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FORMATION AND DECAY OF ACCELERATION WAVES IN STEADY HYPERSONIC FLOWS WITH RELAXATION EFFECTS

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The effect of non-linearity on the wave propagation has been a subject of great interest from mathematical and physical point of view. Considerable attention has been given to the development of non-equilibrium gas flow theories which have applications in the field of entry physics, combustion and reaction. The non-equilibrium effects can occur in any of the molecular processes like translation, rotation, vibration, chemical transformation etc. The law which governs the evolution of the amplitude of a weak discontinuity in an one dimensional medium obeys, as known, an ordinary differential equation of Bernoulli's type (Boillat [1], Varley and Cumberbatch [2], Boillat and Ruggeri [3], Becker [4], Bowen [5], Ram [6]). A different procedure given by Jeffrey and Taniutti [7], in one dimensional case, has led to an evolutionary law which has undergone a troubled history in the course of the last two decades. Relaxation effects on the evolution of non-linear waves play a decisive role in the stability of waves. Rarity [8] used Jeffrey's technique [7] to study the problem of breakdown of characteristic solutions in flows with vibrational relaxation. The analysis of the corresponding problem in two-dimensional steady supersonic flow is given. The solution in the neighbourhood of the first disturbed forward characteristic is discussed. In this study it is not clear that when and how a characteristic solution breaksdown. He has also overlooked the effects of relaxation on the global behaviour of the wave amplitude. Here an attempt is made to study the evolution of an acceleration wave propagating along characteristic lines in a steady two-dimensional hypersonic flows of a general class of relaxing fluids. Also, investigations are made to study the critical stages when a week wave breaksdown and a shock wave will be formed under relaxation effects. The fluid is assumed to move along a plane wall with a speed greater than the frozen speed of sound and encounters a smooth compressive corner. Making use of the concept of the stream line characteristics, it is assumed that along a stream line all flow quantities are continuous and have continuous derivatives in the stream direction; across the stream line all flow quantities are continuous but the derivatives in the normal direction suffer a jump.

The hydrodynamical equations governing the two-dimensional continuous flow of relaxing fluids in a plane steady hypersonic flow are [1]

$$u_i \rho_{,i} + \rho u_{i,i} = 0, \tag{1}$$

$$\rho u_j u_{i,j} + p_{,i} = 0, \tag{2}$$

$$u_i \xi_{,i} - L(\xi, \eta, p) = 0, \tag{3}$$

$$u_i p_{,i} + \rho c_f^2 u_{i,i} + c_f^2 \left\{ \frac{A}{T} \rho_\eta + \rho_\xi \right\} L = 0, \tag{4}$$

where L is the relaxation rate function, A is the affinity of internal chemical transformations, c_f is the frozen speed of sound, ξ is the relaxation parameter and η is the entropy of the fluid. u_i, p and ρ respectively represent fluid velocity components, pressure and density. A comma followed by an index i denotes partial differentiation with respect to a spatial coordinate x_i.

If the curvilinear coordinates S and n are the measure of distances along the normal to the stream line and θ is the angle of deflection of the streamline from a suitable reference direction, the equations of motion (1)-(4) can be transformed into new intrinsic coordinates (S, n) in the following form:

$$q\rho_S + \rho q_S + \rho q \theta_n = 0, \tag{5}$$

$$\rho q q_S + p_S = 0, \tag{6}$$

$$\rho q^2 \theta_S + p_n = 0, \tag{7}$$

$$q p_S + \rho c_f^2 (q_S + q\theta_n) + c_f^2 \left(\frac{A}{T} \rho_\eta + \rho_\xi \right) L = 0, \tag{8}$$

$$q \xi_S - L = 0, \tag{9}$$

where q is the magnitude of fluid velocity and the surffixes S and n denote partial differentiation.

The above set of equations are the basic equations of motion in therms of intrinsic coordinates. To solve the problem of two-dimensional fluid flow, the equations (5)-(9) can be combined in the matrix form:

$$U_S + AU_n + B = 0, \tag{10}$$

where

$$U = [p, q, \theta, \xi, \rho]^T, \qquad B = [qQ, -Q/\rho, 0, -L/q, Q/q]^T,$$

$$A = \begin{bmatrix} 0 & 0 & \frac{c_f^2 \rho M_f^2}{M_f^2 - 1} & 0 & 0 \\ 0 & 0 & -\frac{q}{M_f^2 - 1} & 0 & 0 \\ \frac{1}{\rho q^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\rho M_f^2}{M_f^2 - 1} & 0 & 0 \end{bmatrix},$$

where

$$Q = \frac{1}{M_f^2 - 1}\left(\frac{A}{T}\rho_\eta + \rho_\xi\right)L, \qquad M_f = \frac{q}{c_f}.$$

The system (10) is quasi-linear with five real characteristics. The eigen values of the matrix A are

$$\lambda^{(1)} = \lambda^{(2)} = \lambda^{(3)} = 0, \quad \lambda^{(4)} = \frac{1}{(M_f^2 - 1)^{1/2}}, \quad \lambda^{(5)} = -(M_f^2 - 1)^{-1/2},$$

and the corresponding left eigen vectors are given by,

$$L^{(1)} = \left[\frac{1}{\rho q}, 1, 0, 0, 0\right], \quad L^{(2)} = [0,0,0,1,0], \quad L^{(3)} = \left[-\frac{1}{c_f^2}, 0, 0, 0, 1\right],$$

$$L^{(4)} = \left[1, 0, \frac{\rho q^2}{(M_f^2 - 1)^{1/2}}, 0, 0\right], \quad L^{(5)} = \left[-1, 0, \frac{\rho q^2}{(M_f^2 - 1)^{1/2}}, 0, 0\right].$$

The system (10) of partial differential equations is of hyperbolic nature and thus admits discontinuities which propagate along the forward characteristics.

Now, let us consider a new coordinate system of characteristic coordinates ϕ and ψ by the equations,

$$\phi_S + \lambda^{(4)}\phi_n = 0, \quad \psi = 0. \tag{11}$$

The equations (11) shows that ϕ is a constant along characteristic wave fronts so that we can assume $\phi = 0$ as the leading forward characteristic wave front across which all the flow parameters are themselves continuous but their first partial derivatives with respect to ϕ undergo finite jump discontinuities. Such a wave front is termed as "acceleration wave". The transformation introduced is non-singular, provided the Jacobian transformation $(S, n) \to (\phi, \psi)$ is given by

$$J = \frac{1}{n_\phi} = \frac{1}{\phi_n}, \tag{12}$$

is non-zero and finite.

Let us consider an open region R' bounded by two characteristics $\phi(S, n) = 0$ and $\psi(S, n) = 0$ such that no other characteristic issuing from the origin enters this open region R'. This open region approximation is essential since we are confined to the neighbourhood of $\phi(S, n) = 0$. We assume that U remains smooth in R' at least for a finite time throughout the region R' except for boundaries (figure 1).

Transforming (10) into new coordinates ϕ and ψ and premultiplying by $L^{(j)}$, we get

$$L^{(j)}\{(\lambda^j - \lambda^4)U_\phi + n_\phi U_\psi\} + n_\phi L^j B = 0. \tag{13}$$

In particular, for $\lambda^j = \lambda^4$, we have

$$L^4 U_\psi + L^4 B = 0. \tag{14}$$

The jump conditions at the wave front $\phi = 0$, are

$$\left.\begin{aligned} &U \text{ is continuous; } [U] = 0,\\ &U_\psi \text{ is continuous; } [U_\psi] = 0,\\ &U_\phi \text{ is continuous; } [U_\phi] \neq 0 = F(\psi),\\ &n_\phi \text{ is continuous; } [n_\phi] \neq 0 = R(\psi), \end{aligned}\right\} \tag{15}$$

where the bracket denotes the jump in the enclosed quantity across the wave front $\phi(S,n) = 0$, $[U] = (U)_{\phi=\bar{0}} - (U)_{\phi=\overset{+}{0}}$. Here $(U)_{\phi=\bar{0}}$ means evaluation of U on the wave front $\phi = 0$ from down stream side and $(U)_{\phi=\overset{+}{0}}$ means evaluation of U from the upstream side.

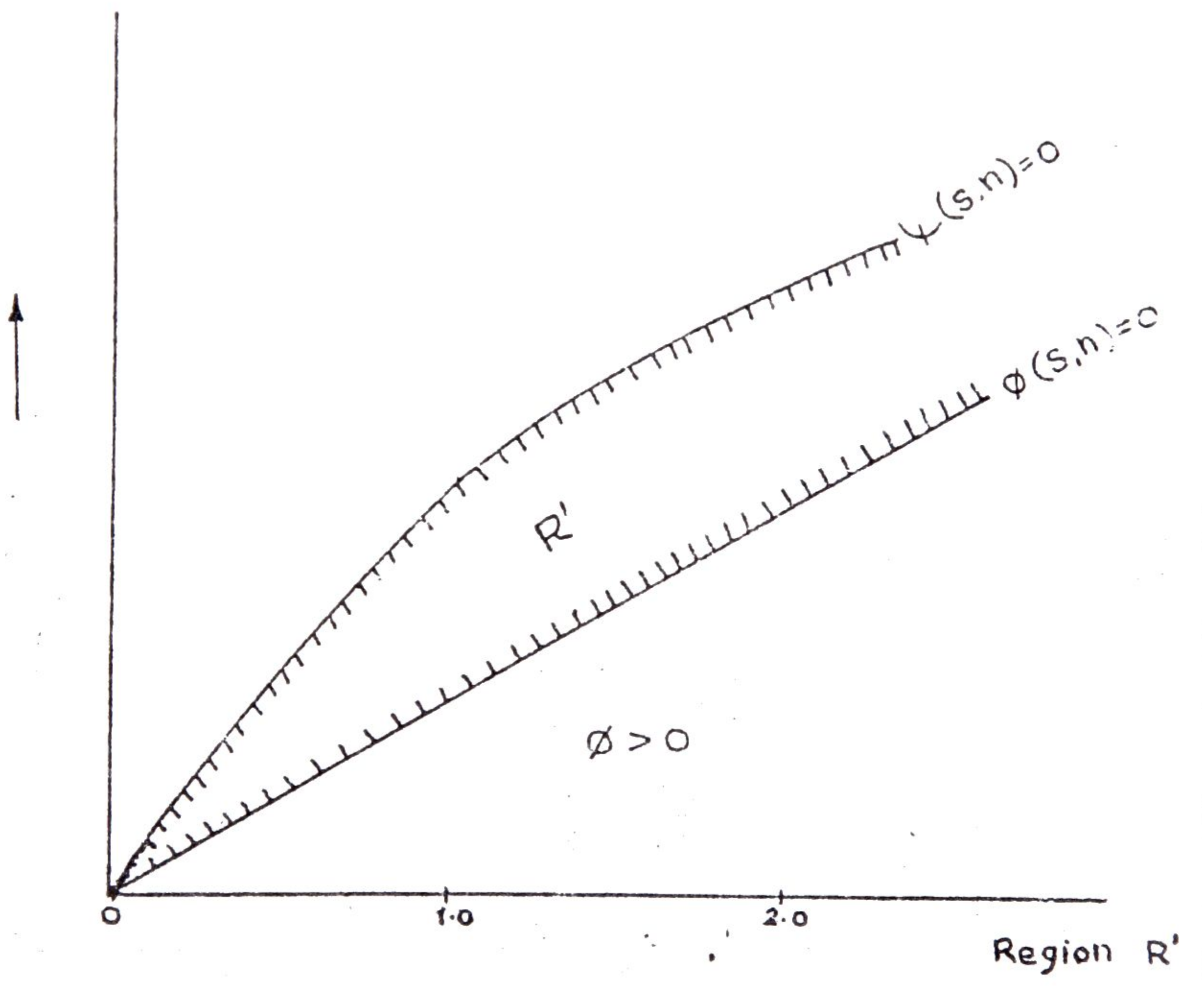

Figure-1 Region R'

From the definition of $R(\psi)$, we see at once that

$$R + (n_\phi)_{\phi=\overset{+}{0}} = (n_\phi)_{\phi=\bar{0}}, \tag{15a}$$

while

$$(n_\phi)_{\phi=\overset{+}{0}} = (n_\phi)_0, \tag{15b}$$

is finite. Hence the condition (12) implies that

$$R + (n_\phi)_0, \tag{15c}$$

is non zero and finite. The quantity $(n_\phi)_{\phi=\bar{0}}$ is associated in the distance between the two neighbouring characteristics $\phi(S,n) = 0$ and $\phi(S,n) = \delta < 0$, we have

$$U_n = U_\phi \phi_n, \tag{15d}$$

which, by virtue of the Jacobian in equation (12), may be written as

$$U_n = \frac{U_\phi}{n_\phi}. \tag{15e}$$

Hence, if n_ϕ becomes zero and U_ϕ remains finite, U ceases to be Lipschitz continuous. By means of constant state condition $B(U_0) = 0$, it follows immediately from (13) that

$$L_0^j F(\psi) = 0, \quad \text{for } \lambda^j \neq \lambda^4. \tag{16}$$

Differentiating (14) with respect to ϕ and then evaluating at the wave front $\phi = 0$, we get

$$L^4 F(\psi) + [\nabla_u(L^4 B)]_0 F = 0, \tag{17}$$

where ∇_u stands for the gradient operator with respect to the components of the vector U.

The equation of outgoing characteristics can be written as

$$n_\psi = \lambda^4. \tag{18}$$

Differentiating (18) with respect to ϕ at any point in the open region R' and allowing this to tend to a point on the wave front $\phi(S,n) = 0$ and using the jump condition, we get $R_\psi = [\nabla_u(\lambda^4)]_0 F$, which provides that

$$R = \overline{R} + \int_0^\psi [\nabla_u(\lambda^4)]_0 F \, d\psi, \tag{19}$$

where $\overline{R} = \lim\limits_{\psi \to 0} R$ and

$$[\nabla_u(\lambda^4)]_0 F = \frac{M_{f_0}}{(M_{f_0}^2 - 1)^{3/2} c_{f_0}} \times \left\{ \frac{1}{\rho_0 q_0} + \frac{q_0}{c_{f_0}^3} \left(\frac{\partial c_f}{\partial \rho} \right)_0 + \frac{q_0}{c_{f_0}^3} \left(\frac{\partial c_f}{\partial p} \right)_0 \right\} F_1.$$

From (16), we get the following relations:

$$F_1 + \rho_0 q_0 F_2 = 0, \tag{20}$$

$$F_4 = 0, \tag{21}$$

$$F_1 - c_{f_0}^2 F_5 = 0, \tag{22}$$

$$(M_{f_0}^2 - 1)^{1/2} F_1 - \rho_0 q_0^2 F_3 = 0. \tag{23}$$

Evaluating (17) on the wave front $\phi = 0$, and using (23), we get

$$qF_{1\psi} + \frac{q_0}{M_{f_0}^2 - 1}(\rho_\psi)\left(\frac{\partial L}{\partial p}\right)_0 F_1 = 0. \tag{24}$$

Integrating (24), we get

$$F_1 = F_1^* \exp(-C_1\psi), \tag{25}$$

where $c_1 = \dfrac{q_0}{2(M_{f_0}^2 - 1)}(\rho_\psi)_0\left(\dfrac{\partial L}{\partial p}\right)_0$, $F_1^* = F(0)$. It is shown in reference [9] that $(\rho_\psi)_0\left(\dfrac{\partial L}{\partial p}\right)_0 > 0$. Since $M_f \gg 1$ for hypersonic flows under study, we conclude that $C_1 > 0$. Substituting from (25) in (19), we get

$$R = \overline{R} + C_2 F_1^* \int_0^\psi e^{-C_1\psi'}\, d\psi', \tag{26}$$

where $C_2 = \dfrac{M_{f_0}}{c_{f_0}(M_{f_0}^2 - 1)^{3/2}}\left\{\dfrac{1}{\rho_0 q_0}\left(\dfrac{\partial c_f}{\partial \rho}\right)_0 + \dfrac{q_0}{c_{f_0}^3}\left(\dfrac{\partial c_f}{\partial p}\right)_0\right\}$. Now we define the wave amplitude $a(\psi)$ of the two dimensional weak wave as

$$a(\psi) = [q_n] = \frac{[q_\phi]}{(n_\phi)_{\overline{0}}} = \frac{F_2}{(n_\phi)_0 + R}. \tag{27}$$

Using (20), (25) and (26), we get

$$a(\psi) = \frac{\exp(-c_1\psi)}{\dfrac{1}{a^*} - \dfrac{c_2\rho_0 q_0}{c_1}(1 - \exp(-c_1\psi))}, \tag{28}$$

where $a^* = a(0) = \dfrac{-F_1^*}{\rho_0 q_0\{\overline{R} + (n_\phi)_{\overset{+}{0}}\}}$. The equation (28) determines the global behaviour of the wave amplitude $a(\psi)$.

Thus the following conclusions can be drawn:

THEOREM 1. *If $a^* < 0$, then $a(\psi) < 0$ for all $\psi \in [0, \infty)$ and $|a(\psi)|$ decreases monotonically to zero and the wave will be damped out ultimately.*

Proof. We can write $a(\psi)$ in the form

$$|a(\psi)| = \left\{ \left(\frac{1}{|a^*|}\right) + \frac{c_2\rho_0 q_0}{c_1} e^{-c_1\psi} - \frac{c_2\rho_0 q_0}{c_1} \right\}^{-1}.$$

Since both c_1 and c_2 are positive constants from physical considerations the r.h.s. of (28) decreases monotonically and tends to zero as $\psi \to \infty$. This shows that the wave will ultimately decay out.

THEOREM 2. *If $a^* > 0$, then there exists the critical value $a_c = \left(\frac{c_1}{c_2\rho_0 q_0}\right)$ of a^* and a critical value ψ_c of ψ such that*

(1) *if $a^* > a_c$, then $\lim_{\psi \to \psi_c} a(\psi) = \infty$, where $\psi_c = \frac{1}{c_1} \log\left(\frac{a^*}{a^* - a_c}\right)$.*

(2) *if $a^* < a_c$, then $\lim_{\psi \to \infty} a(\psi) = 0$,*

(3) *if $a^* = a_c$, then $a(\psi) = a_c$.*

Proof. (1) When $a^* > a_c$, then the denominator of $a(\psi)$ decreases from $(a^*)^{-1}$ and approaches zero within range ψ_c of ψ given by $\frac{1}{a^*} - \frac{\rho_0 q_0 c_2}{c_1}(1 - e^{-c_1\psi_c}) = 0$, which provides that $\psi_c = \frac{1}{c_1} \log\left(\frac{a^*}{a^* - a_c}\right)$ and $\lim_{\psi \to \psi_c} a(\psi) = \infty$.

This result shows two things. Firstly, there is a breakdown in the weak wave phenomenon and secondly a shock type discontinuity appears due to infinitely large gradients of flow parameters.

(2) When $a^* < a_c$, then the denominator of $a(\psi)$ will never vanish but the numerator tends to zero as $\psi \to \infty$. Thus, we have $\lim_{\psi \to \infty} a(\psi) = 0$. This shows that the wave will decay out and will be damped out.

(3) When $a^* = a_c$, then $a(\psi) = a_c$. This shows that the wave is of stable form and the wave amplitude does not undergo distortion.

THEOREM 3. *The relaxation effects either disallows or delays the process of shock formation.*

Proof. We have $\frac{\partial a_c}{\partial c_1} = \frac{1}{c_2\rho_0 q_0} > 0$. This shows that a_c increases with c_1 so that the critical amplitude a_c increases under relaxation effects. This means that the relaxation effect in gasflows has a stabilizing effect.

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FORMATION AND DECAY OF ACCELERATION WAVES IN STEADY HYPERSONIC FLOWS WITH RELAXATION EFFECTS

The present communication is devoted to the study of characteristic solutions in the neighbourhood of the leading frozen characteristics in steady hypersonic flows of fluids. The effects of relaxation on the global behaviour of the wave amplitude have been studied. It is concluded that all compressive waves with initial amplitude greater than the critical one will grow and terminate into shock waves due to non-linear steepenning, while all expansion waves will decay out. A critical stage is also discussed when the compressive wave will either grow or decay.

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