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ELASTIC PARAMETERS OF THE BODY WEAKENED BY THE ELLIPTICAL VOIDS

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1. Introduction

It is well known that all engineering materials are, in the natural state, with the different kind of damages like pores, cracks and other type of microdefects. All of them have decisive influence on mechanical behavior of such materials during exploitation, and as well, to the ultimate load carrying capacity. The behavior of those type of materials is the subject of relatively new branch of Continuum Mechanics, Damage Mechanics (see Krajcinovic, 1989 [7]). The initial damage occurs more often in the so called brittle materials such as: rocks, concrete, dry clays etc., than in the case of ductile materials like steel. Also new type of materials such as composites and ceramics with high strengths, are very sensitive to the existence of defects. The Damage Mechanics (if the defects are distributed) or Fracture Mechanics (if the defects coalesce making main crack), are the research areas where those materials are analyzed.

There are three problems of the body containing voids (the term void in this paper is actually associated with the microvoid). First the evaluation of the elastic parameters which describe the overall elastic behavior. Second the growth of the governing defects under the increasing load above the fracture resistance. Finally the third problem is determination of the rupture force for the considered material. All those problems were analyzed in the paper Sumarac and Krajcinovic 1987, [11] in the case of brittle body containing cracks.

In this paper, first problem will be considered, in the case of the body containing elliptical voids. Obtained results are checked, with already existing in the literature, for circular voids and cracks, taking the ratio of the half axes of ellipse to be equal to one for circle or zero for cracks.

The elastic parameters of the composite material which are made from the matrix with inclusions, has been a long time research topic (see for example Hill 1965 [5], Budiansky and O'Connell 1976 [1], Hashin 1983 [8], Horii and Nemat-Nasser 1983 [6], Sumarac and Krajcinovic 1987 [11], Nemat-Nasser and Hori 1990 [10]). In the literature three main methods were proposed: Taylor model, Self-consistent model and Differential method. In this paper first two will be applied for determination of the elastic constants for isotropic elastic matrix containing elliptical voids.

2. The equivalent inclusion method

Consider the problem of the elliptic cylinder, ($a_3 \to \infty$) (Fig. 1.), embedded in the elastic isotropic material with the same elastic parameters E (Young's modulus) and ν (Poisson's ratio).

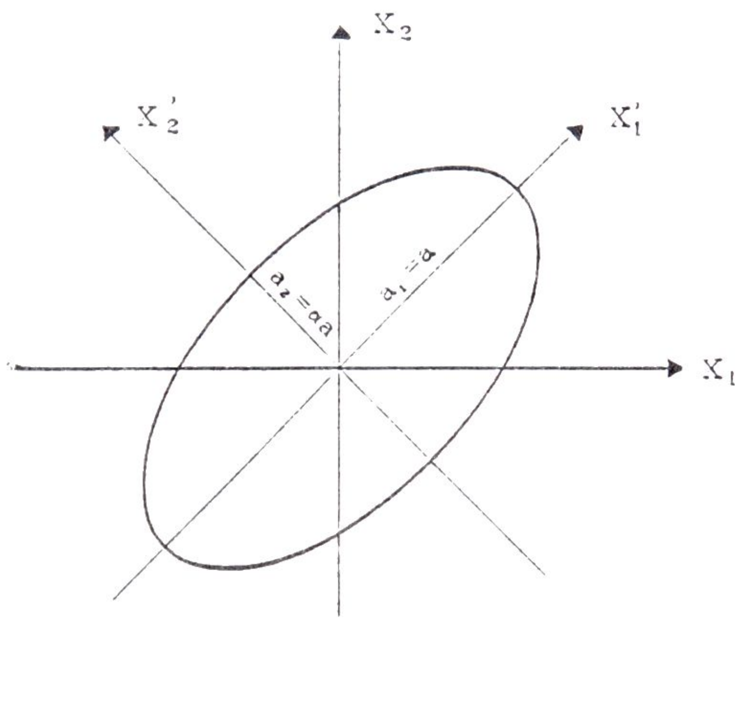

Fig. 1. Elliptical void (inclusion) in the global and local (primed) coordinate system

Let $\varepsilon_{ij}^{*'}$ are "eigenstrains" given within the elliptical cylinder. The solution for the stresses ("eigenstresses") inside or outside of the cylinder are based on the Eshelby's celebrated papers 1957 [2], and 1959 [3]. "Eigenstrain" is a generic name given by Mura 1987 [9], to such nonelastic strains as thermal expansion, phase transformation, initial strains, plastic strains and misfit strains. Eshelby in his, 1957 [2], paper referred to "eigenstrains" as stress-free transformation strains. "Eigenstress" is self equilibrated internal stress, or engineers have used the term residual stress. Eshelby proved taht the uniform "eigenstrain" $\varepsilon_{ij}^{*'}$ within the elliptical inclusion, cause the uniform "eigenstresses" $\sigma_{ij}^{*'}$ in the same region (see also Mura, 1987 [9]):

$$
\left.
\begin{aligned}
\sigma^{*'}_{11} &= \frac{\mu}{1-\nu}\left\{-2+\frac{a_2^2+2a_1a_2}{(a_1+a_2)^2}+\frac{a_2}{a_1+a_2}\right\}\varepsilon^{*'}_{11}\\
&\quad+\frac{\mu}{1-\nu}\left\{\frac{a_2^2}{(a_1+a_2)^2}-\frac{a_2}{a_1+a_2}\right\}\varepsilon^{*'}_{22}-\frac{2\mu\nu}{1-\nu}\frac{a_1}{a_1+a_2}\varepsilon^{*'}_{33},\\
\sigma^{*'}_{22} &= \frac{\mu}{1-\nu}\left\{-2+\frac{a_1^2+2a_1a_2}{(a_1+a_2)^2}+\frac{a_1}{a_1+a_2}\right\}\varepsilon^{*'}_{22}\\
&\quad+\frac{\mu}{1-\nu}\left\{\frac{a_1^2}{(a_1+a_2)^2}-\frac{a_1}{a_1+a_2}\right\}\varepsilon^{*'}_{11}-\frac{2\mu\nu}{1-\nu}\frac{a_2}{a_1+a_2}\varepsilon^{*'}_{33},\\
\sigma^{*'}_{33} &= -\frac{2\mu\nu}{1-\nu}\frac{a_1}{a_1+a_2}\varepsilon^{*'}_{11}-\frac{2\mu\nu}{1-\nu}\frac{a_2}{a_1+a_2}\varepsilon^{*'}_{22}-\frac{2\mu}{1-\nu}\varepsilon^{*'}_{33},\\
\sigma^{*'}_{12} &= -\frac{2\mu}{1-\nu}\frac{a_1a_2}{(a_1+a_2)^2}\varepsilon^{*'}_{12},\\
\sigma^{*'}_{23} &= -2\mu\frac{a_2}{a_1+a_2}\varepsilon^{*'}_{23},\\
\sigma^{*'}_{31} &= -2\mu\frac{a_1}{a_1+a_2}\varepsilon^{*'}_{31}.
\end{aligned}
\right\}
\tag{2.1}
$$

In the above expression $a_1 = a$ and $a_2 = \alpha a$ are half axes of the elliptical region, while μ and ν are the shear modulus and Poisson's ratio respectively. According to the equivalent inclusion method, Mura 1987 [9], the total stress within the elliptical region, under far field stresses σ'_{ij}, and one that is caused by the "eigenstrain" given by expressions (2.1) should be zero everywhere in the elliptical region if the region should represent the void:

$$
\sigma'_{11}+\sigma^{*'}_{11}=0, \quad \sigma'_{22}+\sigma^{*'}_{22}=0, \quad \sigma'_{12}+\sigma^{*'}_{12}=0. \tag{2.2}
$$

For simplicity, from now on, the plane stress condition will be considered, that is already taken into account in the expression (2.2). If the region is loaded with the far field stresses σ'_{11}, σ'_{22} and σ'_{12}, then $\varepsilon^{*'}_{ij}=0$, except $\varepsilon^{*'}_{11}$, $\varepsilon^{*'}_{22}$ and $\varepsilon^{*'}_{12}$. With this considerations and with the restriction to the plane stress conditions ($E/(1-\nu^2)$ in expressions (2.1) should be replaced by E), substitution of eq. (2.1) into (2.2) leads to the system of equations with respect to unknown "eigenstrains" $\varepsilon^{*'}_{11}$, $\varepsilon^{*'}_{22}$ and $\varepsilon^{*'}_{12}$. The solution of this system of equations is:

$$
\left.
\begin{aligned}
\varepsilon^{*'}_{11} &= \frac{1-\nu}{2\mu}[(1+2\alpha)\sigma'_{11}-\sigma'_{22}],\\
\varepsilon^{*'}_{22} &= \frac{1-\nu}{2\mu}\left[\frac{2+\alpha}{\alpha}\sigma'_{22}-\sigma'_{11},\right.\\
\varepsilon^{*'}_{12} &= \frac{1-\nu}{2\mu}\frac{(1+\alpha)^2}{\alpha}\sigma'_{12}.
\end{aligned}
\right\}
\tag{2.3}
$$

Once the $\varepsilon^{*'}_{ij}$ are known, the increase of the strain energy of the body due to presence of the elliptical void is obtained as:

$$
\Delta W = -\frac{1}{2}V\sigma'_{ij}\varepsilon^{*'}_{ij} = -\frac{1}{2}\pi a_1 a_2\sigma'_{ij}\varepsilon^{*'}_{ij}, \tag{2.4}
$$

where $V = \pi a_1 a_2 = \pi \alpha a^2$ is the volume of the void under the plane stress condition ($a_3 = 1.0$). Substituting (2.3) into (2.4) it is obtained:

$$\Delta W = \frac{\pi \alpha a^2}{2E}\left[(1+2\alpha)(\sigma'_{11})^2 - 2\sigma'_{11}\sigma'_{22} + \frac{(2+\alpha)}{\alpha}(\sigma'_{22})^2 + \frac{2(1+\alpha)^2}{\alpha}(\sigma'_{12})^2\right]. \quad (2.5)$$

Differentiating expression (2.5) twice with respect to stresses yields to the compliances:

$$S'^{(k)^*}_{ij} = \frac{\partial^2 W}{\partial \sigma'_i \partial \sigma'_j}, \quad (2.6)$$

where the Voigt notation, $\sigma'_1 = \sigma'_{11}$, $\sigma'_2 = \sigma'_{22}$ and $\sigma'_6 = \sigma'_{12}$ is used. Also in the expression (2.6) (k) refers to a single elliptical void and $(^*)$ stands for the increase of the governing value of the compliance due to presence of the void. In particular expression (2.6) reads:

$$\left.\begin{aligned} S'^{(k)^*}_{11} &= \frac{\pi a^2 \alpha}{E}(1+2\alpha), \quad S'^{(k)^*}_{12} = S'^{(k)^*}_{21} = -\frac{\pi a^2 \alpha}{E}, \\ S'^{(k)^*}_{22} &= \frac{\pi a^2 \alpha}{E}\frac{2+\alpha}{\alpha}, \quad S'^{(k)^*}_{66} = \frac{2\pi a^2}{E}(1+\alpha)^2. \end{aligned}\right\} \quad (2.7)$$

Expressions (2.7) could be written in the condensed form:

$$\begin{aligned} S'^{(k)^*}_{ij} = \frac{\pi a^2 \alpha}{E}\Big[&(1+2\alpha)\delta_{1i}\delta_{1j} - (\delta_{1i}\delta_{2j} + \delta_{2i}\delta_{1j}) + \frac{2+\alpha}{\alpha}\delta_{2i}\delta_{2j} \\ &+ \frac{2(1+\alpha)^2}{\alpha}\delta_{6i}\delta_{6j}\Big] \qquad (i,j = 1,2,6). \end{aligned} \quad (2.8)$$

In the case of the circle ($\alpha = 1$) expression (2.7) becomes:

$$S'^{(k)^*}_{11} = S'^{(k)^*}_{22} = \frac{3\pi a^2}{E}, \quad S'^{(k)^*}_{12} = -\frac{\pi a^2}{E}, \quad S'^{(k)^*}_{66} = \frac{8\pi a^2}{E}. \quad (2.9)$$

On the other hand for $\alpha = 0$, ellipse degenerates to crack, and from the expressions (2.7) it follows:

$$S'^{(k)^*}_{22} = \frac{2\pi a^2}{E}, \quad S'^{(k)^*}_{66} = \frac{2\pi a^2}{E}, \quad S'^{(k)^*}_{ij} = 0 \quad (otherwise). \quad (2.10)$$

The expression (2.10) are the same as those derived in the Sumarac and Krajcinovic 1987 [11], and 1989 [12] by quite different approaches. Once the compliances $S'^{(k)^*}_{ij}$, in the local coordinate system, are determined, using the transformation rule, Horii and Nemat-Nasser 1983 [6], the compliances in the global coordinate system are:

$$S^{(k)^*}_{ij} = S'^{(k)^*}_{mn} g_{mi} g_{nj}, \quad (2.11)$$

where the transformation matrix g_{ij} is given by:

$$g_{ij} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & \sin 2\theta \\ \sin^2\theta & \cos^2\theta & -\sin 2\theta \\ -\frac{1}{2}\sin 2\theta & \frac{1}{2}\sin 2\theta & \cos 2\theta \end{bmatrix}. \tag{2.12}$$

Substituting (2.12) and (2.8) into (2.11) it follows:

$$S_{ij}^{(k)^*} = \frac{\pi a^2 \alpha}{E}\Big[(1+2\alpha)g_{1i}g_{1j} - (g_{1i}g_{2j} + g_{2i}g_{1j}) + \frac{2+\alpha}{\alpha}g_{2i}g_{2j} + \frac{2(1+\alpha)^2}{\alpha}g_{6i}g_{6j}\Big] \qquad (i,j = 1,2,6). \tag{2.13}$$

Using (2.12) and (2.13) it is possible to calculate the particular values:

$$\left.\begin{aligned} S_{11}^{(k)^*} &= \frac{\pi a^2\alpha}{E}\Big[(1+2\alpha)\cos^4\theta + \frac{2+\alpha}{\alpha}\sin^4\theta + \frac{1+\alpha+\alpha^2}{2\alpha}(\sin 2\theta)^2\Big], \\ S_{12}^{(k)^*} &= S_{21}^{(k)^*} = -\frac{\pi a^2\alpha}{E}\Big[\cos^4\theta + \sin^4\theta + \frac{1}{2}(\sin 2\theta)^2\Big], \\ S_{22}^{(k)^*} &= \frac{\pi a^2\alpha}{E}\Big[(1+2\alpha)\sin^4\theta + \frac{2+\alpha}{\alpha}\cos^4\theta + \frac{1+\alpha+\alpha^2}{2\alpha}(\sin 2\theta)^2\Big], \\ S_{66}^{(k)^*} &= \frac{2\pi a^2}{E}(1+\alpha)^2. \end{aligned}\right\} \tag{2.14}$$

In the case of circle ($\alpha = 1$) expressions (2.14) become:

$$S_{11}^{(k)^*} = S_{22}^{(k)^*} = \frac{3\pi a^2}{E}, \quad S_{12}^{(k)^*} = -\frac{\pi a^2}{E}, \quad S_{66}^{(k)^*} = \frac{8\pi a^2}{E}. \tag{2.15}$$

On the other hand in the case of crack ($\alpha = 0$), from (2.14) it follows:

$$\left.\begin{aligned} S_{11}^{(k)^*} &= \frac{2\pi a^2}{E}\sin^2\theta, \quad S_{12}^{(k)^*} = 0, \\ S_{22}^{(k)^*} &= \frac{2\pi a^2}{E}\cos^2\theta, \quad S_{66}^{(k)^*} = \frac{2\pi a^2}{E}. \end{aligned}\right\} \tag{2.16}$$

If the increase of the compliance due to presence of a single void (crack) is known, then it is possible to find the compliance due to presence of many cracks.

3. Mean field theory (uniform distribution of voids)

In the case of many voids (which is going to be considered as the first example), the total compliance would be, Horii and Nemat-Nasser 1983 [6], Kunin 1983 [8], Sumarac and Krajcinovic 1987 [11]:

$$\overline{S}_{ij} = S_{ij} + \widehat{S}_{ij}^*, \tag{3.1}$$

where $(\ )^*$ refers to the increase of the value due to presence of voids, and S_{ij} is the compliance matrix of the undamaged (virgin) material, which is in the case of the plane stress and isotropic matrix given by:

$$S_{ij} = \begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & 0 \\ 0 & 0 & \frac{2(1+\nu)}{E} \end{bmatrix}. \tag{3.2}$$

Assuming the uniform distribution of voids with respect to orientation (angle θ has the uniform distribution density function in the range $0 < \theta < \pi$, and taking the deterministic value (constant radius) a_1 and a_2, compliances $\widehat{S}^*_{ij}$ will be determined as:

$$\widehat{S}^*_{ij} = \frac{N}{\pi} \int_0^{\pi} S^{(k)^*}_{ij}\, d\theta. \tag{3.3}$$

In the above expression N is the number of voids per unit area. Substituting expression (2.14) into (3.3) leads to:

$$\widehat{S}^*_{11} = \widehat{S}^*_{22} = \frac{N\pi a^2}{\widehat{E}}(\alpha^2+\alpha+1), \quad \widehat{S}^*_{12} = -\frac{N\pi a^2 \alpha}{\widehat{E}}, \quad \widehat{S}^*_{66} = \frac{2N\pi a^2}{\widehat{E}}(1+\alpha)^2. \tag{3.4}$$

Introducing:

$$\omega = N\pi a^2 \tag{3.5}$$

as the damage parameter (the total area of the govering circle lacunity per unit area of the body) expressions (3.4) become:

$$\widehat{S}^*_{11} = \widehat{S}^*_{22} = \frac{\omega}{\overline{E}}(\alpha^2+\alpha+1), \quad \widehat{S}^*_{12} = -\frac{\omega}{\overline{E}}\alpha, \quad \widehat{S}^*_{66} = \frac{2\omega}{\overline{E}}(1+\alpha)^2. \tag{3.6}$$

3.1. Taylor model

According to Taylor, 1934 [14], idea about the plastic strain of polycristaline materials, if the response of the body with many voids is determined without interaction, i.e. if the representative void is embedded in the virgin (undamaged) material than $\widehat{E}$ should be replaced by E in the expression (3.4). Then from the expression (3.1) it follows:

$$\frac{1}{\overline{E}} = \frac{1}{E} + \frac{\omega}{E}(\alpha^2+\alpha+1), \quad -\frac{\overline{\nu}}{\overline{E}} = -\frac{\nu}{E} - \frac{\omega}{E}\alpha. \tag{3.7}$$

Finally, solving the above system of equations it is obtained:

$$\frac{\overline{E}^{tm}}{E} = \frac{1}{1+\omega(\alpha^2+\alpha+1)}, \tag{3.8}$$

$$\frac{\overline{\nu}^{tm}}{\nu} = \frac{1+\frac{\omega\alpha}{\nu}}{1+\omega(\alpha^2+\alpha+1)}. \tag{3.9}$$

Expressions (3.8) and (3.9) represent the overall Young's modulus and Poisson's ratio, obtained from Taylor model, as the function of the damage measure ω and the ratio α. This results are the same for $\alpha = 1$ with those obtained in [10] and for $\alpha = 0$ with those obtained in [11].

3.2. Self-consistent model

According to the idea of self-consistency, introduced by Hill (1965) [5], Hashin 1983 [4], Bodiansky and O'Conell 1976 [1], the response of the body with many voids can be represented as the response of the continuum with the elastic parameters that still have to be determined. On the other hand the response of the void is influenced by the responce of others through the decrease of elastic parameters, i.e. the so called "weak interaction" is then taken into account. Then instead of $\widehat{E}$ in the expression (3.4), $\overline{E}$ should be introduced, and from (3.1) it follows:

$$\frac{1}{\overline{E}} = \frac{1}{E} + \frac{\omega}{\overline{E}}(\alpha^2 + \alpha + 1), \quad -\frac{\overline{\nu}}{\overline{E}} = -\frac{\nu}{E} - \frac{\omega}{\overline{E}}\alpha. \tag{3.10}$$

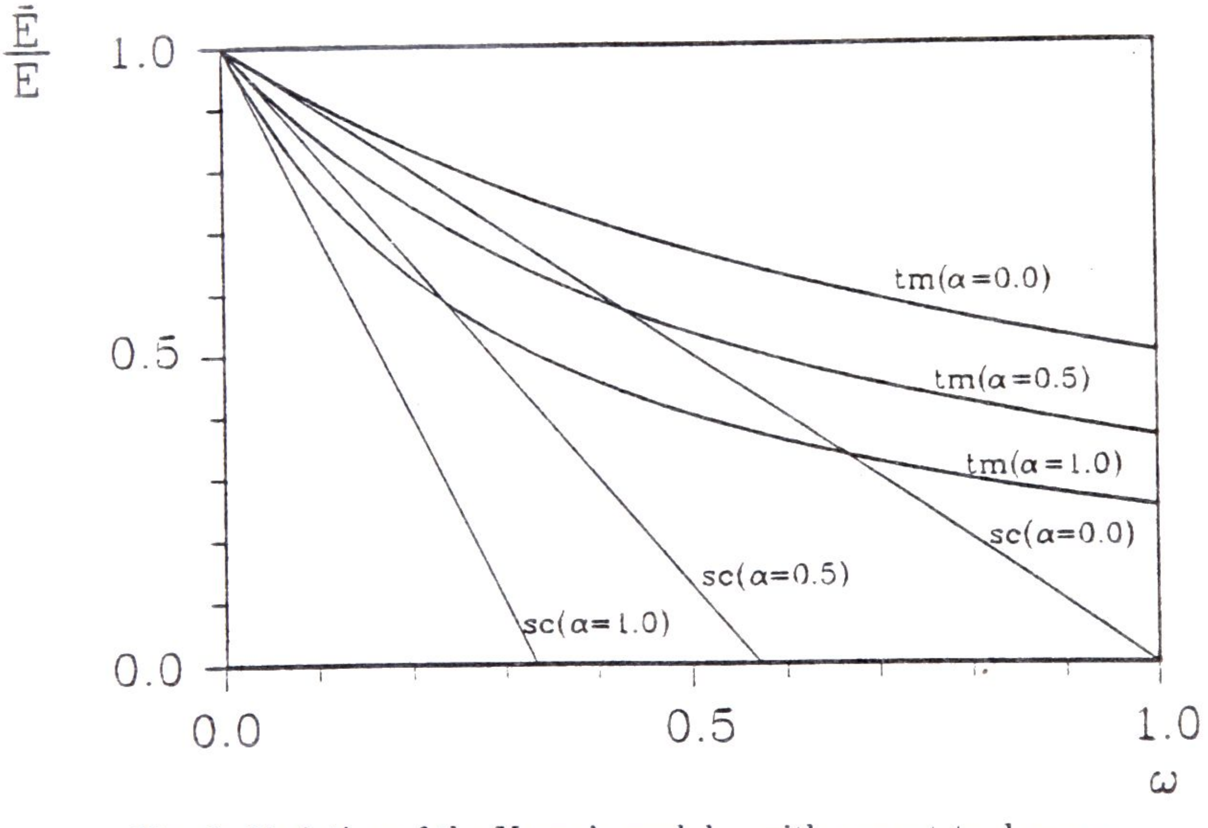

Fig. 2. Variation of the Young's modulus with respect to damage for the uniform distribution of voids

Solving equations (3.10) yields to:

$$\frac{\overline{E}^{sc}}{E} = 1 - \omega(\alpha^2 + \alpha + 1), \tag{3.11}$$

$$\frac{\overline{\nu}^{sc}}{\nu} = 1 - \omega(\alpha^2 + \alpha + 1) + \frac{\omega\alpha}{\nu}. \tag{3.12}$$

Expressions (3.11) and (3.12) represent the overall elastic constants according to the self-consistent approximation. They are the same as the results in [10], for $\alpha = 1$, and [11] for $\alpha = 0$. In Fig. 2 the $\overline{E}/E$ versus ω is plotted for both Taylor and Self-consistent aproximations and for different values of α.

From Fig. 2. it can be seen that, with the increase of the damage measure, the Young's modulus obtained according to Taylor and the self-consistent model are decreasing. Also from expressions (3.11) and (3.12) it can be seen that for the uniform distributions of voids (no meter weather they are circles, ellipses or cracks) the overall response of the body is isotropic. The statement is valid also for both, Taylor and Self-consistent approximation. The total overall compliance for matrix in the case of Self-consistent approximation for uniform distribution of elliptical voids is:

$$\frac{\overline{S}_{ij}^{sc}}{S_{ij}} = \begin{bmatrix} \dfrac{1}{1-\omega A} & -1-\dfrac{\omega\alpha}{\nu(1-\omega A)} & 0 \\ -1-\dfrac{\omega\alpha}{\nu(1-\omega A)} & \dfrac{1}{1-\omega A} & 0 \\ 0 & 0 & \dfrac{1+\nu-\omega\nu A+\omega\alpha}{(1+\nu)(1-\omega A)} \end{bmatrix}, \tag{3.13}$$

where is $A = \alpha^2 + \alpha + 1$. In the case of circle (for $\alpha = 1$ and $A = 3$) matrix (3.13) becomes:

$$\frac{\overline{S}_{ij}^{sc}}{S_{ij}} = \begin{bmatrix} \dfrac{1}{1-3\omega} & -1-\dfrac{\omega}{\nu(1-3\omega)} & 0 \\ -1-\dfrac{\omega}{\nu(1-3\omega)} & \dfrac{1}{1-3\omega} & 0 \\ 0 & 0 & \dfrac{1+\nu-3\omega\nu+\omega}{(1+\nu)(1-3\omega)} \end{bmatrix}. \tag{3.14}$$

Finally in the case of cracks from (3.13) for $\alpha = 0$ and $A = 1$ it is obtained:

$$\frac{\overline{S}_{ij}^{sc}}{S_{ij}} = \begin{bmatrix} \dfrac{1}{1-\omega} & -1-\dfrac{\omega}{\nu(1-\omega)} & 0 \\ -1-\dfrac{\omega}{\nu(1-\omega)} & \dfrac{1}{1-\omega} & 0 \\ 0 & 0 & \dfrac{1+\nu-\omega\nu}{(1+\nu)(1-\omega)} \end{bmatrix}. \tag{3.15}$$

All those matrices represent isotropic overall response. From diagrams in Fig. 2., however, it can be seen that the Young's modulus determined by the self-consistent model is lower than one determined with the Taylor method. The body behaves stiffer according to Taylor than according to the self-consistent model. This is the consequence of the main assumption that the Taylor model neglects interaction of voids at all. Also both approximations give the same result for small concentration of damage. From Fig. 2 it can be seen that it is for $\omega < 0.1$. In this region, the results obtained by those two methods, are appropriate. For larger concentration, the stronger interaction has to be taken into account (see Sumarac, Krajcinovic and Kaushik 1992 [13]). The result for Young's modulus obtained from the self-consistent method has upper value for ω. In the case of cracks it is $\omega_c = 1.0$. This value is bellow the value obtained by percolation theory. On the other hand the Taylor model doesn't have the limitation.

4. Aligned elliptical voids parallel to the x_1 axis

As the second example consider the elliptical voids all parallel to one axis, let say to the x_1 axis. Then:

$$\theta = 0 \tag{4.1}$$

should be introduced into (2.14), from where it follows:

$$\left.\begin{aligned} S_{11}^{(k)^*} &= \frac{\pi a^2 \alpha}{E}(1+2\alpha), \quad S_{12}^{(k)^*} = -\frac{\pi a^2 \alpha}{E}, \\ S_{22}^{(k)^*} &= \frac{\pi a^2}{E}(2+\alpha), \quad S_{66}^{(k)^*} = \frac{2\pi a^2}{E}(1+\alpha)^2. \end{aligned}\right\} \tag{4.2}$$

Expressions (4.2) are increase of the governing compliance due to presence of one void. For many voids (N per unit area), instead of integration (3.3) expressions (4.2) should be multiplied by N, i.e.:

$$\left.\begin{aligned} \widehat{S}_{11}^* &= \frac{\omega\alpha}{\widehat{E}}(1+2\alpha, \quad \widehat{S}_{12}^* = -\frac{\omega\alpha}{\widehat{E}}, \\ \widehat{S}_{22}^* &= \frac{\omega}{\widehat{E}}(2+\alpha), \quad \widehat{S}_{66}^* = \frac{2\omega}{\widehat{E}}(1+\alpha)^2. \end{aligned}\right\} \tag{4.3}$$

where ω is, as before, given by (3.5).

4.1. Taylor model

Substituting (4.3) into (3.1) and taking $\widehat{E} = E$ and $\widehat{\nu} = \nu$, according to the Taylor model, it is obtained:

$$\left.\begin{aligned} \frac{\overline{E}_1^{tm}}{E} &= \frac{1}{1+\omega\alpha(1+2\alpha)}, \\ \frac{\overline{E}_2^{tm}}{E} &= \frac{1}{1+\omega(2+\alpha)}. \end{aligned}\right\} \tag{4.4}$$

From expression (4.4) it can be seen that the body does not obey the isotropy any more, i.e. it becomes orthotropic. This is so called void induced orthotropy. On the other hand, in the case of circles ($\alpha = 1$) from (4.4) it follows:

$$\frac{\overline{E}_1^{tm}}{E} = \frac{\overline{E}_2^{tm}}{E} = \frac{1}{1+3\omega} \tag{4.5}$$

and the body is again isotropic because of the circular shape of a single void. Finally from (4.4) in the case of cracks (for $\alpha = 0$) it is obtained:

$$\frac{\overline{E}_1^{tm}}{E} = 1, \quad \frac{\overline{E}_2^{tm}}{E} = \frac{1}{1+2\omega} \tag{4.6}$$

and the response is also orthotropic. It has to be noticed that in the case of the parallel cracks to the x_1 axis, there is no change of modulus of elasticity $\overline{E}_1$, i.e. the damage doesn't have influence on this direction.

4.2. Self-consistent model

According to self consistent method, by substitution of (4.3) into (3.1), taking into account that $\widehat{E} = \overline{E}$ and $\widehat{\nu} = \overline{\nu}$ it follows:

$$\left.\begin{aligned} \frac{\overline{E}_1^{sc}}{E} &= 1 - \omega\alpha(1+2\alpha), \\ \frac{\overline{E}_2^{sc}}{E} &= 1 - \omega(2+\alpha). \end{aligned}\right\} \tag{4.7}$$

Expressions (4.7) are for aligned elliptical voids. In the case of circles (for $\alpha = 1$) (4.7) leads to:

$$\frac{\overline{E}_1^{sc}}{E} = \frac{\overline{E}_2^{sc}}{E} = 1 - 3\omega. \tag{4.8}$$

Finally if ellipses degenerate to the cracks (for $\alpha = 0$), from (4.7) it is obtained:

$$\frac{\overline{E}_1^{sc}}{E} = 1, \quad \frac{\overline{E}_2^{sc}}{E} = 1 - 2\omega. \tag{4.9}$$

For the Self-consistent approximation the response, as it was in the Taylor model, in the case of aligned defects is orthotropic for elliptical voids and cracks while it is isotropic in the case of circles.

5. Conclusion

In the present study more general approach, from already existing in the literature, for the determination of the overall elastic parameters for the elastic body weakned by the voids is presented. Starting from the elliptical void the solution for the circles and cracks (that is already obtained in the literature) is recovered. The two aproaches within the mean field theory are considered, the Taylor and Self-consistent model. It is shown that the Taylor model (neglecting interaction between voids at all) gives the more stiffer response, then the Self-consistent method, that introduces the so called weak interaction. It is important to notice, that both approximations, are good in the case of the dilute concentration of voids. For larger concentration of defects, the stronger interaction should be taken into account.

Dedication. This year it is 40th anniversary since my advisor, Professor Natalija Naerlović-Veljković, started her teaching and research work. To that occasion I dedicate this paper.

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ELASTIČNE KONSTANTE TELA OSLABLJENOG ŠUPLJINAMA ELIPTIČNOG OBLIKA

U ovom radu daje se opštiji pristup odredjivanju elastičnih parametara tela oslabljenog šupljinama eliptičnog oblika, u odnosu na postojeće u literaturi. Pokazano je da se postojeći rezultati za kružne šupljine i prsline mogu dobiti kao specijalni slučajevi ovde datog rešenja. Primenjena su dva postupka: Tejlorov i samo-konsistentni model. Po Tejlorovom modelu dobija se krući materijal (jer je po ovom metodu zanemarena interakcija šupljina), za razliku od samo-konsistentnog modela kod koga se uzima u obzir tzv. "slaba interakcija". Obe metode mogu se uspešno primeniti kod male koncentracije šupljina, (za malo oštećenje materijala), dok se kod većih koncentracija oštećenja mora uzeti strožija interakcija defekata.

УПРУГИЕ ПАРАМЕТРИ ТЕЛА ОСЛАБЛЕННОГО ЭЛИПТИЧЕСКИМИ ОТВЕРСТИЯМИ

В настоящей работе изучается более общий метод получения упругих параметров для материялов с малыми элиптическими отверстиями. На примерах кругового отверстия и трещины показанно что из выведеного более общего решения получаются эти специальные случаи. В работе применяются два метода: Тейлоров и самоконсистентний. В первом интеракция между отверстиями пренебрегается, пока же в втором использованна так називаемая "слаба интеракция". В работе показанно что оба метода для малой концентрации отверстий дают удовлетворяющие результаты.

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