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TRANSONIC FLOW SOLUTION OF THE THIN LAYER NAVIER-STOKES EQUATIONS USING IMPLICIT LU FACTORIZATION

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1. Introduction

The basic limitations in the Euler equations applications lie in flow calculation around wings at high angles of attack when viscous effects are not negligible. In cases where flow separation occurs it is necessary to add viscous terms to the Euler equations, which result in the Navier–Stokes equations. An application of the full Navier–Stokes equations requires a large computer storage and strongly increases the computational time needed. Approximative Navier–Stokes equations, known as thin-layer Navier–Stokes equations (TLNS), are described in this paper. Numerical solution of these equations is based on finite volume method (FVM) with flux splitting implicit LU factorization.

2. Theoretical postulation

The three–dimensional unsteady Navier–Stokes equations may be written in Cartesian coordinate system in conservation form as follows :

$$\partial_t q + \partial_x(F - F_v) + \partial_y(G - G_v) + \partial_z(H - H_v) = 0, \tag{2.1}$$

where q is the flow properties vector

$$q = (\rho, \rho u, \rho v, \rho w, e)^T, \tag{2.2}$$

while F, G and H are flux vector projections on Cartesian axes :

$$F = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ \rho uw \\ u(e+p) \end{pmatrix} \quad G = \begin{pmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ \rho vw \\ v(e+p) \end{pmatrix} \quad \text{and} \quad H = \begin{pmatrix} \rho w \\ \rho uw \\ \rho vw \\ \rho w^2 + p \\ w(e+p) \end{pmatrix}. \tag{2.3}$$

The viscous terms F_v, G_v and H_v, present in the equation (2.1), are given by the following relations :

$$\left.\begin{aligned} F_v &= \begin{pmatrix} 0 \\ \tau_{xx} \\ \tau_{yx} \\ \tau_{zx} \\ u\,\tau_{xx} + v\,\tau_{xy} + w\,\tau_{xz} - Q_x \end{pmatrix}, \\ G_v &= \begin{pmatrix} 0 \\ \tau_{xy} \\ \tau_{yy} \\ \tau_{zy} \\ u\,\tau_{yx} + v\,\tau_{yy} + w\,\tau_{yz} - Q_y \end{pmatrix}, \\ H_v &= \begin{pmatrix} 0 \\ \tau_{xz} \\ \tau_{yz} \\ \tau_{zz} \\ u\,\tau_{zx} + v\,\tau_{zy} + w\,\tau_{zz} - Q_z \end{pmatrix}. \end{aligned}\right\} \tag{2.4}$$

In the equations (2.2) and (2.3) quantities ρ, u, v, w, p and e denote density, velocity vector projections, pressure and total energy per unit volume, respectively. The elements of the shear-stress tensor are given by the constitutive equations for a Newtonian fluid as follows :

$$\left.\begin{aligned} &\tau_{xx} = 2\mu\, u_x - \frac{2}{3}\mu(u_x + v_y + w_z), \quad \tau_{yy} = 2\mu\, v_y - \frac{2}{3}\mu(u_x + v_y + w_z), \\ &\tau_{zz} = 2\mu\, w_z - \frac{2}{3}\mu(u_x + v_y + w_z), \quad \tau_{xy} = \tau_{yx} = \mu(u_y + v_x), \\ &\tau_{xz} = \tau_{zx} = \mu(u_z + w_x), \quad \tau_{yz} = \tau_{zy} = \mu(v_z + w_y), \end{aligned}\right\} \tag{2.4.1}$$

while the heat-flux vector is

$$Q_x = -k\,\frac{\partial T}{\partial x}, \quad Q_y = -k\,\frac{\partial T}{\partial y}, \quad Q_z = -k\,\frac{\partial T}{\partial z}, \tag{2.4.2}$$

where the viscosity coefficient μ is dependent on temperature T

$$\mu = \mu_\infty\,(\frac{T}{T_\infty})^{0.67}. \tag{2.4.3}$$

The thermal conductivity coefficient k is defined by the expression

$$k = \frac{\gamma}{\gamma - 1}\,\frac{\mu}{Pr}. \tag{2.4.4}$$

The Prandtl number Pr, present in the equation (2.4.4), can be considered as constant, $Pr = 0.72$, while $\gamma = 1.4$ is adiabatic constant for air. For turbulent flows, the laminar viscosity μ is replaced by $\mu + \mu_t$, where the eddy viscosity μ_t, and the turbulent Prandtl number Pr_t are provided by a turbulence model. In the present work the turbulence model of Baldwin and Lomax [1] is used.

An application of the curvilinear coordinate system transformation, grid generated in physical space, which surfaces are related to the constant values of curvilinear coordinates (ξ, η, ζ), is mapped to the rectangular computational grid. Neglecting diffusion processes parallel to a body surface, keeping only body-normal partial derivatives in the equations, leads to the thin-layer Navier–Stokes approximation. In the transformed space the equation (2.1) becomes

$$\partial_\tau \overline{q} + \partial_\xi \overline{F} + \partial_\eta \overline{G} + \partial_\zeta \overline{H} = \partial_\zeta \overline{H}_v, \tag{2.1.1}$$

where $\overline{q}, \overline{F}, \overline{G}$ and $\overline{H}$ are quantities defined as follows :

$$\overline{q} = Jq \tag{2.2.1}$$

and

$$\left.\begin{array}{c} \overline{F} = J \begin{pmatrix} \rho U \\ \rho u U + \xi_x p \\ \rho v U + \xi_y p \\ \rho w U + \xi_z p \\ U(e+p) - \xi_t p \end{pmatrix}, \quad \overline{G} = J \begin{pmatrix} \rho V \\ \rho u V + \eta_x p \\ \rho v V + \eta_y p \\ \rho w V + \eta_z p \\ V(e+p) - \eta_t p \end{pmatrix}, \\ \overline{H} = J \begin{pmatrix} \rho W \\ \rho u W + \zeta_x p \\ \rho v W + \zeta_y p \\ \rho w W + \zeta_z p \\ W(e+p) - \zeta_t p \end{pmatrix}. \end{array}\right\} \tag{2.3.1}$$

where U, V and W are velocity vector contravariant coordinates, defined by following transformation :

$$\left.\begin{array}{l} U = \xi_t + \xi_x u + \xi_y v + \xi_z w, \\ V = \eta_t + \eta_x u + \eta_y v + \eta_z w, \\ W = \zeta_t + \zeta_x u + \zeta_y v + \zeta_z w, \end{array}\right\} \tag{2.3.2}$$

while ξ_t, η_t and ζ_t are evaluated by equations

$$\left.\begin{array}{l} \xi_t = -x_\tau \xi_x - y_\tau \xi_y - z_\tau \xi_z, \\ \eta_t = -x_\tau \eta_x - y_\tau \eta_y - z_\tau \eta_z, \\ \zeta_t = -x_\tau \zeta_x - y_\tau \zeta_y - z_\tau \zeta_z, \end{array}\right\} \tag{2.3.3}$$

having in mind known the coordinate transformation law

$$\left.\begin{array}{l} \xi = \xi(x, y, z, t), \\ \eta = \eta(x, y, z, t), \\ \zeta = \zeta(x, y, z, t), \\ \tau = t. \end{array}\right\} \tag{2.3.4}$$

The quantities ξ_t, η_t and ζ_t in the equations (2.3.3) are equal to zero, for a stationary, fixed grid. The viscous term on the right hand side of the equation (2.1.1) after the transformation, becomes

$$\overline{H}_v = J \begin{pmatrix} 0 \\ \mu\,(\zeta_x^2 + \zeta_y^2 + \zeta_z^2)\,u_\zeta + \mu/3\,(\zeta_x\,u_\zeta + \zeta_y\,v_\zeta + \zeta_z\,w_\zeta)\,\zeta_x \\ \mu\,(\zeta_x^2 + \zeta_y^2 + \zeta_z^2)\,v_\zeta + \mu/3\,(\zeta_x\,u_\zeta + \zeta_y\,v_\zeta + \zeta_z\,w_\zeta)\,\zeta_y \\ \mu\,(\zeta_x^2 + \zeta_y^2 + \zeta_z^2)\,w_\zeta + \mu/3\,(\zeta_x\,u_\zeta + \zeta_y\,v_\zeta + \zeta_z\,w_\zeta)\,\zeta_z \\ [\mu/2\,(\zeta_x^2 + \zeta_y^2 + \zeta_z^2)\,(u^2+v^2+w^2)_\zeta + \\ +\mu/3\,(\zeta_x\,u + \zeta_y\,v + \zeta_z\,w)\,(\zeta_x\,u_\zeta + \zeta_y\,v_\zeta + \zeta_z\,w_\zeta) + \\ +k\,(\zeta_x^2 + \zeta_y^2 + \zeta_z^2)\,T_\zeta] \end{pmatrix}. \tag{2.4.5}$$

In the equations (2.2.1), (2.3.1) and (2.4.5) Jacobian $J = \partial(x,y,z)/\partial(\xi,\eta,\zeta)$ is evaluated from the expression

$$J = x_\xi(y_\eta z_\zeta - z_\eta y_\zeta) - y_\xi(x_\eta z_\zeta - z_\eta x_\zeta) + z_\xi(x_\eta y_\zeta - y_\eta x_\zeta). \tag{2.4.6}$$

For a perfect gas the system of equations (2.1) is completed by the definition of total fluid energy

$$e = \frac{1}{\gamma - 1}\,p + \frac{1}{2}\rho(u^2 + v^2 + w^2). \tag{2.5}$$

The unknown in the system of equations (2.1) and (2.1.1) is the flow property vector $\overline{q}$. The quantities $\overline{F}$, $\overline{G}$ and $\overline{H}$ are nonlinear functions of $\overline{q}$ at the time level $n+1$. These functions were linearized about time level n :

$$\left.\begin{aligned} \overline{F}^{n+1} &= \overline{F}^{n} + [\frac{D\overline{F}}{D\overline{q}}]^n\,\Delta\overline{q}^n, \\ \overline{G}^{n+1} &= \overline{G}^{n} + [\frac{D\overline{G}}{D\overline{q}}]^n\,\Delta\overline{q}^n, \\ \overline{H}^{n+1} &= \overline{H}^{n} + [\frac{D\overline{H}}{D\overline{q}}]^n\,\Delta\overline{q}^n, \end{aligned}\right\} \tag{2.6}$$

where is

$$\Delta\overline{q}^n = \overline{q}^{n+1} - \overline{q}^n. \tag{2.7}$$

In the equations (2.6) matrices $[D\overline{F}/D\overline{q}]^n$, $[D\overline{G}/D\overline{q}]^n$ and $[D\overline{H}/D\overline{q}]^n$, denoted by $\overline{\mathbf{A}}$, $\overline{\mathbf{B}}$ and $\overline{\mathbf{C}}$, respectively, are defined as follows :

$$\overline{\mathbf{A}},\quad \overline{\mathbf{B}},\quad \overline{\mathbf{C}} = \tag{2.8}$$

$$\begin{pmatrix} k_t & k_x & k_y & k_z & 0 \\ k_x\phi^2 - u\theta & k_t + \theta - k_x bu & k_y u - k_x av & k_z u - k_x aw & k_x a \\ k_y\phi^2 - v\theta & k_x v - k_y au & k_t + \theta - k_y bv & k_z v - k_y aw & k_y a \\ k_z\phi^2 - w\theta & k_x w - k_z au & k_y w - k_z av & k_t + \theta - k_z bw & k_z a \\ \theta(\phi^2 - \omega) & k_x\omega - au\theta & k_y\omega - av\theta & k_z\omega - aw\theta & k_t + \gamma\theta, \end{pmatrix}$$

where $a = \gamma - 1$, $b = \gamma - 2$ and $k = (\xi, \eta, \zeta)$ for matrices $\overline{\mathbf{A}}$, $\overline{\mathbf{B}}$ and $\overline{\mathbf{C}}$, respectively. The quantities ϕ^2, θ and ω in the equations (2.8) are evaluated in the following way :

$$\left.\begin{aligned} \phi^2 &= \frac{1}{2}(\gamma - 1)(u^2 + v^2 + w^2), \\ \theta &= k_x u + k_y v + k_z w, \\ \omega &= \gamma e/\rho - \phi^2. \end{aligned}\right\} \tag{2.9}$$

In the numerical approach of the Navier–Stokes equation solution with finite volume method (FVM), the computational domain is discretised by dividing into hexahedral cells and then the system of equations (2.1)–(2.5) is approximated. It is assumed that the value of the dependent variable $\overline{q}$ is known at the point (i, j, k), where each such point is the center of one of the cells, approximative system of equations (2.1.1) may be written in the following form :

$$\left[\mathbf{I} + \beta\Delta t\left(\delta_\xi \overline{\mathbf{A}}^n + \delta_\eta \overline{\mathbf{B}}^n + \delta_\zeta(\overline{\mathbf{C}}^n - \overline{\mathbf{C}}_v^n)\right)\right] \Delta\overline{q}^n + \Delta t\, \overline{R}^n = 0, \tag{2.10}$$

where the residual $\overline{R}^n$ is

$$\overline{R}^n = \delta_\xi \overline{F}(\overline{q}^n) + \delta_\eta \overline{G}(\overline{q}^n) + \delta_\zeta\left(\overline{H}(\overline{q}^n) - \overline{H}_v(\overline{q}^n)\right). \tag{2.10.1}$$

In the equations (2.10) and (2.10.1) δ_ξ, δ_η and δ_ζ denote central difference operators $\frac{\partial}{\partial\xi}$, $\frac{\partial}{\partial\eta}$ and $\frac{\partial}{\partial\zeta}$.

Parameter β in the relation (2.10) defines time accuracy of the applied scheme. If $\beta = 1/2$, the scheme remains second-order accurate in time; for other values of β the time accuracy drops to first order. The unfactored implicit scheme, defined by the equation (2.10), requires huge storage for very large block-banded matrix, which is very costly to invert especially for 3D flow case. An unconditionally stable implicit scheme that has error terms at most of order $(\Delta t)^2$ in any number of space dimensions can be derived by LU factorization [2],[3] and [4]

$$\left.\begin{aligned} &\left[\mathbf{I} + \beta\Delta t\left(\delta_\xi^- \overline{\mathbf{A}}^+ + \delta_\eta^- \overline{\mathbf{B}}^+ + \delta_\zeta^-(\overline{\mathbf{C}}^+ - \overline{\mathbf{C}}_v^+)\right)\right]^n * \\ * &\left[\mathbf{I} + \beta\Delta t\left(\delta_\xi^+ \overline{\mathbf{A}}^- + \delta_\eta^+ \overline{\mathbf{B}}^- + \delta_\zeta^+(\overline{\mathbf{C}}^- - \overline{\mathbf{C}}_v^-)\right)\right]^n \Delta\overline{q}^n + \Delta t\, \overline{R}^n = 0, \end{aligned}\right\} \tag{2.11}$$

where δ_ξ^-, δ_η^- and δ_ζ^- are backward difference operators and δ_ξ^+, δ_η^+ and δ_ζ^+ are forward difference operators

$$\left.\begin{aligned} \delta_\xi^-(\overline{\mathbf{A}}^+ \Delta\overline{q}^n)_{i,j,k} &= \overline{\mathbf{A}}^+_{i+1/2,j,k}\Delta\overline{q}^n_{i,j,k} - \overline{\mathbf{A}}^+_{i-1/2,j,k}\Delta\overline{q}^n_{i-1,j,k}, \\ \delta_\xi^+(\overline{\mathbf{A}}^- \Delta\overline{q}^n)_{i,j,k} &= \overline{\mathbf{A}}^-_{i+1/2,j,k}\Delta\overline{q}^n_{i+1,j,k} - \overline{\mathbf{A}}^-_{i-1/2,j,k}\Delta\overline{q}^n_{i,j,k}. \end{aligned}\right\} \tag{2.11.1}$$

The values of the matrix elements with indices $(i + 1/2, j, k)$ and $(i - 1/2, j, k)$ are evaluated by averaging between the cell points (i, j, k) and $(i + 1, j, k)$, or $(i - 1, j, k)$ and (i, j, k), respectively. The operators for two remaining coordinate directions can be calculated in a similar manner.

The reason for splitting in the equation (2.11) is to ensure the diagonal dominance of lower and upper factors as well as to make possible usage of the built-in implicit dissipation. Flux matrices $\overline{\mathbf{A}}^+$, $\overline{\mathbf{B}}^+$, $\overline{\mathbf{C}}^+$, $\overline{\mathbf{A}}^-$, $\overline{\mathbf{B}}^-$ and $\overline{\mathbf{C}}^-$ in the equation (2.11) are constructed so that the eigenvalues of "+" matrices are nonnegative and those of "−" matrices are nonpositive

$$\left.\begin{aligned}\overline{\mathbf{A}}^+ &= \frac{1}{2}(\overline{\mathbf{A}} + r_A\mathbf{I}), & \overline{\mathbf{A}}^- &= \frac{1}{2}(\overline{\mathbf{A}} - r_A\mathbf{I}),\\ \overline{\mathbf{B}}^+ &= \frac{1}{2}(\overline{\mathbf{B}} + r_B\mathbf{I}), & \overline{\mathbf{B}}^- &= \frac{1}{2}(\overline{\mathbf{B}} - r_B\mathbf{I}),\\ \overline{\mathbf{C}}^+ &= \frac{1}{2}(\overline{\mathbf{C}} + r_C\mathbf{I}), & \overline{\mathbf{C}}^- &= \frac{1}{2}(\overline{\mathbf{C}} - r_C\mathbf{I}),\end{aligned}\right\}$$

where $\mathbf{I}$ is the unit matrix and factors r_A, r_B and r_C are defined as follows :

$$r_A \geq max(|\lambda_A|), \quad r_B \geq max(|\lambda_B|), \quad r_C \geq max(|\lambda_C|). \tag{2.11.3}$$

In the equations (2.11.3) λ_A, λ_B and λ_C represent the eigenvalues of the Jacobian matrices $\overline{\mathbf{A}}$, $\overline{\mathbf{B}}$ and $\overline{\mathbf{C}}$, respectively :

$$\left.\begin{aligned}\lambda_A &= (U, U, U, U + c(\xi_x^2 + \xi_y^2 + \xi_z^2)^{\frac{1}{2}}, U - c(\xi_x^2 + \xi_y^2 + \xi_z^2)^{\frac{1}{2}}),\\ \lambda_B &= (V, V, V, V + c(\eta_x^2 + \eta_y^2 + \eta_z^2)^{\frac{1}{2}}, V - c(\eta_x^2 + \eta_y^2 + \eta_z^2)^{\frac{1}{2}}),\\ \lambda_C &= (W, W, W, W + c(\zeta_x^2 + \zeta_y^2 + \zeta_z^2)^{\frac{1}{2}}, W - c(\zeta_x^2 + \zeta_y^2 + \zeta_z^2)^{\frac{1}{2}}).\end{aligned}\right\} \tag{2.11.4}$$

In the expressions (2.11.4) the quantities U, V and W are contravariant velocity coordinates as defined by the equations (2.3.2), while c represents local speed of sound.

The viscous flux matrices $\overline{\mathbf{C}}_v^+$ and $\overline{\mathbf{C}}_v^-$, that are present in the implicit term of the equation (2.11), can be approximated in the way of Pulliam [5]

$$\overline{\mathbf{C}}_v^+ = \lambda_{C_v}\mathbf{I}, \quad \overline{\mathbf{C}}_v^- = -\lambda_{C_v}\mathbf{I}, \tag{2.11.5}$$

where λ_{C_v} is the eigenvalue of the viscous flux matrix $\overline{\mathbf{C}}_v$

$$\lambda_{C_v} = \mu\,(\zeta_x^2 + \zeta_y^2 + \zeta_z^2)\,\partial_\zeta\left(\frac{1}{\rho}\right). \tag{2.11.6}$$

The equations (2.11) can be inverted in two steps :

$$\left.\begin{aligned}&[\mathbf{I} + \beta\Delta t(\delta_\xi^-\overline{\mathbf{A}}^+ + \delta_\eta^-\overline{\mathbf{B}}^+ + \delta_\zeta^-(\overline{\mathbf{C}}^+ - \overline{\mathbf{C}}_v^+))]^n\,\Delta\overline{q}^{*n} = -\Delta t\,\overline{R}^n,\\ &[\mathbf{I} + \beta\Delta t(\delta_\xi^+\overline{\mathbf{A}}^- + \delta_\eta^+\overline{\mathbf{B}}^- + \delta_\zeta^+(\overline{\mathbf{C}}^- - \overline{\mathbf{C}}_v^-))]^n\,\Delta\overline{q}^n = \Delta\overline{q}^{*n}.\end{aligned}\right\} \tag{2.11.7}$$

Solution of the first system of equations (2.11.7) is done by a simple forward substitution and solution of the second system by a simple back substitution.

At solid surfaces, the no-slip conditions are used for the velocity components, adiabatic condition for the temperature and zero pressure gradients are assumed [6] and [7]

$$u = v = w = 0, \quad \frac{\partial h}{\partial n} = 0, \quad \frac{\partial p}{\partial n} = 0, \qquad (2.11.8)$$

where h is total enthalpy. At the far-field boundaries (upstream, lateral and downstream), the flow was assumed to be undisturbed whenever the freestream Mach number M_∞ was less than one. If the freestream Mach number exceeded unity ($M_\infty \geq 1$), all the five flow variables were extrapolated at the outflow boundary from the nearest inside cells.
At the boundaries of the impermeable surfaces, information about the values of flow variables inside the body are needed, as can be concluded from the equations (2.11) and (2.11.1). In order to overcome the mentioned obstacle it is necessary to modify the applied scheme for cell indices $k = 1$:

$$\overline{\mathbf{C}}^{+}_{i,j,k-1/2}\Delta\overline{q}_{i,j,k-1} = \mathbf{E}\,\overline{\mathbf{C}}^{-}_{i,j,k-1/2}\Delta\overline{q}_{i,j,k}, \qquad (2.11.9)$$

where the matrix $\mathbf{E}$ is defined in the following way :

$$\mathbf{E} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1. \end{pmatrix} \qquad (2.11.10)$$

This modification eliminates flow across the boundary. Similar can be done for cells at the plane of symmetry of aircraft. At the far-field boundaries Whitfield [8] has shown the validity of the assumption $\Delta\overline{q}^n = 0$. Now, solution of the system of equations (2.11.7) can be efficiently solved by inversion of sparse triangular matrices without using large computer storage. Owing to the fact that there are only two factors present in this scheme, factorization error can be reduced significantly. Although the alternating direction implicit (ADI) scheme has been valuable in two-dimensional problems, its inherent limitation in three dimensions suggests the LU approach. If the equation (2.11.7) is solved by a pass through computational space sweeping through diagonal plane, defined by $i + j + k = const$, program code can be fully vectorized and adapted for usage at supercomputers.
Central difference approximations in evaluation of the residual $\overline{R}^n$ in the equation (2.10.1) require an artificial viscosity in order to converge to a steady state of the fluid dynamic equations [9]. Additional dissipative terms, known as artificial viscosity terms, are added in order to inhibit any odd–even decoupling of the numerical solution by introduction of dissipation. On the other hand, artificial dissipation terms are added to eliminate high frequency oscillations in the neighborhood of shock waves. Also, from the mathematical theory for hyperbolic systems of inviscid conservation laws [10], the introduction of artificial dissipation is necessary to

guarantee a unique weak solution. The artificial dissipation employed in this paper is blending of second and fourth differences [11], [12] and [13]

$$R_1^n = (D_\xi^2 + D_\eta^2 + D_\zeta^2 - D_\xi^4 - D_\eta^4 - D_\zeta^4) q_{i,j,k}^n. \tag{2.12}$$

After the dissipation terms are added, the equation (2.11) takes the final form

$$\left.\begin{aligned} &[\mathbf{I} + \beta\Delta t(\delta_\xi^- \overline{\mathbf{A}}^+ + \delta_\eta^- \overline{\mathbf{B}}^+ + \delta_\zeta^- (\overline{\mathbf{C}}^+ - \overline{\mathbf{C}}_v^+))]^n * \\ &* [\mathbf{I} + \beta\Delta t(\delta_\xi^+ \overline{\mathbf{A}}^- + \delta_\eta^+ \overline{\mathbf{B}}^- + \delta_\zeta^+ (\overline{\mathbf{C}}^- - \overline{\mathbf{C}}_v^-))]^n \Delta\overline{q}^n + \Delta t [\overline{R}^n - R_1^n] = 0. \end{aligned}\right\} \tag{2.13}$$

Also, it is very important to determine correctly the time step size Δt, having in mind that the highest acceptable value is determined by a time interval of perturbation propagation from one side of a cell to another. The local time step Δt for the cell with indices (i, j, k) is evaluated in the following way :

$$\Delta t_{i,j,k} = \left[\frac{1}{(\Delta t_\xi)_{i,j,k}} + \frac{1}{(\Delta t_\eta)_{i,j,k}} + \frac{1}{(\Delta t_\zeta)_{i,j,k}}\right]^{-1}, \tag{2.14}$$

where $(\Delta t_\xi)_{i,j,k}$, $(\Delta t_\eta)_{i,j,k}$ and $(\Delta t_\zeta)_{i,j,k}$ are time intervals of perturbation propagation inside a cell in given coordinate directions. Time intervals $(\Delta t_\xi)_{i,j,k}$, $(\Delta t_\eta)_{i,j,k}$ and $(\Delta t_\zeta)_{i,j,k}$ can be evaluated in physical space by the following expressions :

$$\left.\begin{aligned} (\Delta t_\xi)_{i,j,k} &= \left(\frac{1}{|U| + c\sqrt{\xi_x^2 + \xi_y^2 + \xi_z^2}}\right)_{i,j,k}, \\ (\Delta t_\eta)_{i,j,k} &= \left(\frac{1}{|V| + c\sqrt{\eta_x^2 + \eta_y^2 + \eta_z^2}}\right)_{i,j,k}, \\ (\Delta t_\zeta)_{i,j,k} &= \left(\frac{1}{|W| + c\sqrt{\zeta_x^2 + \zeta_y^2 + \zeta_z^2}}\right)_{i,j,k}. \end{aligned}\right\} \tag{2.14.1}$$

In the equations (2.14.1) U, V and W are contravariant velocity coordinates, while c is local speed of sound.

The time step size Δt, evaluated in the equation (2.14), is to be scaled with constant, known as the Courant number. Stability analysis of the applied two–pass implicit LU scheme [14] has shown insensitivity to relatively high values of the Courant number. In order to accelerate convergence to a steady state usage of local time stepping is highly recommended, having in mind that cell sizes may differ very drastically. Since flow properties do not vary rapidly inside one iteration cycle it is not necessary to repeat time step calculation after one single iteration.

It is obvious that shown discussion is not valid for unsteady flows, since constant time step $\Delta t = min(\Delta t_{i,j,k})$ should be used.

3. Results

Computer results are presented for steady inviscid and viscous laminar and turbulent flow past a rectangular wing with constant spanwise NACA 65A010 airfoil distribution. Three-dimensional algebraically generated non-orthogonal "C–H" grids are used in all examples. In the calculations presented here, the convergence was considered to have been achieved when the value of residual was reduced by four orders of magnitude.
Computation are performed for a freestream Mach number $M_\infty = 0.8$ and zero angle of attack, with low Reynolds number $Re_\infty = 50000$ for laminar, and higher Reynolds number value $Re_\infty = 1.2 \times 10^7$, for turbulent flows. An extremely coarse grid $(65 \times 11 \times 15)$ is employed and pressure distribution is presented for the section of the plane of symmetry.

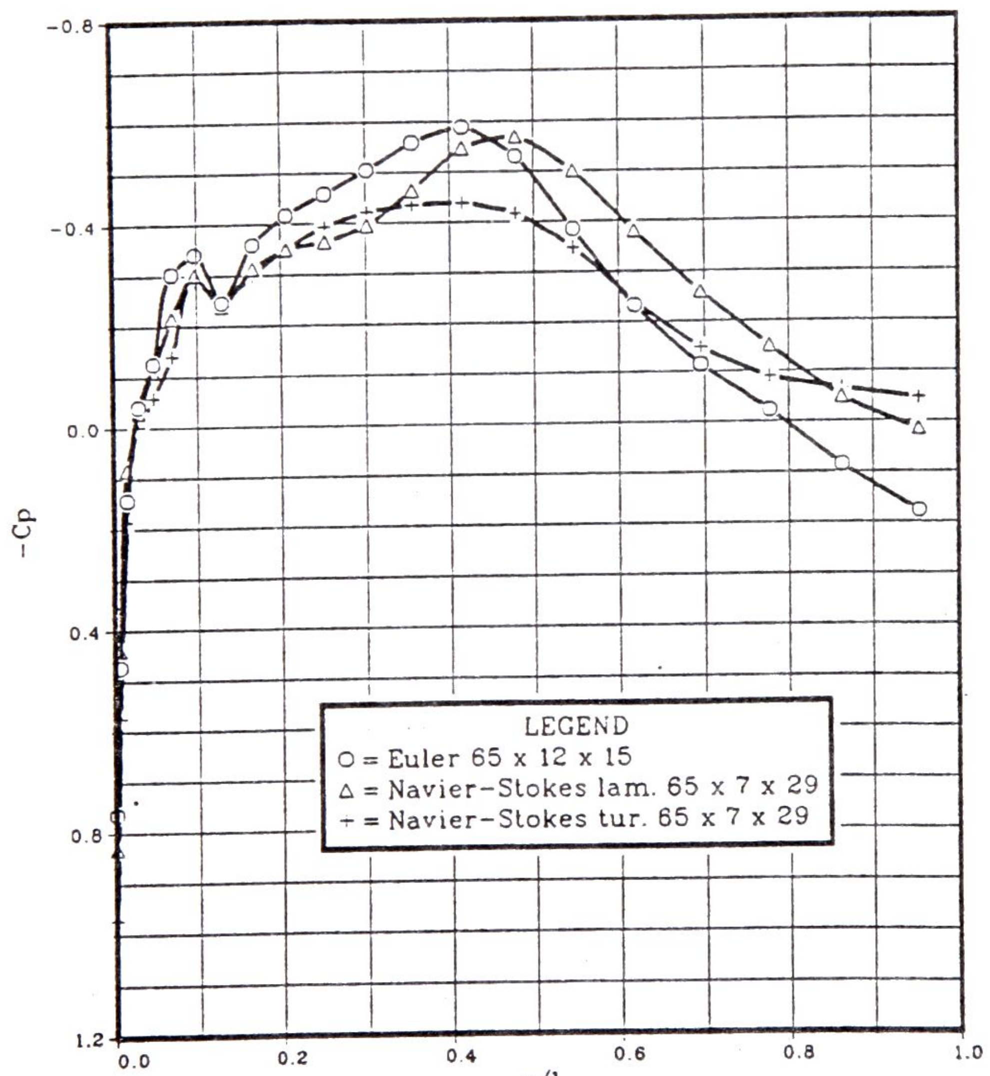

Fig. 1 Calculation of viscous flow over a rectangular wing with NACA 65A010 airfoil

As shown in Fig. 1, in the present coarse-grid solution flow separation occurs very close to the maximum thickness location of the airfoil. By increasing the value of the Reynolds number in the case of turbulent flow, the separation bubble is moved toward the trailing edge. For such coarse grids used in this paper the quantitative aspects of the results, such as the separation point and C_p distribution, can not be reliably predicted. Finer grid applications increase the accuracy but require access to CRAY Y/MP class of computers which was not available to the author.

4. Conclusion

Transonic viscous flow analysis, presented in this paper, provides very accurate aerodynamic load prediction, when the application of potential theory and the Euler equations are practically unacceptable. The numerical stability and fast convergence in differential equation solution make a significant improvement in comparison with explicit scheme employment. Flux splitting LU implicit factorization scheme ensures the numerical stability even for very large time step sizes, i.e. for $CFL \geq 20$. In highly time step dependent flow calculation, such as in the case of 3D unsteady motion, application of the present approach is quite acceptable. Presence of only two factors, even in the case of 3D flow reduces the factorization error. In comparison with classical ADI schemes LU implicit factorization decreases the amount of CPU time required for calculation. The chosen approach does not require large storage in solution of system of equations and employs only inversion of fifth order matrices. Correct program coding supports easy vectorization and usage at supercomputers. High accuracy of a solution is obtained by the modification of the scheme at physical boundaries, improving precise boundary condition definition.

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TRANSONIC FLOW SOLUTION OF THE THIN LAYER NAVIER-STOKES EQUATIONS USING IMPLICIT LU FACTORIZATION

An approach for numerical calculation of the transonic three dimensional viscous fluid flow, based on finite volume method, is described in this paper. Stability and fast convergence are obtained in this approach by the introduction of second and fourth order time accurate artificial viscosity terms. A system of differential equations is solved by the application of the LU implicit factorization scheme using Jameson-Yoon flux vector splitting.

PRORAČUN TRANSONIČNOG VISKOZNOG STRUJANJA PRIMENOM LU IMPLICITNE FAKTORIZACIJE NA SISTEM JEDNAČINA NAVIER-STOKES-A

U radu je izložen postupak numeričkog rešavanja jednačina trodimenzionalnog viskoznog transoničnog strujanja, baziran na metodi konačnih zapremina. Stabilnost i brza konvergencija postupka obezbedjena je uvodjenjem članova veštačke viskoznosti drugog i četvrtog reda. Sistem diferencijalnih jednačina rešavan je primenom aproksimativne LU implicitne faktorizacije sa "razdvajanjem" fluksa metodom Jameson-Yoon-a.

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