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THERMAL YIELD OF A HYPERBOLIC DISK

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1. Introduction

If a thin disk of variable thickness is considered, having hyperbolic shape $t(r) = t_0\left(\frac{a}{r}\right)^n$, $a \le r \le b$, $n > 0$, the equlibrium equation between radial and hoop stresses is given by

$$r\frac{d\sigma_R}{dr} + (1-n)\sigma_R = \sigma_\phi. \tag{1}$$

For the thermo-elastic case, bearing in mind Hookean law, this gives rise to the following equation of stresses in radial direction

$$\frac{d^2\sigma_R}{dr^2} + \frac{3-n}{r}\cdot\frac{d\sigma_R}{dr} - \frac{n(1+\nu)}{r^2}\sigma_R + \frac{\alpha E}{r}\cdot\frac{dT}{dr} = 0, \tag{2}$$

while for the plastic range, Tresca yield is encompassed by

$$|\sigma_\phi - \sigma_R| = 2k, \qquad \sigma_\phi\sigma_R < 0, \tag{3}$$

$$|\sigma_\phi| = 2k, \qquad \sigma_\phi\sigma_R > 0. \tag{4}$$

The above equations may be solved subject to boundary conditions

$$\sigma_R(r=a) = -p_a, \qquad \sigma_R(r=b) = -p_b \tag{5}$$

and for known temperature distribution $T(r)$.

2. Elastic case

The solution of the eq. (2) with (5) is known to be [1]

$$\begin{aligned}\sigma_R = {} & \frac{(p_b a^q - p_a b^q)r^p - (p_b a^p - p_a b^p)r^q}{a^p b^q - a^q b^p} + \frac{\alpha E}{q-p}\Bigg\{ r^p\Big[T(r)r^{-p} - T(a)a^{-p} + \\ & + p\int_a^r T(r)r^{-p-1}dr\Big] - r^q\Big[T(r)r^{-q} - T(a)a^{-q} + q\int_a^r T(r)r^{-q-1}dr\Big] + \\ & + \frac{a^q r^p - a^p r^q}{a^q b^p - a^p b^q}\Big[b^q\Big(T(b)b^{-q} - T(a)a^{-q} + q\int_a^b T(r)r^{-q-1}\,dr\Big) - \\ & - b^p\Big(T(b)b^{-p} - T(a)a^{-p} + p\int_a^b T(r)r^{-p-1}dr\Big)\Big]\Bigg\}\end{aligned} \tag{6}$$

and from (1)

$$\sigma_\phi = \frac{(p_b a^q - p_a b^q)(1+p-n)r^p - (p_b a^p - p_a b^p)(1+q-n)r^q}{a^p b^q - a^q b^p} +$$
$$+ \frac{\alpha E}{q-p}\Big\{(1+p-n)r^p\Big[T(r)r^{-p} - T(a)a^{-p} + p\int_a^r T(r)r^{-p-1}dr\Big] -$$
$$- (1+q-n)r^q\Big[T(r)r^{-q} - T(a)a^{-q} + q\int_a^r T(r)r^{-q-1}dr\Big] +$$
$$+ \frac{(1+p-n)a^q r^p - (1+q-n)a^p r^q}{a^q b^p - a^p b^q} \times$$
$$\times \Big[b^q\Big(T(b)b^{-q} - T(a)a^{-q} + q\int_a^b T(r)r^{-q-1}dr\Big) -$$
$$- b^p\Big(T(b)b^{-p} - T(a)a^{-p} + p\int_a^b T(r)r^{-p-1}dr\Big)\Big]\Big\}, \qquad (7)$$

where p and q have values of

$$p, q = \frac{n}{2} - 1 \pm \sqrt{\left(\frac{n}{2}\right)^2 + \nu n + 1}. \qquad (8)$$

Extreme thermoelastic hoop stress values are (Fig. 1) if $p_a = p_b = 0$

$$\sigma_\phi(r=a) = \alpha E T_a(\eta_0 - 1), \qquad (9)$$
$$\sigma_\phi(r=b) = \alpha E T_a(\eta_1 - T_b/T_a), \qquad (10)$$

where

$$\eta_k = \frac{p b^p \left(\frac{b}{a}\right)^{kq} \cdot \int_a^b T(r)r^{-p-1}dr - q b^q \left(\frac{b}{a}\right)^{kp} \cdot \int_a^b T(r)r^{-q-1}dr}{T_a\left[\left(\frac{b}{a}\right)^p - \left(\frac{b}{a}\right)^q\right]} \quad (k=0,1). \qquad (11)$$

If $n = 0$ ($p = 0$, $q = -2$): $\eta_0 = \eta_1$.

3. Plastic yield

Taking into account Tresca case (3) for thermal stresses, with internal pressure p_a, while $p_b = 0$, which reads (Fig. 2)

$$\sigma_\phi - \sigma_R = -2k \qquad (12)$$

and introducing it to the equlibrium (1), it renders with (5) the plastic solution ($a \le r \le c$)

$$\sigma_R = \frac{2k}{n} - \left(p_a + \frac{2k}{n}\right)\left(\frac{r}{a}\right)^n. \qquad (13)$$

From (6) and (7) the corresponding elastic solution is $(c \le r \le b)$

$$\sigma_R = p_c \frac{b^p r^q - b^q r^p}{c^p b^q - c^q b^p} + \frac{\alpha E}{q-p}\Big\{ r^p \Big[T(r) r^{-p} - T(c) c^{-p} + $$
$$ + p \int_c^r T(r) r^{-p-1} dr \Big] - r^q \Big[T(r) r^{-q} - T(c) c^{-q} + q \int_c^r T(r) r^{-q-1} dr \Big] + $$
$$ + \frac{c^q r^p - c^p r^q}{c^q b^p - c^p b^q} \Big[b^q \Big(T(b) b^{-q} - T(c) c^{-q} + q \int_c^b T(r) r^{-q-1} dr \Big) - $$
$$ - b^p \Big(T(b) b^{-p} - T(c) c^{-p} + p \int_c^b T(r) r^{-p-1} dr \Big) \Big] \Big\}, \tag{14}$$

and

$$\sigma_\phi = p_c \frac{(1+q-n) b^p r^q - (1+p-n) b^q r^p}{c^p b^q - c^q b^p} + $$
$$ + \frac{\alpha E}{q-p} \Big\{ (1+p-n) r^p \Big[T(r) r^{-p} - T(c) c^{-p} + p \int_c^r T(r) r^{-p-1} dr \Big] - $$
$$ - (1+q-n) r^q \Big[T(r) r^{-q} - T(c) c^{-q} + q \int_c^r T(r) r^{-q-1} dr \Big] + $$
$$ + \frac{(1+p-n) c^q r^p - (1+q-n) c^p r^q}{c^q b^p - c^p b^q} \Big[b^q \Big(\frac{T(b)}{b^q} - \frac{T(c)}{c^q} + q \int_c^b \frac{T(r)}{r^{q+1}} dr \Big) - $$
$$ - b^p \Big(T(b) b^{-p} - T(c) c^{-p} + p \int_c^b T(r) r^{-p-1} dr \Big) \Big] \Big\}. \tag{15}$$

Using (12) with (13) and (14), the interface pressure is (since $p + q = n - 2$)

$$p_c = \frac{2k(c^p b^q - c^q b^p) + \alpha E c^{n-2} T(c) \left[\left(\frac{b}{c}\right)^p - \left(\frac{b}{c}\right)^q \right]}{(p-n) b^q c^p - (q-n) b^p c^q} + $$
$$ + \frac{\alpha E c^{n-2} \left[q b^q \int_c^b T(r) r^{-q-1} dr - p b^p \int_c^b T(r) r^{-p-1} dr \right]}{(p-n) b^q c^p - (q-n) b^p c^q}. \tag{16}$$

From the continuity of radial stresses at $r = c$ (Fig. 3), required pressure at $r = a$ is

$$p_a = -\frac{2k}{n}[1 + a^n A], \tag{17}$$

where

$$A = \frac{p b^q c^{p-n} - q b^p c^{q-n} - \frac{\alpha E n}{2kc^2} T(c) \left[\left(\frac{b}{c}\right)^p - \left(\frac{b}{c}\right)^q \right]}{(q-n) b^p c^q - (p-n) b^q c^p} - $$
$$ - \frac{\frac{\alpha E n}{2kc^2} \left[q b^q \int_c^b T(r) r^{-q-1} dr - p b^p \int_c^b T(r) r^{-p-1} dr \right]}{(q-n) b^p c^q - (p-n) b^q c^p}.$$

For the first yield ($c = b$) its value becomes

$$p_a^0 = -2k\frac{b^q a^p - b^p a^q - \frac{\alpha E}{2k}a^{n-2}T(a)\left[\left(\frac{b}{a}\right)^p - \left(\frac{b}{a}\right)^q\right]}{(q-n)b^p a^q - (p-n)b^q a^p} + $$
$$+ 2k\frac{\frac{\alpha E}{2k}a^{n-2}\left[qb^q\int_a^b T(r)r^{-q-1}dr - pb^p\int_a^b T(r)r^{-p-1}dr\right]}{(q-n)b^p a^q - (p-n)b^q a^p} \tag{18}$$

and for the ultimate yield ($c = b$) simply

$$p_a^u = -\frac{2k}{n}\left[1 - \left(\frac{a}{b}\right)^n\right], \tag{19}$$

what also follows directly from (13). If alternatively the other Tresca option (4) applies, then

$$\sigma_\phi = -2k \tag{20}$$

and

$$\sigma_R = \frac{2k}{n-1} - \left(p_a + \frac{2k}{n-1}\right)\left(\frac{r}{a}\right)^{n-1} \tag{21}$$

give respective values of the interface pressure

$$p_c = \frac{2k(c^p b^q - c^q b^p) + \alpha E c^{n-2}T(c)\left[\left(\frac{b}{c}\right)^p - \left(\frac{b}{c}\right)^q\right]}{(1+p-n)b^q c^p - (q-n)b^p c^q} + $$
$$+ \frac{\alpha E c^{n-2}\left[qb^p\int_c^b T(r)r^{-q-1}dr - pb^p\int_c^b T(r)r^{-p-1}dr\right]}{(1+p-n)b^q c^p - (1+q-n)b^p c^q}, \tag{22}$$

so that the required at $r = a$ is now

$$p_a = -\frac{2k}{n-1}[1 + a^{n-1}A], \tag{23}$$

where

$$A = \frac{pb^p c^{1+p-n} - qb^p c^{1+q-n} - \frac{\alpha E}{2k}\frac{n-1}{c}T(c)\left[\left(\frac{b}{c}\right)^p - \left(\frac{b}{c}\right)^q\right]}{(1+q-n)b^p c^q - (1+p-n)b^q c^p} - $$
$$- \frac{\frac{\alpha E}{2k}\frac{n-1}{c}\left[qb^p\int_c^b T(r)r^{-q-1}dr - pb^p\int_c^b T(r)r^{-p-1}dr\right]}{(1+q-n)b^p c^q - (1+p-n)b^q c^p}.$$

First yield ($c = a$) occurs with

$$p_a^0 = -2k\frac{b^q a^p - b^p a^q - \frac{\alpha E}{2k}a^{n-2}T(a)\left[\left(\frac{b}{a}\right)^p - \left(\frac{b}{a}\right)^q\right]}{(1+q-n)b^p a^q - (1+p-n)b^q a^p} - $$
$$+ \frac{\alpha E a^{n-2}\left[qb^q\int_a^b T(r)r^{-q-1}dr - pb^p\int_a^b T(r)r^{-p-1}dr\right]}{(1+q-n)b^p a^q - (1+p-n)b^q a^p} \tag{24}$$

and the ultimate yield ($c = b$) for

$$p_a^u = -\frac{2k}{n-1}\left[1 - \left(\frac{a}{b}\right)^{n-1}\right], \tag{25}$$

what also follows directly from (21). For a case without surface pressures ($p_a = p_b = 0$), using eq.(20), the required temperature rise causing the first yield may also be evaluated ($c = a$) from the expression

$$T(a)\left[\left(\frac{b}{a}\right)^p - \left(\frac{b}{a}\right)^q\right] + qb^q \int_a^b T(r)r^{-q-1}dr - pb^p \int_a^b T(r)r^{-p-1}dr = \\ = \frac{2k}{\alpha E}\left[\left(\frac{b}{a}\right)^p - \left(\frac{b}{a}\right)^q\right]. \tag{26}$$

If $n = 0$ ($p = 0$, $q = -2$), the last equation simplifies to

$$T(a) = \frac{2k}{\alpha E} \cdot \frac{1}{1 - \eta_0}, \tag{27}$$

where η_0 is heat exchange efficiency, given by

$$\eta_0 = \frac{2}{T(a)(b^2 - a^2)} \int_a^b T(r)r\, dr. \tag{28}$$

Therefore for $n \neq 0$ equivalent efficiency is defined by (11)

$$\eta_0 = \frac{pb^p \int_a^b T(r)r^{-p-1}dr - qb^q \int_a^b T(r)r^{-q-1}dr}{T(a)\left[\left(\frac{b}{a}\right)^p - \left(\frac{b}{a}\right)^q\right]} \tag{29}$$

enabling the use of eq.(27). However, not only the amplitude $T(a)$ but also the distribution $T(r)$ has to be considered. It is not possible to achive the complete plasticity just by an increase of temperature, unless a decrease of k value due to temperature is taken into account. All presented formulae simply degenerate with constant profile shape ($n = 0$). The only correction required is with eq.(19), where

$$p_a^u = -2k \cdot \ln\left(\frac{b}{a}\right) \approx -2k \cdot \frac{b-a}{a}, \tag{30}$$

what is the same as by eq. (25).

4. Conclusion

Thermal stress elasto-plastic relations have been determined for the assumed hyperbolic disk profile. Thus the known formulae from [2] and [3] have been generalised. No material hardening has been assumed.

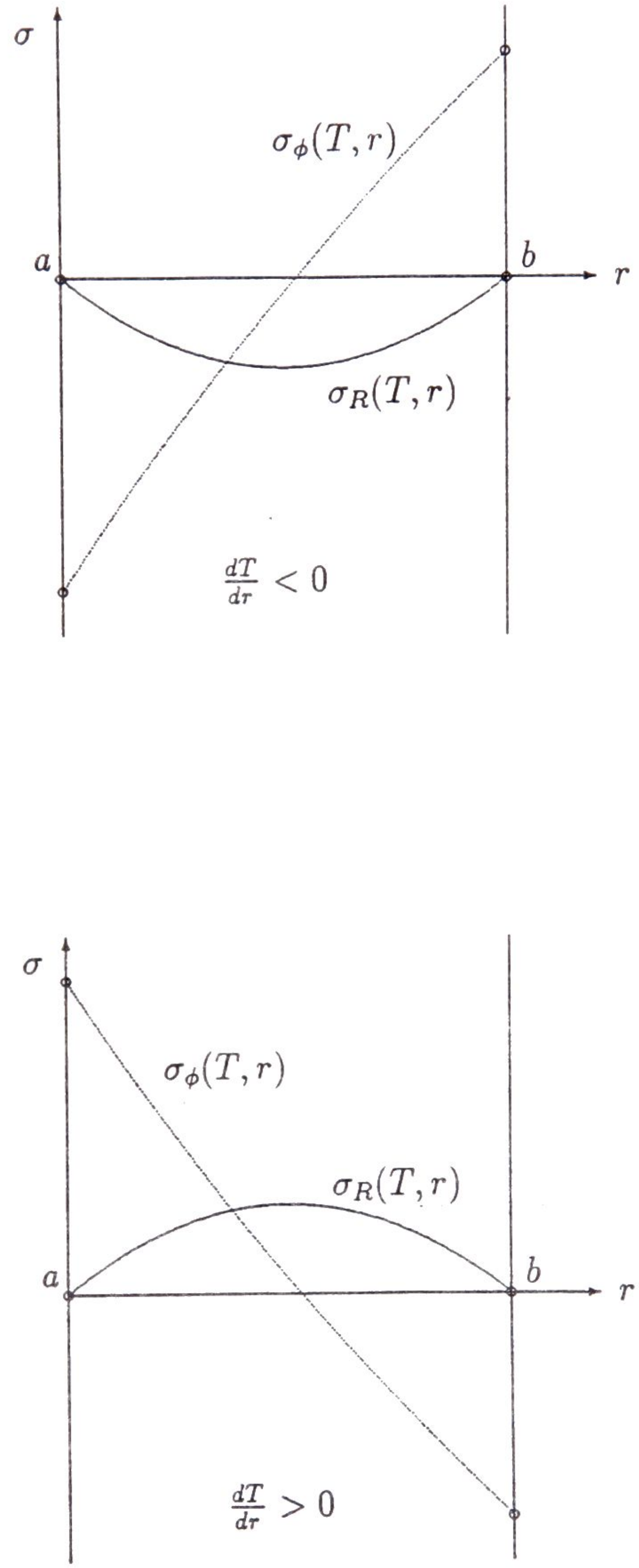

Figure 1: Thermal stresses in a disk

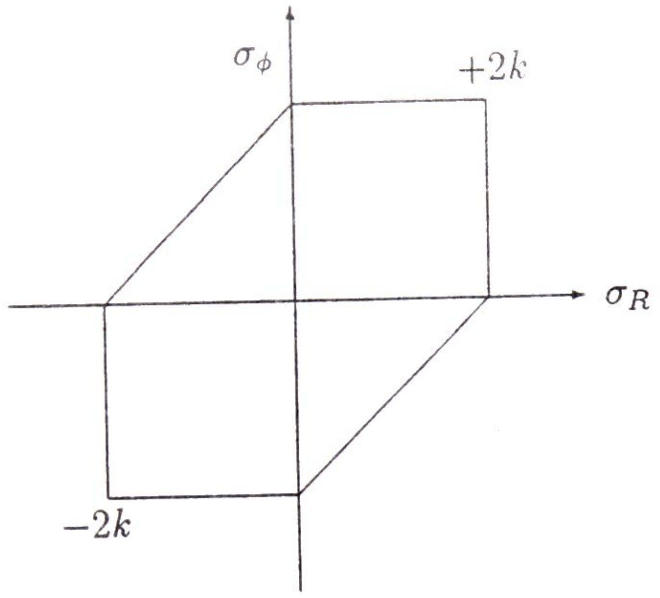

Figure 2: Tresca yield criterion

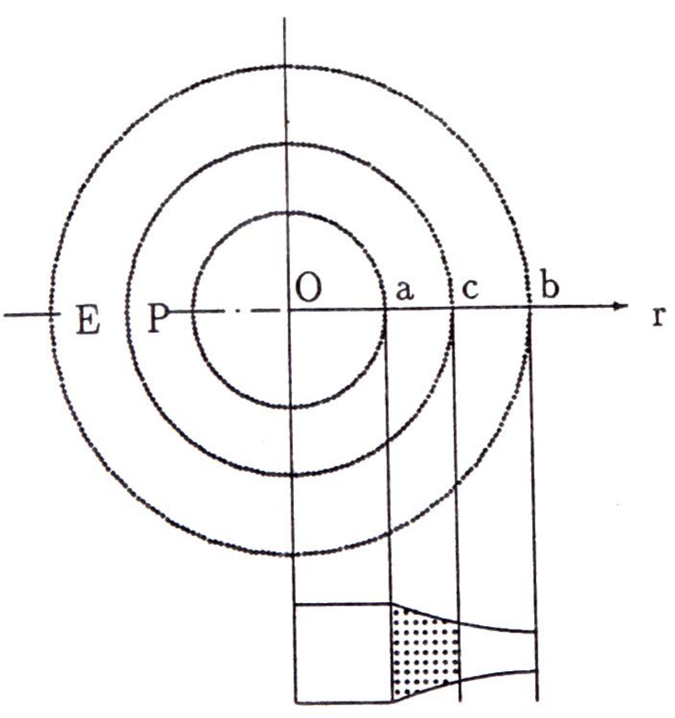

Figure 3: Hyperbolic disk plastification

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THERMOPLASTIZITÄT HYPERBOLISCHER SCHEIBE

In der vorliegenden Arbeit ist das Temperaturspannungsfeld in einer Kreisringscheibe mit variabler Dicke angegeben. Wenn die Flieβspannungsgrenze überschritten ist, gemäss dem Kriterium nach Tresca der Anfang - und der Enddruck an der inneren Fläche, die für die Plastifikation bei konstanter Temperaturverteilung notwending ist bestimmt. Ausserdem behandelt die Arbeit noch das Beispiel ohne oberflächliche Belastung als die zusätzliche Temperatursteigerung das Materialflieβen verursacht. In diesem Fall ist für die Vollplastifikation auch die Festigkeitsverminderung zu berücksichtigen.

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