectral properties of the loads in the stochastically loaded structures, treated as a linear system with a finite number of degrees of freedom, where derived by Ariaratnam and Tam [1].

Tylikowski [9] has studied the asymptotic stability and the almost sure asymptotic stability of viscoelastic beams compressed by time-dependent deterministic, as well as stochastic parametric excitations.

In [6] the stability conditions of the beams, with rotatory inertia taken into account, were derived. The beam is simply supported and subjected to axially time-dependent random loads.

The purpose of this paper is to analyse the influence of rotatory inertia and shear on the dynamic stability of beams. Using the direct Liapunov's method, we have derived the sufficient conditions for the almost-sure asymptotic instability.

2. Problem formulation

Let us consider a simply supported beam, of density ϱ, and of constant cross-section A and length l. Equations of the transverse motion of an axially compressed beam, taking into account rotatory inertia and the shear deformations, have the forms [2],

$$\frac{\varrho}{kG}\left(\frac{\partial^2 W}{\partial T^2}+\beta_1\frac{\partial W}{\partial T}\right)=\frac{\partial}{\partial Z}\left[\left(1-\frac{N_x(T)}{kGA}\right)\frac{\partial W}{\partial Z}-\psi\right] \tag{1}$$

$$\frac{\varrho I_x}{kGA}\left(\frac{\partial^2 \psi}{\partial T^2}+\beta_2\frac{\partial \psi}{\partial T}\right)=\frac{\partial W}{\partial Z}-\psi+\frac{EI_x}{kGA}\frac{\partial^2\psi}{\partial Z^2}, \tag{2}$$

where $W=W(Z,T)$ is the transverse displacement, ψ is the angle of rotation of the cross-section, T - time, $Z\in(0,l)$ is the axial coordinate, β_1, β_2 are the viscous damping coefficients for the transverse and rotatory motions respectively, k - the cross-section shape factor, E - Young's modulus, G is the modulus of rigidity, I_x is the cross-section moment of inertia and $N_x(T)$ is the stochastic axial compression force.

The boundary conditions for simply supported ends are

$$\left.\begin{matrix} Z=0\\ Z=l\end{matrix}\right\}\quad W(Z,T)=0,\quad \frac{\partial\psi(Z,T)}{\partial Z}=0. \tag{3}$$

Assuming that damping in the transverse and rotatory motions is the same, i.e., $\beta_2=\beta_1=\beta$, the following parameters can be used in the non-dimensional Eqs. (1) and (2),

$$W=wl,\quad Z=zl,\quad z\in(0,1);\quad T=k_t t,\quad \zeta=\frac{1}{2}k_t\beta,\quad k_t=l^2\left(\frac{\varrho A}{EI_x}\right)^{1/2},$$
$$\kappa^2=\frac{kGAl^2}{EI_x},\quad f(t)=\frac{N_x(t)l^2}{EI_x},\quad s^2=\frac{Al^2}{I_x}. \tag{4}$$

A substitution of relations (4) into Eqs. (1) and (2) yields

$$\frac{\partial^2 w}{\partial t^2}+2\zeta\frac{\partial w}{\partial t}+f(t)\frac{\partial^2 w}{\partial z^2}-\kappa^2\frac{\partial}{\partial z}\left(\frac{\partial w}{\partial z}-\psi\right)=0, \tag{5}$$

$$\frac{\partial^2 w}{\partial t^2}+2\zeta\frac{\partial w}{\partial t}-s^2\frac{\partial^2 w}{\partial z^2}-\kappa^2 s^2\left(\frac{\partial w}{\partial z}-\psi\right)=0. \tag{6}$$

In the following analysis we shell assume that the solutions of stochastic differential Eqs. (5) and (6) exist, and that Eqs. (5) and (6) have trivial solutions $w(z,t)=0$, $\psi(z,t)=0$ with given zero initial conditions.

3. Stability analysis

In order to apply the Liapunov's method, we can construct a functional by means of the Parks – Pritchard's method [4]. In that sense, let us write Eqs. (5) and (6) in a matrix form

$$\mathcal{L}u = 0, \tag{7}$$

where the differential operator (matrix $\mathcal{L}$) is given by

$$\mathcal{L} = \begin{bmatrix} \dfrac{\partial^2}{\partial t^2} + 2\zeta\dfrac{\partial}{\partial t} + \dfrac{\partial^2}{\partial z^2} - \kappa^2\dfrac{\partial^2}{\partial z^2} & \kappa^2\dfrac{\partial}{\partial z} \\ -\kappa^2 s^2\dfrac{\partial}{\partial z} & \dfrac{\partial^2}{\partial t^2} + 2\zeta\dfrac{\partial}{\partial t} - s^2\dfrac{\partial^2}{\partial z^2} + \kappa^2 s^2 \end{bmatrix}, \tag{8}$$

where u is a column vector of variables w and ψ.

According to the Parks–Pritchard's methods, an operator $\mathcal{N}$ is defined by

$$\mathcal{N} = \begin{bmatrix} 2s^2\dfrac{\partial}{\partial t} + 2s^2\zeta & 0 \\ 0 & 2\dfrac{\partial}{\partial t} + 2\zeta \end{bmatrix}. \tag{9}$$

Using the dot–product of vector $\mathcal{L}u$ and $\mathcal{N}u$, and integrating over the rectangle $C = \{z : 0 \le z \le 1\} \times \{\tau : 0 \le \tau \le t\}$,

$$\int_0^t \int_0^1 (\mathcal{L}u, \mathcal{N}u)\, \mathrm{d}z\, \mathrm{d}\tau = 0. \tag{10}$$

After a partial integration of Eq. (10), it can be written in the form of the sum of two integrals. The first, obtained by an integration over the axial coordinate only, we obtain that it is the Liapunov's functional

$$\mathbb{V} = \int_0^1 \left[s^2\left(\frac{\partial w}{\partial t} + \zeta w\right)^2 + \zeta^2 s^2 w^2 + \right.$$
$$\left. + \kappa^2 s^2\left(\frac{\partial w}{\partial z} - \psi\right)^2 + \left(\frac{\partial \psi}{\partial t} + \zeta\psi\right)^2 + \zeta^2\psi^2 \right] \mathrm{d}z. \tag{11}$$

However, it is evident that

$$\mathbb{V}|_0^t - \int_0^t \frac{\mathrm{d}\mathbb{V}}{\mathrm{d}t}\,\mathrm{d}t = 0, \tag{12}$$

hence the second integral in (10) is the time derivative of the functional (11) together with Eqs. (5) and (6),

$$\frac{\mathrm{d}\mathbb{V}}{\mathrm{d}t} = -\int_0^1 \left[2\zeta s^2\left(\frac{\partial w}{\partial t}\right)^2 + 2s^2 f(t)\frac{\partial^2 w}{\partial z^2}\left(\frac{\partial w}{\partial t} + \zeta w\right) - \right.$$

$$- 2\zeta\kappa^2 s^2 w\left(\frac{\partial^2 w}{\partial z^2} - \frac{\partial\psi}{\partial z}\right) + 2\zeta\left(\frac{\partial\psi}{\partial t}\right)^2 - \tag{13}$$
$$- 2\zeta s^2\psi\frac{\partial^2\psi}{\partial z^2} - 2\zeta\kappa^2 s^2\psi\left(\frac{\partial w}{\partial z} - \psi\right)\Bigg].$$

The functional (11) is evidently positive–definite. In order to estimate the derivative of the functional expresion (13) can be written in the form

$$\frac{\mathrm{d}\mathbb{V}}{\mathrm{d}t} = -2\zeta\mathbb{V} + 2\mathbb{U} \tag{14}$$

where $\mathbb{U}$ is the functional

$$\mathbb{U} = \int_0^1 \left[2\zeta^2 s^2 w\left(\frac{\partial w}{\partial t} + \zeta w\right) + 2\zeta^2\psi\left(\frac{\partial\psi}{\partial t} + \zeta\psi\right) - \right.$$
$$\left. -s^2 f(t)\frac{\partial^2 w}{\partial z^2}\left(\frac{\partial w}{\partial t} + \zeta w\right)\right] \mathrm{d}z. \tag{15}$$

Introduce now a function λ, defined as a minimum over all w, ψ, $v = \dfrac{\partial v}{\partial t}$ and $\omega = \dfrac{\partial\psi}{\partial t}$ of the ratio $\mathbb{U}/\mathbb{V}$

$$\lambda = \min_{w,v,\psi,\omega}\frac{\mathbb{U}}{\mathbb{V}}. \tag{16}$$

As a minimum value in the particular case of a stationary point, we take zero for the variation of the $\mathbb{U}/\mathbb{V}$ value and obtain the equivalent variational equation

$$\delta(\mathbb{U} - \lambda\mathbb{V}) = 0. \tag{17}$$

Using the associated Euler's equations we obtain

$$2\zeta^2(v + 2\zeta w) - \zeta f(t)\frac{\partial^2 w}{\partial z^2} - f(t)\left(\frac{\partial^2 v}{\partial z^2} + \zeta\frac{\partial^2 w}{\partial z^2}\right) -$$
$$-2\lambda\left[\zeta(v + 2\zeta w) - \kappa^2\left(\frac{\partial^2 w}{\partial z^2} - \frac{\partial\psi}{\partial z}\right)\right] = 0,$$
$$\zeta^2(\omega + 2\zeta\psi) - \lambda\left[\zeta(\omega + 2\zeta\psi) - s^2\frac{\partial^2\psi}{\partial z^2} - \kappa^2 s^2\left(\frac{\partial w}{\partial z} - \psi\right)\right] = 0, \tag{18}$$
$$2\zeta^2 w - f(t)\frac{\partial^2 w}{\partial z^2} - 2\lambda(v + \zeta w) = 0,$$
$$\zeta^2\psi - \lambda(\omega + \zeta\psi) = 0.$$

Upon solving Eqs. (16), the function $\lambda(t)$ obtains the form

$$\lambda(t) = \min_m \left\{ \frac{\zeta^4 n_m + l_m F_m^2(t) + \sqrt{(\zeta^4 n_m - l_m F_m^2(t))^2 + 4\zeta^4 \kappa^4 s^2 \alpha_m^2 F_m^2(t)}}{2\,(n_m l_m - \kappa^4 s^2 \alpha_m^2)} \right\}^{1/2} \tag{19}$$

where,

$$\alpha_m = m\pi, \quad n_m = \zeta^2 + \kappa^2 \alpha_m^2$$
$$l_m = \kappa^2 s^2 + \zeta^2 + s^2 \alpha_m^2, \quad F_m(t) = \zeta^2 + \frac{1}{2} f(t) \alpha_m^2. \tag{20}$$

Using the property of the function $\lambda(t)$, we can estimate the time–derivative of the functional $\mathbb{V}$,

$$\frac{\mathrm{d}\mathbb{V}}{\mathrm{d}t} \geq -2(\zeta\mathbb{V} - \lambda\mathbb{V}). \tag{21}$$

Solving the differential inequality (21), we obtain the following estimation of the functional

$$\mathbb{V} \geq \mathbb{V}(0)\exp\left\{2t\left[-\zeta + \frac{1}{t}\int_0^t \lambda(\tau)\,\mathrm{d}\tau\right]\right\}. \tag{22}$$

Therefore, we can postulate that the trivial solution of the Eq. (1) is almost sure asymptotically unstable if

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \lambda(\tau)\,\mathrm{d}\tau \geq \zeta, \tag{23}$$

or, if the process $f(t)$ is ergodic and stationary, i.e. if

$$\mathbb{E}\lambda \geq \zeta, \tag{24}$$

where $\mathbb{E}$ denotes the operator of the mathematical expectation.

4. Numerical Results and Discussion

Expression (19) and the inequalities (23) or (24) give the possibility to obtain the critical viscous damping coefficient, guaranteeing an almost sure asymptotic instability as a function of the statistic characteristics of the axial force. Applying Schwarz's inequality to the relation (24) one obtains

$$\zeta^8 n_m^2 - 2\zeta^4 n_m l_m \left(\zeta^4 + \frac{1}{4}\alpha_m^4 \sigma^2\right) +$$
$$l_m^2 \left(\zeta^8 + 3\zeta^4 \alpha_m^4 \sigma^2 + \frac{3}{16}\alpha_m^8 \sigma^4\right) + 4\zeta^4 \kappa^4 s^2 \alpha_m^2 \left(\zeta^4 + \frac{1}{4}\alpha_m^4 \sigma^2\right) \geq$$

$$\geq \left[2\zeta^2\left(n_m l_m - \kappa^4 s^2 \alpha_m^2\right) - \zeta^4(n_m + l_m) - \frac{1}{4} l_m \alpha_m^4 \sigma^2\right]^2 .$$

The almost sure asymptotic instability region is defined as a set where the damping coefficient ζ is smaller then it's critical value. The instability regions as functions of loading variance, damping coefficients mode number, cross-section shape-factor and the slenderness of beams are shown on the appented diagrams and the Table 1.

Table 1

m = 1, k = 5/6, s = 10.

ζ	0.00	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00	2.25	2.50
σ^2	[illegible]	0.012	0.046	0.097	0.162	0.236	0.317	0.403	0.494	0.590	0.693

ζ	2.75	3.00	3.25	3.50	3.75	4.00	4.25	4.50	4.75	5.00
σ^2	0.802	0.918	1.041	1.172	1.310	1.457	1.611	1.774	1.944	2.123

m = 1, k = 5/6, s = 20.

ζ	0.00	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00	2.25	2.50
σ^2	0.000	0.012	0.046	0.097	0.162	0.236	0.317	0.403	0.494	0.591	0.694

ζ	2.75	3.00	3.25	3.50	3.75	4.00	4.25	4.50	4.75	5.00
σ^2	0.804	0.921	1.046	1.178	1.319	1.467	1.624	1.790	1.964	2.147

m = 1, k = 5/6, s = 100.

ζ	0.00	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00	2.25	2.50
σ^2	0.000	0.012	0.046	0.097	0.162	0.236	0.317	0.403	0.495	0.592	0.695

ζ	2.75	3.00	3.25	3.50	3.75	4.00	4.25	4.50	4.75	5.00
σ^2	0.805	0.922	1.047	1.180	1.321	1.471	1.629	1.795	1.971	2.155

From Table 1, we conclude that the increase of the slenderness s, or the decrease of rotatory inertia would cause small increases of the almost sure asymptotic instability regions. This effect is greater for higher order harmonics, as seen on Fig.1.

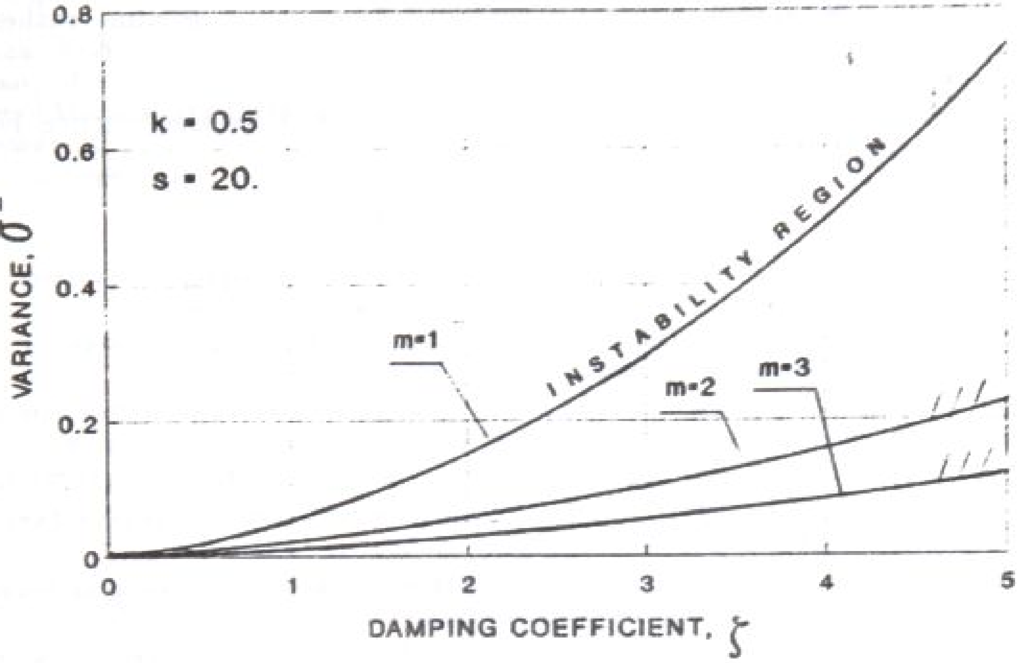

Fig. 1 The influence of the mode number on instability regions

The influence of shear is significant which is evident from Fig.2. When the cross–section shape factor k increases, almost sure asymptotic instability regions would rapidly decrease.

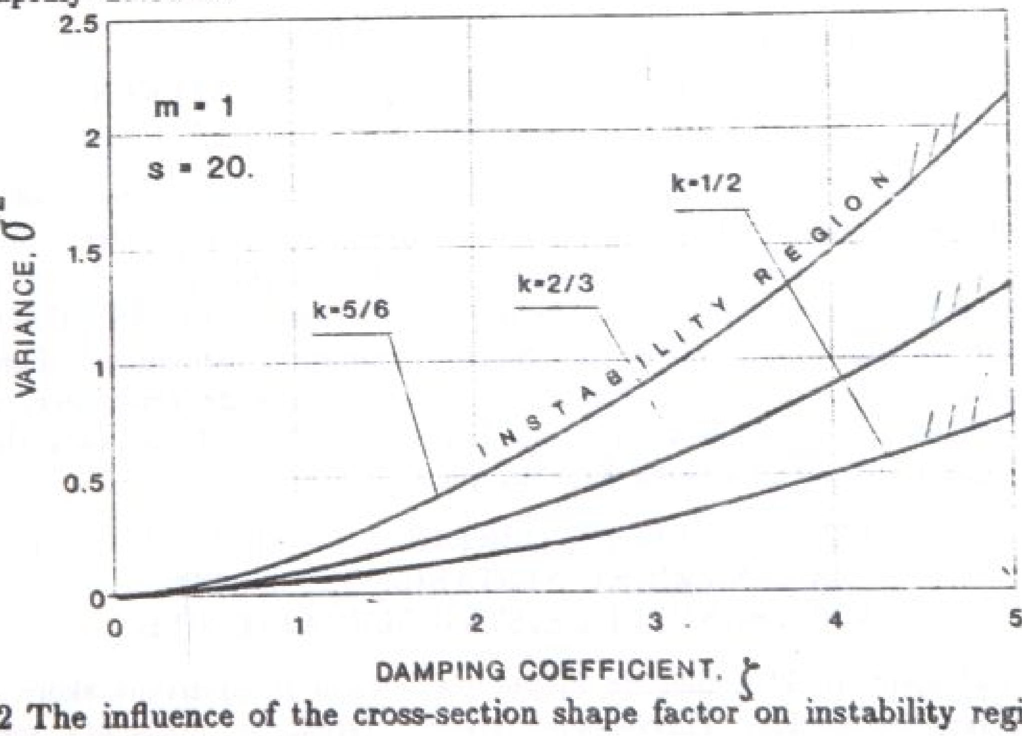

Fig. 2 The influence of the cross-section shape factor on instability regions

5. Conclusions

The applicability of the direct Liapunov's method has been extended to beams where rotatory inertia and shear are included. The beams are subjected to time-dependent axial forces. The major conclusion is that the instability regions change qualitatively when the shear is taken into account rather than rotatory inertia.

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DREHTRÄGHEITS-UND SCHUBEINFLUSS AUF DYNAMISCH INSTABILITÄT VON BALKEN UNTER WIRKUNG EINER STOCHASTISCHEN KRAFT

Anhand von direkten Liapunov - Methode wird in der Arbeit fast sichere asimptotishe Instabilität des Timoschenko -Balkens untersucht. Der Balken wird durch eine zeitlich veränderliche stochastische axiale Kraft under Druck gesetzt. Als Unterschied zu der klasischen Theorie wurden hier in Betracht Einfluss von Drehträgheits des Querschnittes und Schubdeformation genommen. Funktional von Liapunov wurd hier nach Parks - Pritchard Methode konstruiert und erhaltene Ergebnisse stellen eine Generalisierung in bezug auf die, nach klasischer Theorie des Balkens, erhaltenen Ergebuissen.

UTICAJ INERCIJE OBRTANJA I SMICANJA NA DINAMIČKU NESTABILNOST GREDA PODVRGNUTIH DEJSTVU SLUČAJNE SILE

Korišćenjem direktne metode Ljapunova u radu je ispitivana skoro sigurna asimptotska nestabilnost Timošenkove grede. Greda je pritisnuta vremenski

promenljivom stohastičkom aksijalnom silom. Za razliku od klasične teorije, ovde je uzet u obzir uticaj inercije obrtanja poprečnog preseka i smičućih deformacija. Funkcional Ljapunova je konstruisan pomoću Parks - Pritchard-ove metode, a dobijeni rezultati predstavljaju uopštenje rezultata koji se odnose na klasičnu teoriju grede.

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