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ON THE THERMODYNAMICS OF AN INTERLINE

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1. Introduction

The papers dealing with the problems of interline are scarce in the reference literature. Papers [1,5] indicate, from different standpoints of the continuum mechanics, the significance of this problem, which is important both from the theoretical and the practical points of view. The following examples point to the necessity of further considerations of two-dimensional materials containing a material or non-material interline respectively.

A characteristic example in which the interline plays an important part is represented by the boundaries of the transition layer ([2]). Namely, if a system formed by two liquids and a solid (Fig. 1) is observed, the liquids can be considered as three-dimensional continua with the same characterictics in the whole area up to the transition layer, and the transition layer can be modeled as an interface. The mistake made in that way can be compensated for by assigning additional surface properties to the separeting surface, such as the mass density, the surface atress etc.

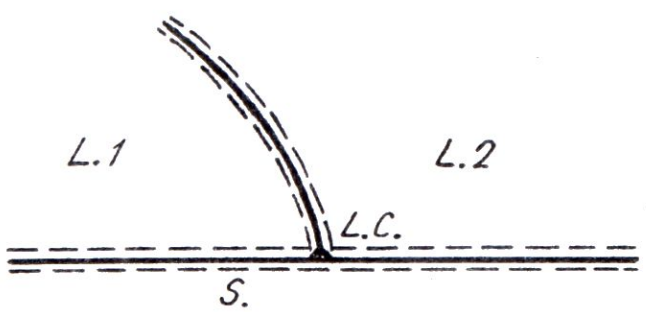

Fig. 1

In that way, the separeting surface is considered as a continuous medium having its own thermodynamics properties and the behaviour of which is not independent of the behaviour of the surrounding liquids. The segment of the transition layer is of particular interest in such problems. This segment appears as a boundary of the transition layer between the liquids and the solid. From the standpoint of the transition layer model, the point in question is the segment of the separeting surface and the solid and it is called the line of contact. The position of the separeting surface is not known in advance and, by the very fact, the position of the line of contact is not defined in advance either.

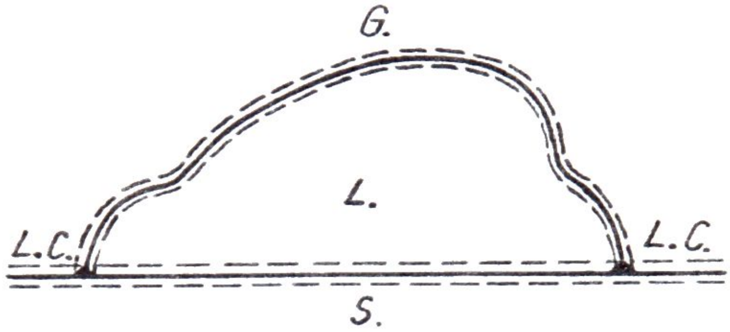

Fig. 2

A similar phenomen can be noticed in the case on the free boundaries of the surface between the stages. The ends of such free boundaries are the lines of contact and, as a rule, their behaviour is not known in advance (Fig. 2). In this case too, the line of contact can be considered as a line of discontinuity which determines the matual action of different stages of the investigated systems. Obviously, the lines of contact are not always plane curves (Fig. 3).

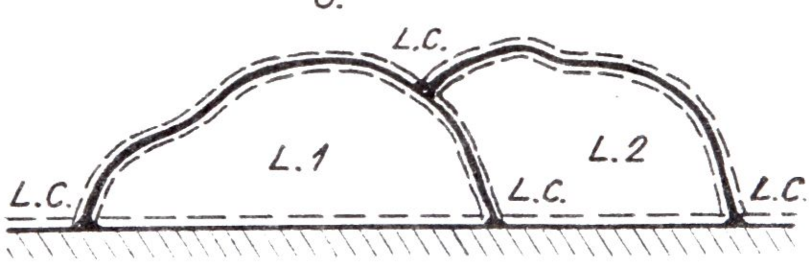

Fig. 3

Applications of this research are foreseen in the investigation of wetting, where droplets touch solid surface (say) along the interline. Also interlines abound in foams and two-phase suspensions of liquids. Because of the interline energy residing in such singular line, a thermodynamic treatment is indicated. Other applications are possible in biology whenever membranes are constrained by muscles or fibers as in the case of the peritoneum.

The group of problems with a nonmaterial interline also comprises the problem of detachment the membrane from the surface of a body. Owing to an important physical motivation, this problem deserves a special attention. This paper analyses a multicomponental surface of an arbitrary shape, considering the interline. It has been demonstrated that a certain kind of analogy can be established with it and the analysis of a three-dimensional continuun containing a discontinuity surface, both in formulating the virtual work principle and in deriving the so-called energy theorem. As an illustration of the theory exposed above, the problem of detachment of the membrane from the surface of a given body has been considered.

2. Preliminary considerations

The body $S(t)$, which changes in time in a definite interval I, and is situated in a three-dimensional Euclidian space E_3, has been considered. Let the boundary $\partial S(t)$, of the body $S(t)$ is smooth enough, so that in each of its points it is a uniquely defined the unit outward normal n^i, which is tangential to $S(t)$, $t \in I$. It has been supposed that the body $S(t)$ is devided by the curve $C(t)$ into two parts $S^+(t)$ and $S^-(t)$. Let the points at the boundary $\partial C(t)$, $t \in I$, are denoted by A and B. The position of a surface practicle in the present configuration at time $t \geq t_0$ refered to a fixed Cartesian system of coordinates x^i $(i = 1, 2, 3)$ is given by

$$x^i = \widehat{x}^i(U^\Gamma, t), \tag{1}$$

where U^Γ $(r = 1, 2)$ are the surface coordinates of the particle, and t is time. Suppose that the surface $S(t)$ is sufficiently smooth so that at each point of the surface there exist the tangent vectors to the coordinate lines $U^\Gamma = const.$

$$\tau^i_\Gamma = \frac{\partial \widehat{x}^i}{\partial U^\Gamma}, \tag{2}$$

and the unit normal vector ν^i, so that

$$\nu^i \nu^i = 1, \quad \text{and} \quad \tau^i_\Gamma \nu^i = 0. \tag{3}$$

The metric tensor on the surface in the present configuration is defined by

$$g_{\Delta\Gamma} = \tau^i_\Delta \tau^i_\Gamma. \tag{4}$$

The velocity of particle U^Γ on a surface $S(t)$ is given by

$$\widehat{V}^i = \frac{\partial \widehat{x}^i(U^\Gamma, t)}{\partial t}. \tag{5}$$

For subsequent derivations are useful to decompose this velocity into normal and tangential components

$$\widehat{V}^i = \underset{\nu}{\widehat{u}} \nu^i + \tau^i_\Gamma \widehat{V}^\Gamma, \tag{6}$$

where $\underset{\nu}{\widehat{u}}$ is the normal speed of the surface. Let a singular curve $C(t)$ be situated in the surface

$$C(t): \quad U^{\Gamma} = U^{\Gamma}(l), \tag{7}$$

where l indicates the length of the arc of the curve $C(t)$. The motion of the curve $C(t)$ referring to the system of coordinates x^i is given by

$$x^i = x^i(l, t). \tag{8}$$

Assume that at each point of the curve the unit tangent vector is defined by

$$\lambda^i = \frac{\partial x^i}{\partial l}. \tag{9}$$

The velocity V^i of a material particle thet belongs to the interline $C(t)$ and the absolute nonmaterial (with regard to the surrounding material) velocity B^i are related by

$$V^i - B^i =^i V^i, \quad (V^i - B^i)\mu^i = 0, \tag{10}$$

where μ^i is a unit vector perpendicular on the curve $C(t)$ lying in a tangential plane of the surface $S(t)$, its direction points into the region $S^+(t)$. For the subsequent derivations, the following theorems will be used

(a) *Transport theorem for the surface containing the interline* [7]

$$\left(\int_{S^+ \cup S^-} \widehat{\Phi}\, da\right)^{\cdot} = \int_{S^+ \cup S^-} \left\{\dot{\widehat{\Phi}} + \widehat{\Phi}(\widehat{V}^{\Delta}_{;\Delta} - 2K_M \underset{\nu}{\widehat{u}})\right\} da + \int_C [\widehat{\Phi}(\widehat{V}^{\Delta} - B^{\Delta})]\, \mu_{\Delta}\, dl, \tag{11}$$

where $[\varphi] = \varphi^+ - \varphi^-$ *indicates the jump of the quantity* φ *across the line* $C(t)$ *at* x_0, $K_M \equiv \frac{1}{2} b^{\Delta}_{\Delta}$ *is the mean curvature of the surface* $S(t)$, $b_{\Gamma\Delta}$ *is the second fundamental form of the surface* $S(t)$, *"." denotes derivative with respect to* U^{Γ} *holding t fixed, and where ";" denotes the covariant differentiation.*

(b) *Transport theorem for a line.*

It can be shown from the consideration given in [4] *that it follows*

$$\left(\int_C \Phi\, dl\right)^{\cdot} = \int_C \{\dot{\Phi} + \Phi(\widehat{V}_{\Gamma;\Delta} - \underset{\nu}{\widehat{u}} b_{\Gamma\Delta}) p^{\Gamma} p^{\Delta}\}\, dl, \tag{12}$$

where

$$p^{\Gamma} = \frac{dU^{\Gamma}}{dt} \tag{13}$$

are coefficient of directions. Equation (12) is the transport theorem for material lines.

(c) *Divergence theorem in the case of the surface containing the interline* [7]

$$\int_{S^+\cup S^-} \widehat{\Phi}^{\Delta}_{;\Delta}\, da = \int_{\partial S^+\cup\partial S^-} \widehat{\Phi}^{\Delta} n_{\Delta}\, dl - \int_C [\widehat{\Phi}^{\Delta}]\mu_{\Delta}\, dl. \tag{14}$$

3. The balance of mass

It has been supposed that the body under consideration is a two dimensional mixture containing a single material interline. The mass of such body is constant in time which can be expressed in the following form

$$\dot{\widehat{M}} + \dot{M} = 0, \tag{15}$$

with

$$\widehat{M} = \sum_{\alpha} \int_{S^+\cup S^-} \widehat{\rho}_{(\alpha)}\, da, \quad M = \int_C \rho\, dl, \tag{16}$$

where $\widehat{\rho}_{(\alpha)}$ is teh density of the α-th constituent of the surrounding material and where ρ is the density of the interline. Taking into account (11) and (12), from (15) and (16) the following equations are obtained

$$\dot{\widehat{\rho}}_{(\alpha)} + \widehat{\rho}_{(\alpha)}(\widehat{V}^{\Delta}_{(\alpha);\Delta} - 2K_M \underset{\nu}{\widehat{u}}) = 0, \quad \text{on } S^+ \cup S^-, \tag{17}$$

$$\dot{\rho} + \rho(V_{\Gamma;\Delta} - \underset{\nu}{u} b_{\Gamma\Delta})p^{\Gamma} p^{\Delta} + \left[\sum_{\alpha} \widehat{\rho}_{(\alpha)}(\widehat{V}^{\Delta}_{(\alpha)} - B^{\Delta}\right]\mu_{\Delta} = 0, \quad \text{on } C, \tag{18}$$

which represent the balance laws of the mass of the α-th constituent of the surrounding material and the interline, respectively.

4. The principle of virtual work and Balance of momentum

The principle of virtual work can be written in the following compact form [7], [12]

$$\delta A^I = \delta A^*. \tag{19}$$

We define the total inertial quantity of the body under consideration in the form

$$I^i = \left(\sum_{\alpha} \int_{S^+\cup S^-} \widehat{\rho}_{(\alpha)} \widehat{V}^i_{(\alpha)}\, da\right)^{\cdot} + \left(\int_C \rho V^i\, dl\right)^{\cdot}. \tag{20}$$

Using (11)-(12) and (17)-(18) in (20) we get

$$I^i = \sum_{\alpha} \int_{S^+\cup S^-} \widehat{\rho}_{(\alpha)} \dot{\widehat{V}}^i{}_{(\alpha)}\, da$$
$$+ \int_C \left\{ \rho \dot{V}^i + \left[\sum_{\alpha} \widehat{\rho}_{(\alpha)}(\widehat{V}^i_{(\alpha)} - V^i)(\widehat{V}^{\Delta}_{(\alpha)} - B^{\Delta})\right]\mu_{\Delta} \right\} dl. \tag{21}$$

The virtual work of inertial forces is

$$\delta A^I = \sum_\alpha \int_{S^+ \cup S^-} \widehat{\rho}_{(\alpha)} \dot{\widehat{V}}^i_{(\alpha)} \delta \widehat{x}^i_{(\alpha)}\, da + \int_C \rho \dot{V}^i \delta x^i\, dl$$
$$+ \int_C \left[\sum_\alpha \widehat{\rho}_{(\alpha)} (\widehat{V}^i_{(\alpha)} - V^i)(\widehat{V}^\Delta_{(\alpha)} - B^\Delta \right] \delta x^i \mu_\Delta\, dl. \tag{22}$$

The virtual work of all other forces can be written in the form

$$\delta A^* = \delta A^*_1(S^+ \cup C \cup S^-) + \delta A^*_2(\delta S^+ \cup \delta C \cup \delta S^-) + \delta A^*_3(\delta_1 C), \tag{23}$$

with

$$\delta A^*_1 = -\sum_\alpha \int_{S^+ \cup S^-} \{ \widehat{S}^{i\Delta}_{(\alpha)} \delta \widehat{x}^i_{(\alpha);\Delta}\, da - (\widehat{\rho}_{(\alpha)} \widehat{f}^i_{(\alpha)} b^i_{(\alpha)}) \delta \widehat{x}^i_{(\alpha)} \}\, da$$
$$- \int_C S^i \delta \left(\frac{\partial x^i}{\partial l} \right) dl + \int_C \rho f^i \delta x^i\, dl, \tag{24}$$

$$\delta A^*_2 = \sum_\alpha \int_{\partial S^+ \cup \partial S^-} \widehat{T}^i_{(\alpha)} \delta \widehat{x}^i_{(\alpha)}\, dl - \int_C \left[\sum_\alpha \widehat{\mathcal{T}}^i_{(\alpha)} (\delta \widehat{x}^i_{(\alpha)} - \delta x^i) \right] dl, \tag{25}$$

$$\delta A^*_3 = \int_C \frac{\partial (S^i \delta x^i)}{\partial l}\, dl, \tag{26}$$

where are: $\widehat{S}^{i\Delta}_{(\alpha)}$ – the surface stree, $\widehat{f}^i_{(\alpha)}$ – the specific surface force, $\widehat{b}^i_{(\alpha)}$ – the momentum supply, $\widehat{T}^i_{(\alpha)}$ – the surface traction and $\widehat{\mathcal{T}}^i_{(\alpha)}$ – the "internal' traction on the -th constituent of the surromunding material respectively. The quantities S^i and f^i are the corresponding fields definied on the interline. Then from (19), (22)-(26), after using (14) the following relation can be obtained

$$\sum_\alpha \int_{S^+ \cup S^-} \widehat{\rho}_{(\alpha)} \dot{\widehat{V}}^i_{(\alpha)} \delta \widehat{x}^i_{(\alpha)}\, da + \int_C \rho \dot{V}^i \delta x^i\, dl +$$
$$+ \int_C \left[\sum_\alpha \widehat{\rho}_{(\alpha)} (\widehat{V}^i_{(\alpha)} - V^i)(\widehat{V}^\Delta_{(\alpha)} - B^\Delta) \right] \delta x^i \mu_\Delta\, dl$$
$$= -\sum_\alpha \int_{S^+ \cup S^-} \widehat{S}^{i\Delta}_{(\alpha)} \delta \widehat{x}^i_{(\alpha)} n_\Delta\, dl + \sum_\alpha \int_C [\widehat{S}^{i\Delta}_{(\alpha)} \delta \widehat{x}^i_{(\alpha)}] \mu_\Delta\, dl \tag{27}$$
$$+ \sum_\alpha \int_{S^+ \cup S^-} \widehat{S}^{i\Delta}_{(\alpha);\Delta}\, da + \sum_\alpha \int_{S^+ \cup S^-} (\widehat{\rho}_{(\alpha)} \widehat{f}^i_{(\alpha)} + \widehat{b}^i_{(\alpha)}) \delta \widehat{x}^i_{(\alpha)}\, da + \int_C \rho f^i \delta x^i\, dl$$
$$+ \sum_\alpha \int_{\partial S^+ \cup \partial S^-} \widehat{T}^i_{(\alpha)} \delta \widehat{x}^i_{(\alpha)}\, dl - \int_C \left[\sum_\alpha \widehat{\mathcal{T}}^i_{(\alpha)} (\delta \widehat{x}^i_{(\alpha)} - \delta x^i) \right] dl + \int_C \frac{\partial S^i}{\partial l} \delta x^i\, dl.$$

From (27) we obtain the following local field equations

$$\widehat{\rho}_{(\alpha)}\dot{\widehat{V}}^i_{(\alpha)} - \widehat{S}^{i\Delta}_{(\alpha);\Delta} = \widehat{\rho}_{(\alpha)}\widehat{f}^i_{(\alpha)} + \widehat{b}^i_{(\alpha)}, \quad \text{on } S^+ \cup S^-, \tag{28}$$

$$\rho\dot{V}^i - \frac{\partial S^i}{\partial l} + \Big[\sum_\alpha \widehat{\rho}_{(\alpha)}(\widehat{V}^i_{(\alpha)} - V^i)(\widehat{V}^\Delta_{(\alpha)} - V^\Delta)\Big]\mu_\Delta = \rho f^i + [\sum_\alpha \widehat{T}^i_{(\alpha)}, \quad \text{on } C, \tag{29}$$

$$\widehat{T}^i_{(\alpha)} = \widehat{S}^{i\Delta}_{(\alpha)} n_\Delta, \quad \text{on } \partial S^+ \cup \partial S^-, \tag{30}$$

$$^{\pm}\widehat{S}^{i\Delta}_{(\alpha)}\mu_\Delta - ^{\pm}\widehat{T}^i = 0, \quad \text{on } C^\pm. \tag{31}$$

Taking into account (31) in (29), it follows

$$\rho\dot{V}^i - \frac{\partial S^i}{\partial l} + \Big[\sum_\alpha \widehat{\rho}_{(\alpha)}(\widehat{V}^i_{(\alpha)} - V^i)(\widehat{V}^\Delta_{(\alpha)} - B^\Delta) - \sum_\alpha \widehat{S}^{i\Delta}_{(\alpha)}\Big]\mu_\Delta = \rho f^i, \quad \text{on } C. \tag{32}$$

The relations (28) and (32) represents the balance laws of momentum of the α-th constituent of the suurounding material and interline, respectively.

5. The balance of moment of momentum

In this paper qe are considering the case of non-polar materials. The balance of moment of momentum can be written in the following form

$$\dot{L}_i = \mathcal{M}^*_i, \tag{33}$$

where

$$L_i = \sum_\alpha \int_{S^+\cup S^-} \varepsilon_{ijk}\widehat{\rho}_{(\alpha)} x^j \widehat{V}^k_{(\alpha)}\, da + \int_C \varepsilon_{ijk}\rho x^j V^k\, dl, \tag{34}$$

$$\mathcal{M}^*_i = \mathcal{M}^1_i(S^+ \cup C \cup S^-) + \mathcal{M}^2_i(\partial S^+ \cup \partial C \cup \partial S^-) + \mathcal{M}^3_i(\partial_1 C). \tag{35}$$

The corresponding quantities in (35) are given by

$$\mathcal{M}^1_i = \sum_\alpha \int_{S^+\cup S^-} (\varepsilon_{ijk}\widehat{\rho}_{(\alpha)} x^j f^k_{(\alpha)} + \varepsilon_{ijk} x^j \widehat{b}^k_{(\alpha)})\, da + \int_C \varepsilon_{ijk}\rho x^j f^k\, dl, \tag{36}$$

$$\mathcal{M}^2_i = \sum_\alpha \int_{\partial S^+\cup S^-} \varepsilon_{ijk} x^j \widehat{S}^{k\Delta}_{(\alpha)} n_\Delta\, dl, \tag{37}$$

$$\mathcal{M}^3_i = \varepsilon_{ijk} x^j S^k |^B_A. \tag{38}$$

Then, from (33)-(38), after using (11)-(12), (14) and (17)-(18) the following can be obtained

$$\sum_\alpha \int_{S^+\cup S^-} \varepsilon_{ijk}\widehat{\rho}_{(\alpha)}x^j \dot{\widehat{V}}{}^k_{(\alpha)}\, da + \int_C \varepsilon_{ijk}\rho x^j \dot{V}^k\, dl$$

$$\sum_\alpha \int_C [\varepsilon_{ijk}\rho_{(\alpha)}x^j (V^k_{(\alpha)} - V^k)(\widehat{V}^\Delta_{(\alpha)} - B^\Delta)]\mu_\Delta\, dl$$

$$= \sum_\alpha \int_{S^+\cup S^-} (\varepsilon_{ijk}\widehat{\rho}_{(\alpha)}x^j \widehat{f}^k_{(\alpha)} + \varepsilon_{ijk}x^j \widehat{b}^k_{(\alpha)})\, da + \int_C \varepsilon_{ijk}\rho x^j f^k\, dl \tag{39}$$

$$+ \sum_\alpha \int_{S^+\cup S^-} (\varepsilon_{ijk}x^j_{;\Delta}\widehat{S}^{k\Delta}_{(\alpha)} + \varepsilon_{ijk}x^j \widehat{S}^{k\Delta}_{(\alpha);\Delta})\, da$$

$$+ \sum_\alpha \int_C [\varepsilon_{ijk}x^j \widehat{S}^{k\Delta}_{(\alpha)}]\mu_\Delta\, dl + \int_C \left(\varepsilon_{ijk}\lambda^j S^k + \varepsilon_{ijk}x^j \frac{\partial S^k}{\partial l}\right) dl.$$

Taking into account (28) and (32) in (39), we obtain the following set of local equations

$$\varepsilon_{ijk}x^j_{;\Delta}\widehat{S}^{k\Delta}_{(\alpha)} = 0, \quad \text{on } S^+\cup S^-, \tag{40}$$

$$\varepsilon_{ijk}\lambda^j S^k = 0, \quad \text{on } C. \tag{41}$$

The relations (40) and (41) represents the balance of moment of momentum of the α-th constituent of the surrounding material and interline, respectively. The, decomposing the surface stress according to

$$S^{k\Lambda} = S^\Lambda \nu^k + S^{\Lambda\Delta}x^k_{;\Delta}, \tag{42}$$

and by using the identies

$$\varepsilon_{ijk}x^j_{;\Lambda}\nu^k = g^{\Delta\Sigma}\varepsilon_{\Sigma\Delta}x^i_{;\Lambda}, \tag{43}$$

$$\varepsilon_{ijk}x^j_{;\Lambda}x^k_{;\Delta} = \varepsilon_{\Lambda\Delta}\nu^i, \tag{44}$$

from (40) we obtain

$$S^\alpha \quad \text{and} \quad S^{\Lambda\Delta} = S^{\Delta\Lambda}, \tag{45}$$

so that (42) is reduced to

$$S^{k\Lambda} = S^{\Lambda\Delta}x^k_{;\Delta}. \tag{46}$$

6. The balance of energy

The first principle of thermodynamics in global form is

$$(\widehat{E} + \widehat{K})^{\cdot} + (E + K)^{\cdot} = \dot{A}^O + Q, \tag{47}$$

with

$$\widehat{E} = \sum_{\alpha} \int_{S^+ \cup S^-} \widehat{\rho}_{(\alpha)} \widehat{e}_{(\alpha)} \, da, \quad E = \int_C \rho e \, dl, \tag{48}$$

$$\widehat{K} = \frac{1}{2} \sum_{\alpha} \int_{S^+ \cup S^-} \widehat{\rho}_{(\alpha)} \widehat{V}^2_{(\alpha)} \, da, \quad K = \frac{1}{2} \int_C \rho V^2 \, dl, \tag{49}$$

$$\begin{aligned} \dot{A}^O = & \sum_{\alpha} \int_{S^+ \cup S^-} \widehat{T}^i_{(\alpha)} \widehat{V}^i_{(\alpha)} \, da + \sum_{\alpha} \int_{S^+ \cup S^-} (\widehat{\rho}_{(\alpha)} \widehat{f}_{(\alpha)} + \widehat{b}^i_{(a)}) \widehat{V}^i_{(\alpha)} \, da \\ & + \int_C \frac{\partial (S^i V^i)}{\partial l} \, dl + \int_C \rho f^i V^i \, dl, \end{aligned} \tag{50}$$

$$\begin{aligned} Q = & \sum_{\alpha} \int_{S^+ \cup S^-} \widehat{\rho}_{(\alpha)} \widehat{r}_{(\alpha)} \, da - \sum_{\alpha} \int_{S^+ \cup S^-} \widehat{q}^i_{(\alpha)} n^i \, dl \\ & + \int_C \rho r \, dl + \int_C \frac{\partial q}{\partial l} \, dl, \end{aligned} \tag{51}$$

where are: $\widehat{e}_{(\alpha)}$ – the specific internal energy, $\widehat{r}_{(\alpha)}$ – the specific external heat supply and $\widehat{q}_{(\alpha)}$ – the beat flux vector of the α-th constituent of the surrounding material, and where e, r and q are corresponding fields defined on the interline. The expresions for $\dot{\widehat{K}} + \dot{K}$ can be conected with (22) written for the real velocity fields

$$\dot{\widehat{K}} + \dot{K} = \dot{A}^I + \frac{1}{2} \int_C \Big[\sum_{\alpha} \widehat{\rho}_{(\alpha)} (\widehat{V}^i_{(\alpha)} - V^i)(\widehat{V}^{\Delta} - B^{\Delta}) \Big] \mu_{\Delta} \, dl. \tag{52}$$

Accounting for (52) in the first principle of thermodynamics (47) we obtain the so-called energy theorem in the following global form

$$\dot{\widehat{E}} + \dot{E} + \dot{A}^I - \dot{A}^O + \dot{K}_R = Q, \tag{53}$$

with

$$\dot{K}_R \equiv \frac{1}{2} \int_C \Big[\sum_{\alpha} \widehat{\rho}_{(\alpha)} (\widehat{V}^i_{(\alpha)} - V^i)^2 (\widehat{V}^{\Delta}_{(\alpha)} - B^{\Delta}) \Big] \mu_{\Delta} \, dl. \tag{54}$$

For subsequent derivations, the principle of virtual work (19), written for real velocity fields, will be used. It follows

$$\begin{aligned} \dot{A}^I - \dot{A}^O = & - \sum_{\alpha} \int_{S^+ \cup S^-} \widehat{S}^{i\Delta}_{(\alpha)} \widehat{V}^i_{(\alpha);\Delta} \, da \\ & - \int_C \Big[\sum_{\alpha} \widehat{T}^i_{(\alpha)} (V^i_{(\alpha)} - V^i) \Big] dl - \int_C S^i \frac{\partial V^i}{\partial l} \, dl \end{aligned} \tag{55}$$

Finaly, by using (14) in (51), we get

$$\begin{aligned} Q = & \sum_{\alpha} \int_{S^+ \cup S^-} \widehat{\rho}_{(\alpha)} \widehat{r}_{(\alpha)} \, da - \sum_{\alpha} \int_{S^+ \cup S^-} \widehat{q}^{\Delta}_{(\alpha);\Delta} \, da \\ & - \int_C \Big[\sum_{\alpha} \widehat{q}^{\Delta}_{(\alpha)} \Big] \mu_{\Delta} \, dl + \int_C \rho r \, dl - \int_C \frac{\partial q}{\partial l} \, dl. \end{aligned} \tag{56}$$

Tacing into account (54)-(56) in (53), it follows

$$\begin{aligned}
&\sum_\alpha \int_{S^+\cup S^-} \widehat{\rho}_{(\alpha)}\dot{\widehat{e}}_{(\alpha)}\, da + \int_C \rho\dot{e}\, dl + \int_C \Big[\sum_\alpha \widehat{\rho}_{(\alpha)}(\widehat{e}_{(\alpha)} - e)(\widehat{V}^\Delta_{(\alpha)} - B^\Delta)\Big]\mu_\Delta\, dl \\
&- \sum_\alpha \int_{S^+\cup S^-} \widehat{S}^{i\Delta}_{(\alpha)}\widehat{V}^i_{(\alpha);\Delta}\, da - \int_C \Big[\sum_\alpha T^i_{(\alpha)}(\widehat{V}^i_{(\alpha)} - V^i)\Big]\, dl - \int_C S^i\frac{\partial V_i}{\partial l}\, dl \\
&+ \frac{1}{2}\int_C \Big[\sum_\alpha \widehat{\rho}_{(\alpha)}(\widehat{V}^i_{(\alpha)} - V^i)^2(\widehat{V}^\Delta_{(\alpha)} - B^\Delta)\Big]\mu_\Delta\, dl \qquad (57)\\
&= \sum_\alpha \int_{S^+\cup S^-} \widehat{\rho}_{(\alpha)}\widehat{r}_{(\alpha)}\, da - \sum_\alpha \int_{S^+\cup S^-} \partial q^\Delta_{(\alpha);\Delta}\, da - \int_C \Big[\sum_\alpha \widehat{q}^\Delta_{(\alpha)}\Big]\mu_D\, dl \\
&+ \int_C \rho r\, dl - \int_C \frac{\partial q}{\partial l}\, dl.
\end{aligned}$$

From (57) we obtain the following local field equations

$$\widehat{\rho}_{(\alpha)}\dot{\widehat{e}}_{(\alpha)} = \widehat{S}^{i\Delta}_{(\alpha)}\widehat{V}^i_{(\alpha);\Delta} + \widehat{\rho}_{(\alpha)}\widehat{r}_{(\alpha)} - \widehat{q}^\Delta_{(\alpha);\Delta}, \quad \text{on } S^+\cup S^-,$$

$$\begin{aligned}
&\rho\dot{e} - S^i\frac{\partial V^i}{\partial l} + \frac{\partial q}{\partial l} + \Big[\sum_\alpha \widehat{\rho}_{(\alpha)}\Big\{\widehat{e}_{(\alpha)} - e + \frac{1}{2}(V^i_{(\alpha)} - V^i)^2\Big\}(\widehat{V}^\Delta_{(\alpha)} - B^\Delta) \\
&- \sum_\alpha \widehat{S}^{i\Delta}_{(\alpha)}(\widehat{V}^i_{(\alpha)} - V^i) + \sum_\alpha \widehat{q}^\Delta_{(\alpha)}\Big]\mu_\Delta = qr, \quad \text{on } C. \qquad (59)
\end{aligned}$$

The relation (58) and (59) represents the balance laws of the internal energy of the α-th constituent of the surronding material and interline, respectively.

7. The balance of entropy

The balance of entropy of the body under consideration is given by

$$\dot{\widehat{N}} + \dot{N} = \mathcal{N}, \qquad (60)$$

with

$$\widehat{N} = \sum_\alpha \int_{S^+\cup S^-} \widehat{\rho}_{(\alpha)}\widehat{\eta}_{(\alpha)}\, da, \qquad N = \int_C \rho\eta\, dl, \qquad (61)$$

$$\mathcal{N} = \sum_\alpha \int_{S^+\cup S^-} \widehat{\rho}_{(\alpha)}\widehat{\kappa}_{(\alpha)}\, da - \sum_\alpha \int_{\partial S^+\cup\partial S^-} \widehat{\varphi}^\Delta_{(\alpha)} n_\Delta\, dl + \int_C \rho\kappa\, da - \int_C \frac{\partial\varphi}{\partial l}\, dl, \qquad (62)$$

where: $\widehat{\eta}_{(\alpha)}$ – the specific entropy, $\widehat{\kappa}_{(\alpha)}$ – the specific supply of entropy and $\widehat{\varphi}^\Delta_{(\alpha)}$ – the entropy flux of α-th constituent of the surrounding material, while η, κ and φ

are corresponding quantities defined on the interline. Tacing into account (11)-(12), (14) and (17)-(18), from (60)-(62) the following local equations can be deduced

$$\sum_{\alpha} \widehat{\rho}_{(\alpha)} \dot{\widehat{\eta}}_{(\alpha)} + \sum_{\alpha} \widehat{\varphi}^{\Delta}_{(\alpha);\Delta} = \sum_{\alpha} \widehat{\rho}_{(\alpha)} \widehat{\kappa}_{(\alpha)}, \quad \text{on } S^+ \cup S^-, \tag{63}$$

$$\rho \dot{\eta} + \frac{\partial \varphi}{\partial l} + \left[\sum_{\alpha} \widehat{\rho}_{(\alpha)} (\widehat{\eta}_{(\alpha)} - \eta)(\widehat{V}^{\Delta}_{(\alpha)} - B^{\Delta}) + \sum_{\alpha} \widehat{\varphi}^{\Delta}_{(\alpha)} \right] \mu_{\Delta} = \rho \kappa, \quad \text{on } C. \tag{64}$$

The above relations represents the balance of the entropy of the surrrounding material and the interline, respectively.

8. Discusion

In the present paper it has been demonstrated that a certain kind of analogy can be established between it and the analysis of the three-dimensional continuum containing discontinuity surface ([7],[12]), as well as between the analysis of a one-dimensional continuum containing a point of singularity [13].

The relation (17), (28), (40), (58) and (63) represent the known balance of the α-th constituent of the two-dimensional surrounding mixture, while the relations (18), (32), (41), (59) and (64) represent the new balance of the material interline separeting this two-dimensional mixture into two parts.

According to the procedure described above, the systems with the nonmaterial interline can also be considered. As it can be seen from the introduction, the problems concerning the nonmaterial interline are numerous and significant. One of them will be analyzed in the following example.

9. Example: The detachment of the membrane from the surface of a given body

We are considering an elastic membrane $\mathcal{M}$ found upon the surface of the body $\mathcal{D}$ (the membrane can be defined as a thin layer which generally separates two materials) (Fig. 4).

Due to the certain effect, the membrane detachts from the surface of a given body along the detachment line $C(t)$ (nonmaterial line of discontinuity). The line $C(t)$ devides the body $S(t)$ into two parts: $S^+(t)$ which is not detached and $S^-(t)$ which is detached from the surface of a given body.

When the membrane is smooth, the transport theorem in the form (11) is used. When the above is not case (Fig. 4), the transport theorem for a nonsmoth surface containing the discontinuity line gets the form

$$\left(\int_{S^+ \cup S^-} \widehat{\Phi}\, da \right)^{\cdot} = \int_{S^+ \cup S^-} \{ \dot{\widehat{\Phi}} + \widehat{\Phi}(\widehat{V}^{\Delta}_{;\Delta} - 2K_M \underset{\nu}{\widehat{u}}) \}\, da + \int_C \langle \widehat{\Phi} \widehat{V}^{\Delta} \mu_{\Delta} \rangle\, dl - \int_C \langle \widehat{\Phi} \mu_{\Delta} \rangle B^{\Delta}\, dl, \tag{65}$$

where $\langle F\rangle = F^+ - F^-$. The divergence theorem in that case reads

$$\int_{S^+\cup S^-} \widehat{\Phi}^{\Delta}_{;\Delta}\, da = \int_{\partial S^+\cup\partial S^-} \widehat{\Phi}^{\Delta} n_\Delta\, dl + \int_C \langle \widehat{\Phi}^{\Delta}\mu_\Delta\rangle\, dl. \tag{66}$$

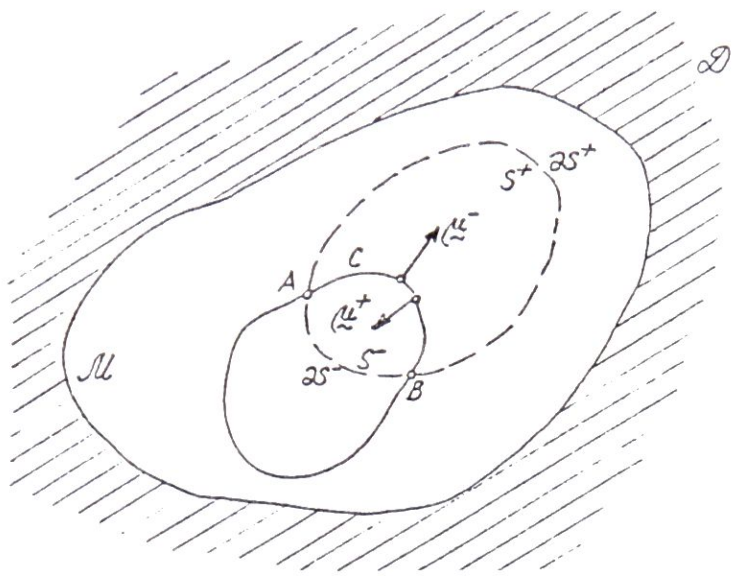

Fig. 4

8.1. The condition of detachment of a membrane expressed over the momentum

The inertion force in this case reads

$$I^i = \left(\sum_\alpha \int_{S^+\cup S^-} \widehat{\rho}_{(\alpha)}\widehat{V}^i_{(\alpha)}\, da\right)^{\cdot}. \tag{67}$$

By using the transport theorem (65) in (67) we obtain

$$I^i = \sum_\alpha \int_{S^+\cup S^-} \widehat{\rho}_{(\alpha)}\dot{\widehat{V}}^i{}_{(\alpha)}\, da + \int_C \left[\sum_\alpha \widehat{\rho}_{(\alpha)}\widehat{V}^i_{(\alpha)}(\widehat{V}^\Delta_{(\alpha)} - B^\Delta)\right]\mu_\Delta\, dl. \tag{68}$$

The virtual work of the inertion force has the form

$$\delta A^I = \sum_\alpha \int_{S^+\cup S^-} \widehat{\rho}_{(\alpha)}\dot{\widehat{V}}^i\delta\widehat{x}^i_{(\alpha)}\, da + \int_C \left[\sum_\alpha \widehat{\rho}_{(\alpha)}\widehat{V}^i_{(\alpha)}(\widehat{V}^\Delta_{(\alpha)} - B^\Delta)\right]\delta x^i\mu_\Delta\, dl. \tag{69}$$

The virtual work of all other forces can be expressed in the form

$$\delta A^1_i = -\sum_a \int_{S^+\cup S^-} \{\widehat{S}^{i\Delta}_{(\alpha)}\delta\widehat{x}^i_{(\alpha);\Delta} - (\widehat{\rho}_{(\alpha)}\widehat{f}^i_{(\alpha)} + \widehat{b}^i_{(\alpha)})\delta\widehat{x}^i_{(\alpha)}\}\, da - \int_C F^i\delta x^i\, dl, \tag{70}$$

$$\delta A^2_i = \sum_\alpha \int_{S^+\cup S^-} \widehat{T}^i\delta\widehat{x}^i_{(\alpha)}\, dl - \int_C \left[\sum_\alpha \widehat{T}^i(\delta\widehat{x}^i_{(\alpha)} - \delta x^i)\right] dl, \tag{71}$$

$$\delta A^3_i = \int_C \frac{\partial(S^i\delta x^i)}{\partial l}\, dl, \tag{72}$$

where F^i is the concentrated cohesive force at the moving boundary $C(t)$. By using the divergence theorem (14) in (71), from (22)-(23), (70) and (72) we obtain

$$\left[\sum_\alpha \widehat{\rho}_{(\alpha)} \widehat{V}^i_{(\alpha)} (\widehat{V}^\Delta_{(\alpha)} - B^\Delta) - \sum_\alpha \widehat{S}^{i\Delta}_{(\alpha)}\right] \mu_\Delta = F^i + \frac{\partial S^i}{\partial l}, \tag{73}$$

which represent the condition of detachment of the smooth membrane from the surface of a given body expressed over the momentum. In analogous way, by using the transport theorem (65), and divergence theorem (66), we obtain

$$\langle (\sum_\alpha \widehat{\rho}_{(\alpha)} \widehat{V}^i_{(\alpha)} \widehat{V}^\Delta_{(\alpha)} - \widehat{S}^{i\Delta}_{(\alpha)}) \mu_\Delta \rangle - \langle \sum_\alpha \widehat{\rho}_{(\alpha)} \widehat{V}^i_{(\alpha)} \mu_\Delta \rangle B^\Delta = F^i + \frac{\partial S^i}{\partial l}, \tag{74}$$

which represents the condition of detachment of the nonsmooth membrane from the surface of a given body expressed over the momentum.

The form of the condition of detachment depends upon the kind of effects causing the detachment of the membrane from the surface of a given body, which has, as a consequence, the expressions of the forms (73)-(74). A further utilization of these conditions depends upon the question what should be solved in a concrete problem. If it is the velocity of the detachment line points, then, in order to complete the system of equations necessary for the solution of the problem, it is also necessary to have the differential equation of motion of the membrane as well as the corresponding boundary and initial conditions.

In addition to the previous detachment conditions, as an ilustration of the exposed theory, in an analogous manner, the detachment conditions can be derived in terms of the power absorbed at the moving boundary by using that part of the theory which referes to the energy balance. The detachment condition (73) can be compared with the condition of detachment of adhesive tape from a plane surface ([14], (A.3))

$$-\sigma_t[\rho \underset{\sim}{x}_t(\sigma, t)] = [\underset{\sim}{\Phi}(\sigma, t)] - \underset{\sim}{\mathbf{F}}, \tag{75}$$

where $\rho(s,t)$ is the linear density, $\underset{\sim}{x}(s,t)$ is the position at time t of the particle at distance S along the one-dimensional continuum in its reference configuration, $\underset{\sim}{\Phi}$ is the force exerted by the part of the continuum with $S' > S$ upon the part with $S' < S$ and $\underset{\sim}{\mathbf{F}}$ is the concentrated cohesive force at the moving boundary $s = \sigma(t)$.

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REFERENCES

[1] Laws, N., *A Simple Dipolar Curve*, Int. J. Engng. Sci., 5 (1967), 653-661.

[2] Davis, S. H., *Contact Line Problems in Fluid Mechanics*, Trans. ASME, Ser. E. J. Appl. Mech. 50 (1983), 977-982.

[3] Suhubi, E. S., *Balance Laws of Continuum Physics*, Lett. Appl. Engng. Sci., 21 (1983), 283-288.

[4] Jarić, J., Golubović, Z., *The Balance of the Interline and the Bulk Material*, ZAMM, 71 (1991), 518-522.

[5] Golubović, Z., *On the Basic Equations of the Thermodynamics of an Interline*, in press.

[6] Müller, I., *Thermodynamics*, Pitman Advan. Publ. Program, Boston, London, Melburn, (1985).

[7] Daher, N., Maugin, G. A., *The Method of Virtual Power in Continuum Mechanics: Application to the Media Presenting Singular Surfaces and Interfaces*, Acta Mech. 60 (1986), 217-240.

[8] Daher, N., Maugin, G. A., *Deformabile Semiconductors with Interfaces: Basic Continuum Equations*, Int. J. Engng. Sci., 25 (1987), 1093-1129.

[9] Müller, I., *A systematic Derivations of the Constitutive Properties of Mixtures of Viscous-Heat Conducting Fluids*, Lectures in Appl. Math., 24 (1986) 357-374.

[10] Müller, I., *Thermodynamics of Diffusive and Reacting Mixtures of Fluids*, Physica 2OD (1986), 35-66.

[11] Moeckel, G. P., *Thermodynamics of an Interface*, Archive for Rational Mechanics and Analysis, 57 (1975), 255-280.

[12] Golubović, Z., Cvetković, P., *On the Principle of Virtual Work in the Theory of Immiscible Mixtures Containing Interface*, Theoretical and Applied Mechanics, 15, Beograd (1989), 31-43.

[13] Bulatović, R., Golubović, Z., *On the Condition of Detachment in Unidimensional Problems of the Continuum Mechanics with a Free Boundary*, in press.

[14] Burridge, R., Keller, B. J., *Peeling, Slipping and Cracking - Some One-dimensional Problems in Mechanics*, SIAM Review, Vol. 20, No. 1 (1978) 31-61.

ÜBER THERMODYNAMIK DER ZWISCHENLINIE

In der Arbeit wird mehrkomponentale Oberfläche der belibigen Form betrachtet, die eine Zwischenlinie enthält. Es wurde gezeigt, daβ eine Gewisse Analogie mit der Analyse der dreidimensionalen Kontinuum hergestellt wird, die eine Zwischenfläche enthält, nicht nur bei der Formulirung des Prinzip's der wirtualen Arbeit, sondern auch bei der Ausführung der sgn. Energietheoreme. Als Ilustration der ausgelegter Theorie wird das Abteilen der Membrane von der Fläche der gegebener Körper betrachtet.

O TERMODINAMICI MEĐULINIJE

U radu je razmatrana međupovrš proizvoljnog oblika koja sadrži međuliniju. Pokazano je da se može uspostaviti izvesna analogija sa analizom trodimenzionalnog kontinuuma koji sadrži međupovrš, kako pri formulisanju principa virtualnog rada, tako i pri izvođenju tzv. teoreme energije. Kao ilustracija izložene teorije, razmatran je problem odvajanja membrane od površine datog tela.

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