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A PARAMETRIC FORMULATION OF THE MECHANICS OF RHEONOMIC SYSTEMS

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1. Introduction

The usual parametric formulation of classical mechanics considers the time t as an additional generalized coordinate, and instead of time introduces a new independent variable $\tau = \tau(t)$. From physical point of view this formalism was introduced and developed by P. Dirac [1], starting from parametric form of Hamilton's variational principle. Later, several authors used it to formulate either a dynamic theory in the space of events [2], or so called homogeneous formalism of mechanics [3]. In this way the appropriate Lagrangian and Hamiltonian (canonical) formalism were obtained and this was of particular interest for covariant formulation of mechanics in the theory of relativity.

However, one other approach to the parametric formulation of mechanics is possible as well, which is formally opposite to Dirac's one. Such type of an extended analytical formalism is recently published by V. Vujičić [4-9], which suggested a modification of classical mechanics of rheonomic systems. In his approach the additional generalized coordinate is suitably chosen function of time $q_0 = \tau(t)$, retaining the time as independent variable, and the constraints as well as the Lagrangian are expressed in therms of this function. On this basis the extended system of Lagrange's equations and modified Hamilton's principle are formulated, where these equations here are mutually independent. Furthermore, a law of energy is obtained in integral form $\mathcal{E} = T + U + P = const$, which differs from Jacobi-Painlevé integral and contains an additional term P, the same originating from the nonstationarity of constraints. Finally, the coresponding system of Hamilton's equations is also formulated, although without any proof, where the Hamiltonian represents mentioned extended energy of system.

In this paper it is given one parametric formulation of the mechanics of systems of this type. It is based on the family of varied paths and on the transition to a new parameter depending on the selected path. It is shown that so formulated theory includes the results obtained from modified analytical formalism of Vujičić and gives a better understanding and the interdependence of the results.

2. The varied paths and introduction of a new parameter

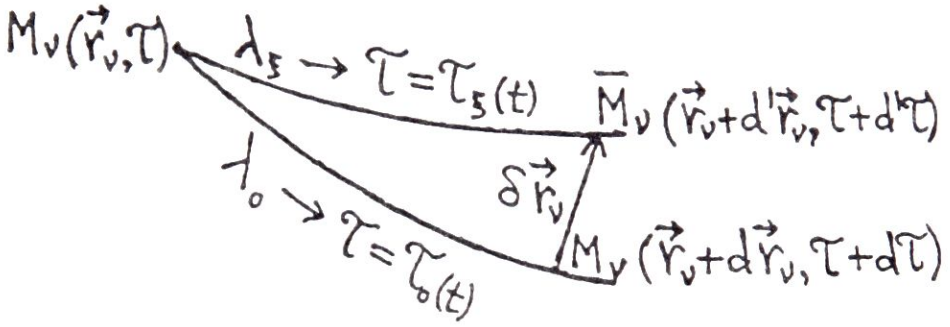
Consider a mechanical system of N particles, under the influence of some arbitrary forces, and let the motion of system be limited by k nonstationary holonomic constraints

$$f_\mu(\vec{r}_\nu, t) = 0 \quad (\mu = 1, 2, \ldots, k). \tag{2.1}$$

The position of this system can be described by a set of generalized coordinates $(q_1, q_2, \ldots, q_n)$, where $n = 3N - k$, and the system itself by a Lagrangian

$$L = L(q_i, \dot{q}_i, t) \quad (i = 1, 2, \ldots, n). \tag{2.2}$$

In the case of these rheonomic systems the time t has double role. On one hand, this is an independent variable, just as in mechanics in general, and on the other it represents one kind of parameter, whose change causes the change in the form of constraints and of Lagrangian.



Imagine a family of neighbouring possible varied paths of particles of the system, drawn from their initial positions (see the graph)

$$\vec{r}_\nu = \vec{r}_\nu(t, \lambda) \quad (\nu = 1, 2, \ldots, N). \tag{2.3}$$

Let M_{ν_0} denotes the position of ν-th particle at instant t, and M_ν and $\overline{M}_\nu$ its position at instant $t + \Delta t$ on the actual and a varied path respectively. Then, instead of time t considered as a parameter, introduce a new parameter τ in dependence on chosen path of the system

$$\tau = \tau(t, \lambda) \Longleftrightarrow t = t(\tau, \lambda) \tag{2.4}$$

and keep the time t as independent variable. This implies that to each varied path, which corresponds to the value λ_ξ corresponds some function of time

$$\textit{varied path}\,(\lambda_\xi) \longrightarrow \tau = \tau(t, \lambda_\xi) \equiv \tau_\xi(t) \tag{2.5}$$

and the function $\tau = \tau_0(t)$ for the actual path of the system is of particular interest. The form of this function in general is arbitrary, but it can be suggested by the form of constraints (2.1), for example if the time t enters in these equations through certain function $\tau(t)$, we can take it for $\tau_0(t)$.

Owing to different functions $\tau_0(t)$ and $\tau_\xi(t)$, on the actual and varied path to the same instant $t + dt$ correspond different values of this parameter. Therefore, the virtual displacement $\delta\vec{r}_\nu$, as a difference of two possible displacements for same interval dt, here is given by

$$\delta\vec{r}_\nu \overset{def}{=} \frac{\partial\vec{r}_\nu}{\partial\lambda}\delta\lambda = \vec{r}_\nu(\overline{M}_\nu)_{\tau+d'\tau} - \vec{r}_\nu(M_\nu)_{\tau+d\tau}. \tag{2.6}$$

In analogous way, if a family of possible varied paths is expressed in generalized coordinates

$$q_i = q_i(t,\lambda), \quad (i = 1, 2, \ldots, n) \tag{2.7}$$

the variation of the generalized coordinate can be represented by

$$\delta q_i \overset{def}{=} \frac{\partial q_i}{\partial\lambda}\delta\lambda = q_i(\overline{P})_{\tau+d'\tau} - q_i(P)_{\tau+d\tau}, \tag{2.8}$$

where P and $\overline{P}$ are corresponding points on the actual and varied path in the configuration space.

The transition from time t to previously introduced parameter τ in relations where the former is considered as a parameter can be effectuated in the following way. If the time t in equations (2.1) is substituted by appropriate expression $t = t(\tau,\lambda)$ from (2.4), we shall obtain the corresponding relations in this formalism, which are valid in general. However, for each particular value $\lambda = \lambda_\xi$ they are valid only along the corresponding path

$$\left.\begin{aligned} &f_\mu[\vec{r}_\nu, t(\tau,\lambda)] \equiv f^*_\mu(\vec{r}_\nu,\tau,\lambda) = 0,\\ &\text{for } \lambda = \lambda_\xi: \quad f^{*(\xi)}_\mu(\vec{r}_\nu,\tau) = 0. \end{aligned}\right\} \tag{2.9}$$

Furthermore, if τ is taken as additional generalized coordinate denoted by $q_0 = \tau$, then the position vector along particular path can be expressed as

$$\vec{r}_\nu = \vec{r}_\nu[q_i, t(\tau,\lambda_\xi)] \equiv \vec{r}^{*(\xi)}_\nu(q_\alpha) \quad (\alpha = 0, 1, \ldots, n). \tag{2.10}$$

In similar manner, the Lagrangian (2.2) for this path transforms into a function of the form

$$L = L[q_i, \dot{q}_i, t(\tau,\lambda_\xi)] \equiv L^{*(\xi)}(q_\alpha, \dot{q}_\alpha). \tag{2.11}$$

Here the dependence of all these functions on q_0 depends on chosen varied path, in contrast to the usual formulation. In further work we shall omit the asterisks, assuming that these quantities are expressed in appropriate variables. So introduced function $q_0 = \tau(t)$, which represents the relation (2.4) for the actual path of the

system, corresponds to additional generalized coordinate in modified analytical formalism of V. Vujičić [5-9]. In this way a definite sense of this function is obtained, from viewpoint of this parametric formulation of mechanics.

3. Nonstationary constraints and ideal reaction forces

Let us consider the actual and one nearby varied path, whose parameters are λ_0 and $\lambda_0 + \delta\lambda$ respectively, and denote the equations of the constraints along each of these paths by

$$\left.\begin{aligned} f_\mu[\vec{r}_\nu, t(\tau, \lambda_0)] &\equiv f_\mu^{(0)}(\vec{r}_\nu, \tau, \lambda_0) = 0, \\ f_\mu[\vec{r}_\nu, t(\tau, \lambda_0 + \delta\lambda)] &\equiv f_\mu^{(1)}(\vec{r}_\nu, \tau, \lambda_0 + \delta\lambda) = 0. \end{aligned}\right\} \tag{3.1}$$

If we differentiate the first of these equations along the actual path and the second one along the varied path, one obtains

$$\frac{\partial f_\mu^{(0)}}{\partial \vec{r}_\nu} \cdot d\vec{r}_\nu + \frac{\partial f_\mu^{(0)}}{\partial \tau} d\tau = 0, \quad \frac{\partial f_\mu^{(1)}}{\partial \vec{r}_\nu} \cdot d'\vec{r}_\nu + \frac{\partial f_\mu^{(1)}}{\partial \tau} d'\tau = 0, \tag{3.2}$$

where the summing is understood over the repeated indices.

The function $f_\mu^{(1)}$ can be approximated by its Taylor's polynomial, neglecting the terms of higher order

$$f_\mu^{(1)}(\vec{r}_\nu, \tau, \lambda_0 + \delta\lambda) \approx f_\mu^{(0)}(\vec{r}_\nu, \tau, \lambda_0) + \delta\lambda \left(\frac{\partial f_\mu^{(0)}}{\partial \lambda}\right)_0. \tag{3.3}$$

By inserting this expression into (3.2b), one yields

$$\left[\frac{\partial f_\mu^{0)}}{\partial \vec{r}_\nu} + \frac{\partial}{\partial \vec{r}_\nu}\left(\frac{\partial f_\mu^{(0)}}{\partial \lambda}\right)\delta\lambda\right] \cdot d'\vec{r}_\nu + \left[\frac{\partial f_\mu^{0)}}{\partial \tau} + \frac{\partial}{\partial \tau}\left(\frac{\partial f_\mu^{(0)}}{\partial \lambda}\right)\delta\lambda\right] \cdot d'\tau = 0,$$

which can be written as

$$\frac{\partial f_\mu^{(0)}}{\partial \vec{r}_\nu} \cdot d'\vec{r}_\nu + \frac{\partial f_\mu^{(0)}}{\partial \tau} d'\tau + d'\left(\frac{\partial f_\mu^{(0)}}{\partial \lambda}\right)\delta\lambda = 0. \tag{3.4}$$

If we subtract the equation (3.2a) from (3.4), omiting the index 0, we obtain

$$\frac{\partial f_\mu}{\partial \vec{r}_\nu} \cdot \delta\vec{r}_\nu + \frac{\partial f_\mu}{\partial \tau}\delta\tau + d'\left(\frac{\partial f_\mu}{\partial \lambda}\right)\delta\lambda = 0, \tag{3.5}$$

with the usual definitions of $\delta\vec{r}_\nu$ and $\delta\tau$, or in the first approximation

$$\frac{\partial f_\mu}{\partial \vec{r}_\nu} \cdot \delta\vec{r}_\nu + \frac{\partial f_\mu}{\partial \tau}\delta\tau \approx 0. \tag{3.6}$$

These are the conditions which the virtual displacements here must satisfy.

In order to introduce the reaction forces in consideration, let us separate them into ideal and nonideal ones

$$\overrightarrow{R}_\nu = \overrightarrow{R}_\nu^{id} + \overrightarrow{R}_\nu^{neid} = \lambda_\mu \frac{\partial f_\mu}{\partial \vec{r}_\nu} + \overrightarrow{R}_\nu^{*} \quad (\nu = 1, 2, \ldots, N), \tag{3.7}$$

where the later are denoted by asterisk. The total work of $\overrightarrow{R}_\nu^{id}$ on all the displacements $\delta\vec{r}_\nu$, according to (3.6) has the form

$$\overrightarrow{R}_\nu^{id} \cdot \delta\vec{r}_\nu = \lambda_\mu \frac{\partial f_\mu}{\partial \vec{r}_\nu} \cdot \delta\vec{r}_\nu = -\lambda_\mu \frac{\partial f_\mu}{\partial \tau}\delta\tau,$$

or concisely

$$\overrightarrow{R}_\nu^{id} \cdot \delta\vec{r}_\nu = \tilde{R}_0 \delta\tau, \quad \tilde{R}_0 = -\lambda_\mu \frac{\partial f_\mu}{\partial \tau}. \tag{3.8}$$

Consequently, the total work of ideal reaction forces on arbitrary virtual displacements in the case of the nonstationary constraints in this formalism is different from zero, what represents a characteristic difference in comparison with the usual formulation.

The meaning of so introduced quantity $\tilde{R}_0$ can be perceived from (3.6)

$$\frac{\partial f_\mu}{\partial \vec{r}_\nu} \cdot \frac{\partial \vec{r}_\nu}{\partial \tau} + \frac{\partial f_\mu}{\partial \tau} = 0 \implies \tilde{R}_0 = -\lambda_\mu \frac{\partial f_\mu}{\partial \tau} = \lambda_\mu \frac{\partial f_\mu}{\partial \vec{r}_\nu} \cdot \frac{\partial \vec{r}_\nu}{\partial \tau},$$

which, by the definition of $\overrightarrow{R}_\nu^{id}$ gives

$$\tilde{R}_0 = \overrightarrow{R}_\nu^{id} \cdot \frac{\partial \vec{r}_\nu}{\partial q_0} \overset{def}{=} R_0. \tag{3.9}$$

Therefore, $\tilde{R}_0$ is equal to the generalized force which corresponds to the ideal reaction force for the generalized coordinate $q_0 = \tau$ and this agree with Vujičić's papers [4,7].

4. D' Alembert-Lagrange principle

Let us start from the fundamental equation of dynamics applied to each particle of the system

$$m_\nu \vec{a}_\nu = \overrightarrow{F}_\nu + \overrightarrow{R}_\nu^{id} + \overrightarrow{R}_\nu^{*} \quad (\nu = 1, 2, \ldots, N). \tag{4.1}$$

If we multiply it with $\delta\vec{r}_\nu$ and sum it over ν, one obtains

$$(\overrightarrow{F}_\nu + \overrightarrow{R}_\nu^{*} - m_\nu \vec{a}_\nu) \cdot \delta\vec{r}_\nu = -\overrightarrow{R}_\nu^{id} \cdot \delta\vec{r}_\nu, \tag{4.2}$$

and here we can substitute the work of ideal reaction forces by the expression (3.8)

$$(\overrightarrow{F}_\nu + \overrightarrow{R}^*_\nu - m_\nu \vec{a}_\nu) \cdot \delta \vec{r}_\nu = -R_0 \delta\tau. \tag{4.3}$$

This is the modified d'Alembert-Lagrange principle in this τ-formulation of the mechanics. The difference between it and the usual formulation is in the term on the right-hand side, which explicitly expresses the influence of the nonstationarity of constraints.

We can present this principle also in the generalized coordinates. By the differentiation of the relation (2.10) along the actual and varied path and by their subtraction, one yields

$$\delta \vec{r}_\nu = \frac{\partial \vec{r}_\nu}{\partial q_\alpha} \delta q_\alpha, \quad \delta q_\alpha = d'q_\alpha - dq_\alpha, \tag{4.4}$$

so that d'Alembert-Lagrange principle (4.3) passes into

$$(\overrightarrow{F}_\nu + \overrightarrow{R}^*_\nu) \cdot \frac{\partial \vec{r}_\nu}{\partial q_\alpha} \delta q_\alpha - m_\nu \vec{a}_\nu \cdot \frac{\partial \vec{r}_\nu}{\partial q_\alpha} \delta q_\alpha = -R_0 \delta q_0. \tag{4.5}$$

The second term can be transformed in the usual way, with the difference that the summing must be here done from 0 to n

$$\begin{aligned} m_\nu \vec{a}_\nu \cdot \frac{\partial \vec{r}_\nu}{\partial q_\alpha} \delta q_\alpha &= \left[\frac{d}{dt} \left(m_\nu \vec{v}_\nu \cdot \frac{\partial \vec{r}_\nu}{\partial q_\alpha} \right) - m_\nu \vec{v}_\nu \cdot \frac{d}{dt} \left(\frac{\partial \vec{r}_\nu}{\partial q_\alpha} \right) \right] \delta q_\alpha \\ &= \left(\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_\alpha} - \frac{\partial T}{\partial q_\alpha} \right) \delta q_\alpha, \end{aligned}$$

where T is the kinetic energy of the system. Then, this principle gets the form

$$\left(Q_\alpha + R^*_\alpha - \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_\alpha} + \frac{\partial T}{\partial q_\alpha} \right) \delta q_\alpha = -R_0 \delta q_0 \tag{4.6}$$

with the corresponding generalized forces

$$Q_\alpha = \overrightarrow{F}_\nu \cdot \frac{\partial \vec{r}_\nu}{\partial q_\alpha}, \quad R^*_\alpha = \overrightarrow{R}^*_\nu \cdot \frac{\partial \vec{r}_\nu}{\partial q_\alpha}. \tag{4.7}$$

Starting from this form of d'Alembert-Lagrange principle one can obtain the differential equations of motion in generalized coordinates. To this aim, let us write it concisely in the form

$$\left(\tilde{Q}_\alpha - \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_\alpha} + \frac{\partial T}{\partial q_\alpha} \right) \delta q_\alpha = 0, \tag{4.8}$$

where

$$\tilde{Q}_\alpha = \begin{cases} Q_i + R_i^* & \text{for } \alpha = i, \\ Q_0 + R_0^* + R_0 & \text{for } \alpha = 0. \end{cases} \tag{4.9}$$

Because of the independence of variations δq_α each expression in parenthesis must be equal to zero, or explicitly

$$\left.\begin{aligned} \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} &= Q_i + R_i^* \quad (i = 1, 2, \ldots, n), \\ \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_0} - \frac{\partial T}{\partial q_0} &= Q_0 + R_0^* + R_0. \end{aligned}\right\} \tag{4.10}$$

This extended system of Lagrange's equations and the modified d' Alembert-Lagrange principle were first formulated by *V*. Vujičić [4-5,7], though in less general form and without connecting them with the work of ideal reaction forces. The quantity R_0 was named generalized reaction force of rheonomic constraints and the easist way to find it is following. If one solves the first n Lagrange's equations (4.10), we get the generalized coordinates as functions of time $q_i = q_i(t)$ and then insert them into the last Lagrange's equation. In this manner one finds this quantity in the form $R_0(q_0, \dot{q}_0, \ddot{q}_0)$ and after putting $q_0 = \tau(t)$ it becomes a function only of q_0, i.e. $R_0 = R_0^*(q_0)$.

5. Central Lagrange's equation

This d' Alembert-Lagrange principle can be transformed so to make it possible to obtain the corresponding general integral principles as well as the conservation laws of mechanics. To this aim transforme the third term of (4.3), applying the commutativity of operations d/dt and δ

$$m_\nu \vec{a}_\nu \cdot \delta \vec{r}_\nu = \frac{d}{dt}(m_\nu \vec{v}_\nu \cdot \delta \vec{r}_\nu) - \delta T. \tag{5.1}$$

Then the d' Alembert-Lagrange principle (4.3), by emphasing the term containing time derivative, obtains the form

$$\delta T + (\vec{F}_\nu + \vec{R}_\nu^*) \cdot \delta \vec{r}_\nu + R_0 \delta \tau = \frac{d}{dt}(m_\nu \vec{v}_\nu \cdot \delta \vec{r}_\nu) \tag{5.2}$$

and this is the corresponding central Lagrange's equation.

To express this equation in the generalized coordinates, let us introduce them by (4.4) and present the expression $m_\nu \vec{v}_\nu \cdot \delta \vec{r}_\nu$ in these coordinates

$$m_\nu \vec{v}_\nu \cdot \delta \vec{r}_\nu = m_\nu \vec{v}_\nu \cdot \frac{\partial \vec{r}_\nu}{\partial q_\alpha} \delta q_\alpha = \frac{\partial T}{\partial \dot{q}_\alpha} \delta q_\alpha. \tag{5.3}$$

Besides, we can separate the generalized forces into the potential and nonpotential ones

$$Q_\alpha = Q_\alpha^{pot} + Q_\alpha^{nepot} = -\frac{\partial V}{\partial q_\alpha} + \frac{d}{dt}\frac{\partial V}{\partial \dot{q}_\alpha} + Q_\alpha^*, \tag{5.4}$$

where the later are denoted by asterisk. Then the relation (5.2) transforms into

$$\delta T - \frac{\partial V}{\partial q_\alpha}\delta q_\alpha + \frac{d}{dt}\left(\frac{\partial V}{\partial \dot{q}_\alpha}\delta q_\alpha\right) - \frac{\partial V}{\partial \dot{q}_\alpha}\delta \dot{q}_\alpha + (Q^*_\alpha + R^*_\alpha)\delta q_\alpha + R_0\delta q_0 = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\delta q_\alpha\right)$$

or, by grouping the similar terms

$$\delta L + (Q^*_\alpha + R^*_\alpha)\delta q_\alpha + R_0\delta q_0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\delta q_\alpha\right), \tag{5.5}$$

where

$$L(q_\alpha, \dot{q}_\alpha) = T - V \tag{5.6}$$

is the Lagrangian of the system.

Here we can still pass from the synchronic to the total variations of the generalized coordinates and the Lagrangian

$$\Delta q_\alpha = \delta q_\alpha + \dot{q}_\alpha \Delta t, \quad \Delta L = \delta L + \dot{L}\Delta t. \tag{5.7}$$

If we substitute δq_α and δL by corresponding expressions from here, after identical transformations the previous relation (5.6) can be written in the form

$$\left.\begin{aligned} \Delta L + L\frac{d}{dt}(\Delta t) + (Q^*_\alpha + R^*_\alpha)(\Delta q_\alpha - \dot{q}_\alpha \Delta t) \\ + R_0(\Delta q_0 - \dot{q}_0\Delta t) = \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_\alpha}(\Delta q_\alpha - \dot{q}_\alpha \Delta t) + L\Delta t\right]. \end{aligned}\right\} \tag{5.8}$$

This relation represents the most general form of central Lagrange's equation in generalized coordinates.

6. Transition to the integral principles

In order to enable the transition to the integral principles, let us start from the central Lagrange's equation (5.2). If we integrate its both sides with respect to time over an interval (t_0, t_1), one yields

$$\int_{t_0}^{t_1} [\delta T + (\overrightarrow{F}_\nu + \overrightarrow{R}^*_\nu)\cdot \delta \overrightarrow{r}_\nu + R_0\delta\tau]\, dt = \left| m_\nu \overrightarrow{v}_\nu \cdot \delta \overrightarrow{r}_\nu \right|_{t_0}^{t_1}.$$

From here the corresponding Hamilton's integral principle follows immediately: if all $\delta \overrightarrow{r}_\nu = 0$ for $t = t_0$ and $t = t_1$, then

$$\int_{t_0}^{t_1} [\delta T + (\overrightarrow{F}_\nu + \overrightarrow{R}^*_\nu)\cdot \delta \overrightarrow{r}_\nu + R_0\delta\tau]\, dt = 0 \tag{6.1}$$

along the actual path of the system. Notice that this principle, like d'Alembert-Lagrange one, has an additional term, which arises from the nonstationarity of constraints.

Here we can also separate the forces $\overrightarrow{F}_\nu$ into the potential and nonpotential ones, as in (5.4)

$$\overrightarrow{F}_\nu = \overrightarrow{F}_\nu^{pot} + \overrightarrow{F}_\nu^{nepot} = -\frac{\partial V}{\partial \vec{r}_\nu} + \frac{d}{dt}\frac{\partial V}{\partial \vec{v}_\nu} + \overrightarrow{F}_\nu^{*}. \tag{6.2}$$

Then, the elementary work of these potential forces on the virtual displacements can be expressed as

$$\overrightarrow{F}_\nu^{pot} \cdot \delta\vec{r}_\nu = -\frac{\partial V}{\partial \vec{r}_\nu} \cdot \delta\vec{r}_\nu + \frac{d}{dt}\left(\frac{\partial V}{\partial \vec{v}_\nu} \cdot \delta\vec{r}_\nu\right) - \frac{\partial V}{\partial \vec{v}_\nu} \cdot \frac{d}{dt}(\delta\vec{r}_\nu)$$

or, since the operations d/dt and δ are comutative

$$\overrightarrow{F}_\nu^{pot} \cdot \delta\vec{r}_\nu = -\delta V + \frac{d}{dt}\left(\frac{\partial V}{\partial \vec{v}_\nu} \cdot \delta\vec{r}_\nu\right). \tag{6.3}$$

By inserting this in (6.1), bearting in mind that the second term of (6.2) by integration disappears, Hamilton's integral principle (6.1) passes into

$$\int_{t_0}^{t_1} [\delta L + (\overrightarrow{F}_\nu^{*} + \overrightarrow{R}_\nu^{*}) \cdot \delta\vec{r}_\nu + R_0\delta\tau]\, dt = 0. \tag{6.4}$$

To express this principle in the generalized coordinates, we must start from central Lagrange's equation in general form (5.8) and integrate it with respect to time. Since $\Delta q_\alpha = 0$ and $\Delta t = 0$ for $t = t_0$ and $t = t_1$, the integrated part will be equal to zero and in this way one obtains

$$\int_{t_0}^{t_1} \left[\Delta L + L\frac{d}{dt}(\Delta t) + (Q_\alpha^{*} + R_\alpha^{*})(\Delta q_\alpha - \dot{q}_\alpha \Delta t) + R_0(\Delta q_0 - \dot{q}_0 \Delta t)\right] dt = 0. \tag{6.5}$$

This is the general Hamilton's principle in generalized coordinates in this τ-formulation of mechanics, which in the case of synchronic variations gets the simpler form

$$\int_{t_0}^{t_1} [\delta L + (Q_\alpha^{*} + R_\alpha^{*})\delta q_\alpha + R_0\delta q_0]\, dt = 0, \tag{6.6}$$

but always with the characteristic term $R_0\delta q_0$.

7. Hamilton's variational principle

This Hamilton's integral principle in general does not have the variational character, since the expression on the left-hand site cannot always be presented in

the form of the variation of some integral. However, the same can happen if certain conditions are satisfied, and then this principle becomes a variational one, which can be realized in the following manner.

When we find the quantity R_0 from the last Lagrange's equation (4.10) in the form $R_0(q_0, \dot{q}_0, \ddot{q}_0)$, let us suppose that $R_0 dq_0$ is a total differential of some function and in this case introduce its negative value

$$P(q_0, \dot{q}_0) \stackrel{def}{=} -\int R_0\, dq_0 \iff R_0 = -\frac{dP}{dq_0}. \tag{7.1}$$

Putting $q_0 = \tau(t)$ and passing from t to q_0 along the actual path, one obtains this quantity as a function of q_0 only $P = P^*(q_0)$, and $R_0\delta\tau$ can be then expressed in terms of the same $R_0\delta\tau = -d'P + dP = -\delta P$. Then Hamilton's integral principle (6.4) transforms into

$$\int_{t_0}^{t_1} [\delta\mathcal{L} + (\overrightarrow{F}^*_\nu + \overrightarrow{R}^*_\nu) \cdot \delta\overrightarrow{r}_\nu]\, dt = 0, \tag{7.2}$$

where

$$\mathcal{L}(q_0, \dot{q}_0) = L - P = T - V - P \tag{7.3}$$

has the role of the Lagrangian in this formulation of mechanics. From (7.2) one obtains immediately the variational Hamilton's principle: if all $\delta\overrightarrow{r}_\nu = 0$ for $t = t_0$ and $t = t_1$, and if the condition $(\overrightarrow{F}^*_\nu + \overrightarrow{R}^*_\nu) \cdot \delta\overrightarrow{r}_\nu = 0$ is satisfied, then

$$\delta\mathcal{W} = \int_{t_0}^{t_1} \delta\mathcal{L}\, dt = 0, \quad \mathcal{W} = \int_{t_0}^{t_1} \mathcal{L}\, dt \tag{7.4}$$

along the actual path. We should point out that the condition under which Hamilton's integral principle transforms into the corresponding variational one does not assume that all forces are potential, these could be nonpotential active and nonideal reaction forces as well, but they all must to be gyroscopic.

In order to pass to the generalized coordinates, we start from (6.5) and transform the last term, introducing the quantity (7.1)

$$R_0(\Delta q_0 - \dot{q}_0 \Delta t) = -\Delta P + \frac{dP}{dt}\Delta t.$$

In this manner, the Hamilton's integral principle gets the form

$$\int_{t_0}^{t_1} \Big[\Delta L + L\frac{d}{dt}(\Delta t) + (Q^*_\alpha + R^*_\alpha)(\Delta q_\alpha - \dot{q}_\alpha \Delta t) - \Delta P + \frac{d}{dt}(P\Delta t) - P\frac{d}{dt}(\Delta t)\Big]\, dt = 0, \tag{7.5}$$

which by simple manipulation, grouping the similar terms and keeping in mind that the term $d/dt(P\Delta t)$ vanishes by integration, transforms into

$$\int_{t_0}^{t_1} [\Delta(\mathcal{L}\Delta t) + (Q_\alpha^* + R_\alpha^*)(\Delta q_\alpha - \dot{q}_\alpha \Delta t)]\, dt = 0. \tag{7.6}$$

From this relation, if the condition $(Q_\alpha^* + R_\alpha^*)(\Delta q_\alpha - \dot{q}_\alpha \Delta t) = 0$ is satisfied, we obtain the Hamilton's variational principle in the form

$$\Delta \mathcal{W} = \int_{t_0}^{t_1} \Delta(\mathcal{L}\, dt) = 0. \tag{7.7}$$

The corresponding differential equations of motion in generalized coordinates can be obtained in the following manner. If we transform the total variation of action in the usual way ([3], p.13), except to do summing from 0 to n, one yields

$$\left.\begin{aligned} \Delta \mathcal{W} = &\int_{t_0}^{t_1} \left(\frac{\partial \mathcal{L}}{\partial q_\alpha} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha}\right)(\Delta q_\alpha - \dot{q}_\alpha \Delta t)\, dt \\ &+ \left|\frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha}\Delta q_\alpha\right|_{t_0}^{t_1} - \left|\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha}\dot{q}_\alpha - \mathcal{L}\right)\Delta t\right|_{t_0}^{t_1}. \end{aligned}\right\} \tag{7.8}$$

Because of the boundary conditions the last two terms are equal to zero and since the variations Δq_α and Δt are independent, from Hamilton's variational principle (7.4) follows

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha} - \frac{\partial \mathcal{L}}{\partial q_\alpha} = 0 \quad (\alpha = 0, 1, \ldots, n). \tag{7.9}$$

These are the corresponding Lagrange's equations, which by passing to the initial Lagrangian and utilizing (7.1), obtain the form

$$\left.\begin{aligned} &\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \quad (i = 1, 2, \ldots, n), \\ &\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_0} = R_0. \end{aligned}\right\} \tag{7.10}$$

This Hamilton's variational principle, as well as the corresponding Lagrange's equations were first formulated by V. Vujičić [5-6,8-9] and the quantity P was named rheonomic potential of the system. In his approach the previous results were obtained in another way, either from the invariance of general principles or in an insufficiently consistent manner, but without connecting this variational principle with other concepts, especially with d'.Alembert-Lagrange principle via general Hamilton's integral principle.

8. Formulation of the Hamiltonian formalism

On the basis of so developed τ-formulation of the mechanics one can formulate the appropriate Hamiltonian canonical formalism in the case when the general Hamilton's principle has the form of a variational one. To this aim, let us define the generalized momenta in the usual way

$$p_\alpha \overset{def}{=} \frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha} \quad (\alpha = 0, 1, \ldots, n) \tag{8.1}$$

corresponding to all $n+1$ generalized coordinates q_α. For the usual potential forces this is equivalent to their purely mechanical definition, according to which those are the sums of the covariant components of momentum vector $\vec{p}_\nu = m_\nu \vec{v}_\nu$ [4,8]

$$p_\alpha = \vec{p}_\nu \cdot \frac{\partial \vec{r}_\nu}{\partial q_\alpha} \quad (\alpha = 0, 1, \ldots . n). \tag{8.2}$$

In this formulation the velocity of a particle is of the form $\vec{v}_\nu = (\partial \vec{r}_\nu / \partial q_\alpha)\dot{q}_\alpha$, so that the kinetic energy of the system is a homogeneous quadratic function of the generalized velocities

$$T = \frac{1}{2} a_{\alpha\beta} \dot{q}_\alpha \dot{q}_\beta, \quad a_{\alpha\beta} = m_\nu \frac{\partial \vec{r}_\nu}{\partial q_\alpha} \cdot \frac{\partial \vec{r}_\nu}{\partial q_\beta}. \tag{8.3}$$

On the other hand, the generalized potential of the system in clasical mechanics must to be linear with respect to $\dot{q}_\alpha$

$$V = b_\alpha \dot{q}_\alpha + U(q_\alpha) \equiv V_1 + U, \tag{8.4}$$

what implies that the corresponding Lagrangian (7.3) is

$$\mathcal{L}(q_\alpha, \dot{q}_\alpha) = L - P = \frac{1}{2} a_{\alpha\beta} \dot{q}_\alpha \dot{q}_\beta - b_\alpha \dot{q}_\alpha - U - P \tag{8.5}$$

and hence the generalized moments (8.1) are

$$p_\alpha = a_{\alpha\beta} \dot{q}_\beta - b_\alpha \quad (\alpha = 0, 1, \ldots, n). \tag{8.6}$$

In order to investigate whether this system of equations can be always solved with respect to the generalized velocities $\dot{q}_\beta$, let us suppose that it is not true. Then the corresponding Jacobian would be equal to zero

$$\Delta = \left| \frac{\partial p_\alpha}{\partial \dot{q}_\beta} \right| = |a_{\alpha\beta}| = 0,$$

and this is equivalent to assuming that system of homogeneous equations

$$a_{\alpha\beta} \dot{q}_\beta = \frac{\partial T}{\partial \dot{q}_\alpha} = 0 \quad (\alpha = 0, 1, \ldots, n) \tag{8.7}$$

has nontrivial solutions $\dot{q}_\beta \neq 0$. However, since the kinetic energy is a homogeneous function of $\dot{q}_\alpha$, according to Euler's theorem and (8.7)

$$\frac{\partial T}{\partial \dot{q}_\alpha}\dot{q}_\alpha = 2T = 0$$

and this is possible only if all the generalized velocities $\dot{q}_\beta = 0$. This contradicts the previous conclusion that $\dot{q}_\beta \neq 0$ and therefore

$$\Delta = \left| \frac{\partial^2 \mathcal{L}}{\partial \dot{q}_\alpha \partial \dot{q}_\beta} \right| = |a_{\alpha\beta}| \neq 0, \tag{8.8}$$

what proves that the system of equations (8.6) can be always solved with respect to all the $\dot{q}_\beta$.

This points out a characteristic difference between this and the usual parametric formulation, called the homogeneous formalism [1], where the time is an additional generalized coordinate and a new independent variable $\tau = \tau(t)$ is introduced instead of time. In this case the corresponding Jacobian is equal to zero, and therefore the mechanical system formulated in such a way is a degenerate system in the sense of Dirac [10]. Because of that, the corresponding Lagrange's equations are mutually dependent, and there exists an additional relation between canonical variables.

The transition from the Lagrangian to the Hamiltonian formalism can be effectuated in analogous manner as in the habitual formulation, except that the summing is always from 0 to n. Actually, if we start from the variation of the Lagrangian

$$\delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial q_\alpha}\delta q_\alpha + \frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha}\delta \dot{q}_\alpha,$$

according to the definition of the generalized momenta (8.1) and to the Lagrange's equations (7.9), this relation can be written in the form

$$\delta(p_\alpha \dot{q}_\alpha - \mathcal{L}) = -\dot{p}_\alpha \delta q_\alpha + \dot{q}_\alpha \delta p_\alpha. \tag{8.9}$$

This shows that the expression on the left-hand side, which represents the corresponding Hamiltonian in this case, must to be considered as a function of variables q_α and p_α. Owing to the property (8.8) the generalized velocities can be always presented as functions of generalized momenta, and the passage from variables $(q_\alpha, \dot{q}_\alpha)$ to (q_α, p_α) in the cited expression here is always possible. From the above it follows immediately

$$\dot{p} = -\frac{\partial \mathcal{H}}{\partial q_\alpha}, \quad \dot{q}_\alpha = \frac{\partial \mathcal{H}}{\partial p_\alpha} \quad (\alpha = 0, 1, \ldots, n) \tag{8.10}$$

where

$$\mathcal{H}(q_\alpha, p_\alpha) = p_\alpha \dot{q}_\alpha - \mathcal{L} \tag{8.11}$$

and these are the corresponding Hamilton's canonical equations.

The same result can be reached if we start from the variation of the action, passing from $\mathcal{L}$ to $\mathcal{H}$ in the usual way

$$\delta \mathcal{W} = \delta \int_{t_0}^{t_1} (p_\alpha \dot{q}_\alpha - \mathcal{H})\, dt = \int_{t_0}^{t_1} (p_\alpha \delta \dot{q}_\alpha + \dot{q}_\alpha \delta p_\alpha - \delta \mathcal{H})\, dt$$

and transforming the first integral by the integration by parts. In this manner, by utilizing the boundary conditions, the variation of the action passes into

$$\delta \mathcal{W} = \int_{t_0}^{t_1} \left[-\left(\dot{p}_\alpha + \frac{\partial \mathcal{H}}{\partial q_\alpha} \right) \delta q_\alpha + \left(\dot{q}_\alpha - \frac{\partial \mathcal{H}}{\partial p_\alpha} \right) \delta p_\alpha \right] dt. \tag{8.12}$$

From this relation and Hamilton's variational principle the Hamilton's equations (8.10) follow directly, due to independence of the variations δq_α and because of $\dot{q}_\alpha = \partial \mathcal{H} / \partial p_\alpha$ arising from (8.11).

These canonical equations were formulated also by V. Vujičić [5,8-9], though without any proof for them. Also, in these papers he gave the physical meaning to so introduced Hamiltonian as the complete mechanical energy of the potential systems, but with the inversion of its definition and quoted property. Namely, by (8.3), (8.4) and Euler's theorem

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha} \dot{q}_\alpha = \frac{\partial T}{\partial \dot{q}_\alpha} \dot{q}_\alpha - \frac{\partial V_1}{\partial \dot{q}_\alpha} \dot{q}_\alpha = 2T - V_1$$

and hence the Hamiltonian, according to (7.3), obtains the form

$$\mathcal{H} = T + U + P = \mathcal{E}^{ext}. \tag{8.13}$$

From there one can deduce that $\mathcal{H}$ is the complete mechanical energy of the system, extended by the rheonomic potential P. This proof is certain generalization of the Vujičić's one to the forces with generalized potential, from where follows that the same conclusion remains valid even for the most general case.

REFERENCES

[1] Dirac, P., *Homogeneous variables in classical dynamics* Proc. Camb. Phil. Soc., v. 29 (1933), p.389-400.

[2] Synge, J., *Classical dynamics*, Handbuch der Physik, v.III/1, Springer Verlag, Berlin 1960, p. 210-259.

[3] Mercier, A., *Analytical and canonical formalism in physics*, North-Holland, Amsterdam 1959, p. 177-192.

[4] Vujičić, V., *Ob integrale energii sistem stesnenih nestacionarnimi svjazjami*, Teor. i prim. Mehanika, v. 6 (1980), p. 133-143.

[5] Vujičić, V., *Ob invariantnosti principov v mehanike*, Teor. i prim. mehanika, v. 11 (1985), p. 155-168.

[6] Vujičić, V., *The modification of analytical dynamics of rheonomic systems*, Tensor (N.S.), v. 46 (1987), p. 418-431.

[7] Vujičić, V., *K principe vozmožnih peremeščenij dlja reonomnih sistem*, Prikl. Mehanika, v. 24 (1988), p. 125-127.

[8] Vujičić, V., *On Hamilton's principle for the rheonomous system*, Bull. Acad. Serbe, Cl. Sci. math. (N.S.), v. 16 (1988), p. 37-50.

[9] Vujičić, V., *Dynamics of rheonomic systems*, Matematički institut, Beograd 1990.

[10] Dirac, P., *On generalized Hamiltonian dynamics*, Can. J. Math., v. 2 (1950), p. 129-148.

UNE FORMULATION PARAMETRIQUE DE LA MECHANIQUE DES SYSTEMES RHEONOMES

Cet article contient une formulation paramètrique de la méchanique des systèmes rhéonomes. Son but est de comprendre et exprimer explicitement l'influence du nonstationairité des liaisosns sur le mouvement du système. C'est basé sur la famille des possibles trajectoires variées, tirées à partir de leur initiale position et sur la transition à un nouvel parametre, qui dépend de la trajectoire choisie.

En partant de l'équation fundamentale de la dynamique, on a montré comment de cette façon on peut obtenir le principe d'Alembert-Lagrange correspondant et passer de celui-ci au principe général d'Hamilton, qui devient variationel sous certaines conditions. Sur cette base on a montré comment on peut formuler le formalisme d'Hamilton convenable, avec une hamiltonienne étendue, ce qui est ici toujours possible.

Les résultats obtenus sont dans l'accordance avec le formalisme analitique modifié par V. Vujičić (1980), dans lequel une fonction de temps est choisie comme une coordonée généralisée additionnelle, et un potentiel appellé rhéonomique est introduit aux relations d'énergie.

JEDNA PARAMETARSKA FORMULACIJA MEHANIKE REONOMNIH SISTEMA

Ovaj rad sadrži jednu parametarsku formulaciju mehanike reonomnih sistema. Njegov cilj je da se obuhvati i eksplicitno izrazi uticaj nestacionarnosti veza na kretanje sistema. To je zasnovano na familiji mogućnih variranih putanja, povučenih iz njihovog početnog položaja i na prelazu na jedan novi parametar, koji zavisi od izabrane putanje.

Polazeći od osnovne jednačine dinamike, pokazano je kako se na ovaj način može dobiti odgovarajući d'Alambert-Lagrange-ev princip i potom preći sa ovoga na opšti Hamilton-ov princip, koji pod izvesnim uslovima postaje varijacioni. Na toj osnovi pokazano je kako se može formulisati svojstven Hamilton-ov formalizam sa proširenim Hamiltonijanom, što je ovde uvek moguće.

Dobijeni rezultati su u saglasnosti sa modifikovanim analitičkim formalizmom koji je dao V. Vujičić (1980), u kome je kao dopunska generalisana koordinata izabrana izvesna funkcija vremena, i takozvani reonomni potencijal je uveden u energijske odnose.

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