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BIFURCATION OF AGING VISCOELASTIC COLUMNS

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Introduction

The creep buckling problem of columns has been studied in a large number of publications, refering to linear or power creep laws. There have been included columns in compression or beam-columns, initially straight or imperfect, subjected to simultaneous action of axial compressive forces, bending moments and lateral forces. Depending on the subject of research these analysis have been based on the assumption of small deformation, implying the geometrical linearity, or on the assumption of nonlinear deflection theory [1–19].

The present paper deals with the bifurcation of the equilibrium of aging linearly viscoelastic columns in compression, having different end supports. Based on the assumptions of small deflection and small creep rates, i.e. on the geometrical linearity, the exact linear eigenvalue problem is formulated as an eigenvalue problem of a homogeneous integral equation. This approach leads to the determination of the critical force as a time function wherefrom we obtain the safe load limit value.

In the linear viscoelasticity of hereditary type the bifurcation concept has been already applied in the stability analysis of spherical shells, circular arches and columns in compression [20–22].

Equation of the Problem

We consider a slender column of uniform bending stiffness, having different end supports, loaded by axial compressive forces affecting the column ends. Their directions do not change during the column deformation. The loading is subjected instantaneously at time $\tau = t_0$ and it does not imply rates sufficiently great to cause the excitation of a dynamic response of the column.

The bifurcation of the equilibrium of the aging viscoelastic column will be discussed by means of a non-trivial solution of the integral equation of the problem. For creating this equation it is necessary first to determine the influence function of an aging viscoelastic beam $K^*_{\varphi m}(x,\xi,t,\tau)$ for the deflection slope at time t at point x due to the unit moment at point ξ subjected at time $\tau \geq t_0$.

We start from the known differential equation referring to the corresponding elastic beam:

$$E_0 I \frac{d^4 v}{dx^4} = 1'(x,\xi), \tag{1}$$

(see Appendix 1) and the corresponding end conditions. The solution of equation (1) is the influence function $K^e_{vv}(x,\xi)$, representing the lateral deflection at point x due to the unit lateral force at point ξ. On the basis of [23], [24] and [30] the following is valid

$$K^e_{\varphi m}(x,\xi) = \frac{\partial^2 K^e_{vv}(x,\xi)}{\partial x \partial \xi}. \tag{2}$$

Using the analogy of aging creep and elasticity, after some transformations, we arrive at the equation referring to the aging viscoelastic beam:

$$E_0 I \frac{\partial^4 v^*}{\partial x^4} = 1'(x,\xi) F^*(t,\tau), \tag{3}$$

$F^*(t,\tau)$ being the nondimensional creep function. The solution of Eq (3) is related to the solution of Eq (1) as follows:

$$K^*_{vv}(x,\xi,t,\tau) = K^e_{vv}(x,\xi) F^*(t,\tau), \tag{4}$$

including the end and initial conditions of the beam. The latter is associated with the fact that at time of the load application $t = \tau = t_0$ the response of the beam is instantaneously elastic. Using Eq (2) we finally get:

$$K^*_{\varphi m}(x,\xi,t,\tau) = \frac{\partial^2 K^e_{vv}(x,\xi)}{\partial x \partial \xi} F^*(t,\tau). \tag{5}$$

In order to obtain the integral equation we apply the known method used in elasticity. We determine the deflection slope of an undeformated column. The influence of the column deformation on the equilibrium conditions we introduce by an additional load, representing distributed moments defined as a product of both the axial force of an undeformated column and the real deflection slope. The latter being an unknown function [23], [24].

The magnitudes of the compressive forces are introduced as $\alpha^* P$ where $\alpha^* = \alpha^*(t,\tau)$ is the unknown nondimensional time function which is the load parameter; $P =$ const. $(P > 0)$. The axial compressive force of an undeformated column is:

$$N^* = N^*(t,\tau) = \alpha^*(t,\tau) P, \qquad \tau \geq t_0. \tag{6}$$

It is a time function as well as the deflection slope $\varphi^* = \varphi^*(x,t,\tau)$. That is why we apply the Boltzmann-Volterra principle to obtain the additional load (see Appendix 2):

$$m^* = m^*(x,t,\tau) = \widetilde{N}'(t,\theta)\varphi^*(x,\theta,\tau) = P\widetilde{\alpha}'(t,\theta)\varphi^*(x,\theta,\tau). \tag{7}$$

On the basis of the Bolztmann-Volterra principle the deflection slope φ^*, due to the distributed moments m^*, as a unique column loading, is given by expression:

$$\varphi^*(x,t,\tau) = \int_0^l \widetilde{K}'_{\varphi m}(x,\xi,t,\theta) m^*(\xi,\theta,\tau)\, d\xi. \tag{8}$$

Substituting Eqs (5) and (7) into Eq (8) we finally arrive at the equation of the problem;

$$\varphi^*(x,t,\tau) - P\widetilde{F}'(t,\theta)\widetilde{\alpha}'(\theta,\omega)\int_0^l \frac{\partial^2 K^e_{vv}(x,\xi)}{\partial x\partial\xi}\varphi^*(\xi,\omega,\tau)\,d\xi = 0. \tag{9}$$

For linear elastic materials the nondimensional creep and relaxation function F^* and R^* reduce to the Heaviside step function 1^*, so that the corresponding operators become unity operators (see Appendix 2). The load parameter becomes a constant. $\alpha = \alpha 1^*$, while the deflection slope depends on the coordinate x only: $\varphi = \varphi(x)1^*$. Under these conditions Eq (9) reduces to the well known elastic bifurcation equation [24]:

$$\varphi(x) - \alpha P\int_0^l \frac{\partial^2 K^e_{vv}(x,\xi)}{\partial x\partial\xi}\varphi(\xi)\,d\xi = 0. \tag{10}$$

Analytical Solution

The aim is to determine the conditions for the existence of a non-trivial solution of Eq (9).

Let $\varphi^e_k(x)$ denote the k-th eigenfunction of the homogeneous integral equation of the elastic bifurcation problem, Eq (10), satisfying the expressions:

$$P\int_0^l \varphi^e_k(x)\varphi^e_j(x)\,dx = \delta_{kj}, \qquad \delta_{kj} = \begin{cases} 1 & \text{for } k=j \\ 0 & \text{for } k\neq j, \end{cases} \tag{11}$$

and let α^e_k denote the k-th eigenvalue of the kernel of the same equation, ($k = 1,2,\ldots$).

We consider the columns having the following end supports: a) column with both pin ends, b) column with clamped end at $x=0$ and free end at $x=l$, c) column with both clamped ends, and d) column with clamped end at $x=0$ and pin end at $x=l$. The corresponding end conditions, since they are well-known, will be omitted here. The eigenfunctions for four considered cases are as follows:

$$\begin{aligned}
&\text{a) } \varphi^e_k(x) = \cos k\pi\frac{x}{l},\\
&\text{b) } \varphi^e_k(x) = \sin\frac{2k-1}{2}\pi\frac{x}{l}\\
&\text{c) } \varphi^e_k(x) = \sin 2k\pi\frac{x}{l},\\
&\text{d) } \varphi^e_k(x) = \cos n_k x - \cos n_k l,\ \operatorname{tg} nl = nl,\ n^2 = \frac{P}{E_0 I},\ k = 1,2,\ldots,
\end{aligned} \tag{12}$$

and they fulfill the end conditions.

It is known that the kernel of Eq (10) can be expressed in $\varphi^e_k(x)$ series [24]:

$$\frac{\partial^2 K^e_{vv}(x,\xi)}{\partial x\partial\xi} = \sum_k \frac{1}{\alpha^e_k}\varphi^e_k(x)\varphi^e_k(\xi). \tag{13}$$

The non-trivial solution of Eq (9) is assumed to be expressed in $\varphi_k^e(x)$ series, too:

$$\varphi^*(x,t,\tau) = \sum_k C_k^*(t,\tau)\varphi_k^e(x), \qquad \tau \geq t_0, \tag{14}$$

where the unknown coefficients $C_k^*(t,\tau)$ satisfy the following expressions:

$$C_k^*(t,\tau) = P\int_0^l \varphi^*(x,t,\tau)\varphi_k^e(x)\,dx, \qquad k = 1,2,\dots\,. \tag{15}$$

Substituting Eqs (13)–(15) into Eq (9) we conclude that the following operator expressions have to be fulfilled:

$$[\tilde{1}'(t,\omega) - (1/\alpha_k^e)\widetilde{F}'(t,\theta)\widetilde{\alpha}'(\theta,\omega)]C_k^*(\omega,\tau) = 0, \quad k = 1,2,\dots\,. \tag{16}$$

The conditions for the existence of the non-trivial solution are represented in the operator form, too:

$$(1/\alpha_k^e)\widetilde{F}'(t,\theta)\widetilde{\alpha}'(\theta,\omega) = \tilde{1}'(t,\omega), \qquad k = 1,2,\dots\,. \tag{17}$$

Premultiplying Eq (17) by the relaxation operator $\widetilde{R}'$ ($\widetilde{R}'\widetilde{F}' = \tilde{1}'$) and postmultiplying by $1^*(\omega,t_0)$, the latter means an integration, we determine k time functions $\alpha_k^*(t,t_0)$ ($k = 1,2,\dots$), being the eigenvalues of the kernel of Eq (9):

$$\alpha_k^*(t,t_0) = \widetilde{\alpha}'(t,\omega)1^*(\omega,t_0) = \alpha_k^e(t_0)\widetilde{R}'(t,\omega)1^*(\omega,t_0) = \alpha_k^e(t_0)R^*(t,t_0),$$
$$k = 1,2,\dots, \tag{18}$$

where the initial conditions $\alpha_k^*(t=t_0,\tau=t_0) = \alpha_k^e(t_0)$ ($k = 1,2,\dots$) are used.

We notice that this result refers to the columns with four earlier quoted end supports and to any given creep function of the aging linear viscoelasticity. The corresponding nondimensional relaxation function R^*, appearing in Eq (18), is the solution of the well-known Volterra integral equation of the second kind:

$$F^*(t,t)R^*(t,\tau) - \int_\tau^t \frac{\partial F^*(t,\theta)}{\partial\theta}R^*(\theta,\tau)\,d\tau = 1, \quad t \geq \tau \geq t_0. \tag{19}$$

For concrete, as an aging linear viscoelastic material, there are numerous proposals for the creep function. The corresponding relaxation functions can be generally obtained by a numerical procedure.

Regarding to the argument t, the relaxation function R^* is a decreasing function the values of which lie between 1, for $t = \tau = t_0$, and $R^{*\infty}(t_0) \geq 0$ for $t \to \infty$ and $\tau = t_0$. (If the creep is limited $R^{*\infty}(t_0) > 0$, while for unlimited creep $R^{*\infty}(t_0) = 0$.) Therefore, in this time interval the eigenvalues α_k^* have the following limits:

$$\alpha_k^e(t_0) \geq \alpha_k^*(t,t_0) \geq \alpha_k^e(t_0)R^{*\infty}(t_0), \qquad k = 1,2,\dots\,. \tag{20}$$

In the elastic bifurcation of the equilibrium the eigenvalues $\alpha_k^e(t_0)$ ($k = 1,2,\dots$) of the kernel of Eq (10) represent the elastic critical load parameters of the compressive force. Similarly, in the aging viscoelastic bifurcation of the equilibrium the eigenvalues $\alpha_k^*(t,t_0)$ $(k=1,2,\dots)$ of the kernel of Eq (9) represent the critical load parameters, too.

The first eigenvalue α_1^* is significant for determination of the critical force P^*:

$$P^* = P^*(t,t_0) = \alpha_1^*(t,t_0)P = \alpha_1^e(t_0)PR^*(t,t_0) = P_E(t_0)R^*(t,t_0), \qquad (21)$$

$P_E = P_E(t_0) = \alpha_1^e(t_0)P$ being the corresponding Euler elastic critical force.

According to Eq (20) the magnitudes of the force lie between the following boundaries:

$$P_E(t_0) \geq P^*(t,t_0) \geq P_E(t_0)R^{*\infty}(t_0). \qquad (22)$$

Obviously, through the nondimensional relaxation function R^* in the aging viscoelastic problem the critical load parameters α_k^* $(k=1,2,\dots)$, Eq (18), the critical force P^*, Eq (21), as well as their limits, Eqs (20) and (22), depend on the time of the load application t_0, on the creep law and on the parameters appearing in the accepted creep function.

The upper limits correspond to the values of the elastic problem. They refer to the corresponding elastic beam the elastic modulus of which $E_0 = E(t_0)$ depends on the time of the load application.

The lower limits depend on the time of the load application, too, through the elastic modulus and the limit value of the nondimensional relaxation function R^* when $t \to \infty$.

The safe load limit represents the lowest value of the critical force:

$$P_S^* = P_S^*(t_0) = P_E(t_0)R^{*\infty}(t_0), \qquad (23)$$

determining the force limit below which the aging viscoelastic column remains asymptotically stable.

Special case: hereditary creep. In this special case all considerations and results are valid. Time functions: $F^*(t,\tau)$, $R^*(t,\tau)$, $\alpha^*(t,\tau)$, etc., become the functions of the time argument $t-\tau$, where we adopt $\tau = t_0 = 0$; $E(t) \equiv E_0 = \text{const}$. For solving the problem the Laplace transformation technique applies. The results obtained are known and they do not depend on the time of the load application.

Example

Starting from the aging viscoelastic beam-column having the initial deflection Distefano [15] have derived the expression for the safe load limit according to the Aroutiounian creep function [26] and $E(t) \equiv E_0 = \text{const}$. The stability criterion have been based on the existence of a bound for the ultimate deflection when $t \to \infty$. The safe load limit obtained does not consistently depend on the time of the load application and, except the reversible part of the creep coefficient, the rest of the parameters of the Aroutiounian creep function do not appear.

As an example of application of the results developed here we shall examine the critical force as a time function, Eq (21), using Aroutiounian's law and $E(t) \equiv$

$E_0 = \text{const.}$ for different times of the load application and for different values of the parameters. The nondimensional creep function will be represented in the following form:

$$F^* = F^*(t,\tau) = 1^* + \left(\Phi_R + \Phi_I \frac{t_0}{\tau}\right)[1 - e^{-\gamma(t-\tau)}], \tag{24}$$

where Φ_R and Φ_I are reversible and irreversible parts of the creep coefficient Φ_0 [31]:

$$\Phi_R + \Phi_I = \Phi_0. \tag{25}$$

The values of the coefficient γ are determined from the condition that the limit values of the relaxation functions when $t \to \infty$ corresponding to Aroutiounian's and Dishinger's creep functions are equal for $\Phi_R = 0$ and $\Phi_I = \Phi_0$ [**31**].

The change of the magnitudes of the critical force $P^* = P^*(t, t_0)$ in the time interval $t_0 \leq t < \infty$ are represented in Figs. 1 and 2. Fig. 1 refers to $t_0 = 5$ days, $l/\gamma = 6{,}95751424$ days, $\Phi_0 = 3$, and Fig. 2 to $t_0 = 28$ days. $1/\gamma = 41{,}11306586$ days, $\Phi_0 = 2$, both for different values of the Φ_R and Φ_I satisfying Eq (25).

We notice that the curves $\Phi_I = 0$ (Figs 1 and 2) correspond to the hereditary creep, see Eq (24). Their ultimate values when $t \to \infty$ representing the safe load limit are approximate to the corresponding ones obtained from the mentioned Distefano expression for $\Phi_R = \Phi_0 = 3$ and $\Phi_R = \Phi_0 = 2$, respectively.

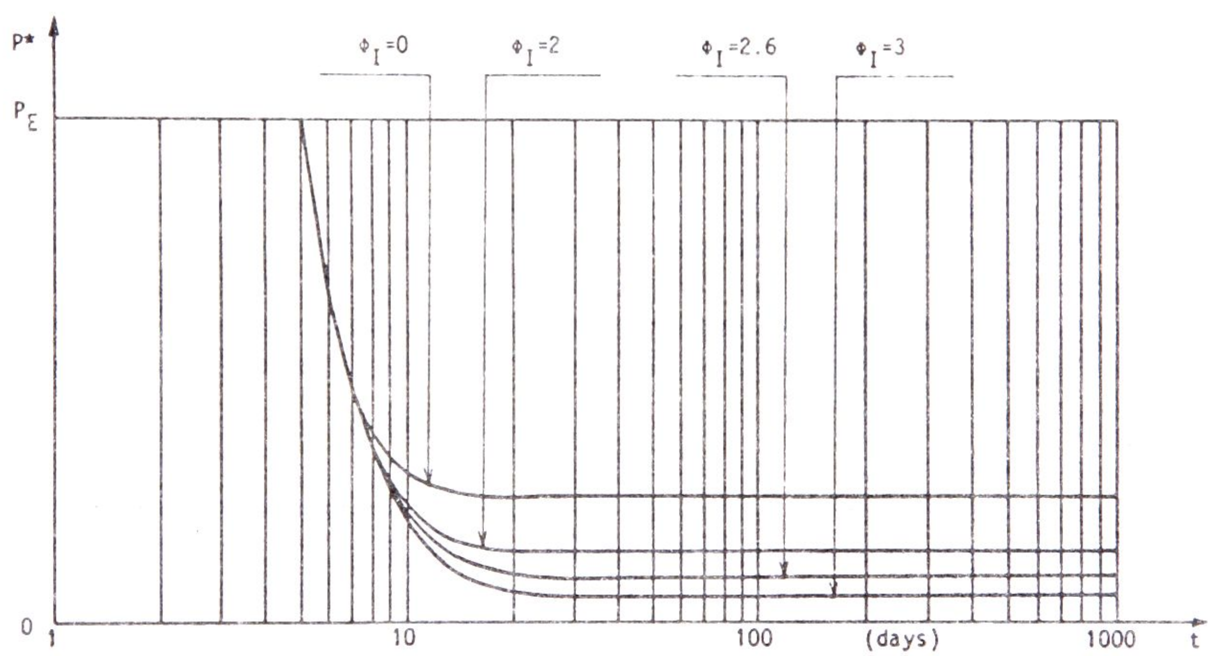

Fig. 1. The magnitudes of the critical force $P^* = P^*(t, t_0)$ according to Aroutiounian law for $t_0 = 5$ days, $1/\gamma = 6.95751424$ days, $\Phi_0 = 3$, for different values of the reversible and irreversible parts of the creep coefficient Φ_0.

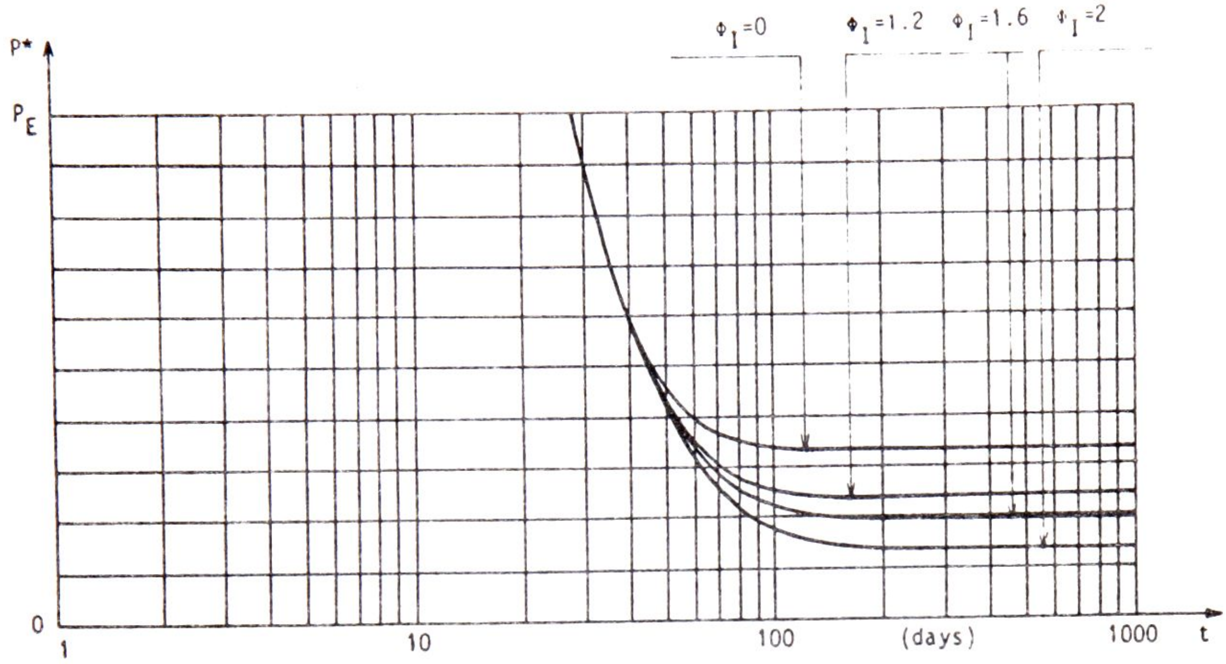

Fig. 2. The magnitudes of the critical force $P^* = P^*(t, t_0)$ according to Aroutiounian law for $t_0 = 28$ days, $1/\gamma = 41.11306585$ days, $\Phi_0 = 2$, for different values of the reversible and irreversible parts of the creep coefficient Φ_0.

Conclusions

In this paper the bifurcation of the equilibrium of slender columns made of aging linearly viscoelastic materials, having different and supports, is analysed. Two generalizations were included in this analysis.

The first regards the mechanical properties of materials where the solution is developed for any given aging creep function $F^*(t, \tau)$.

The second refers to a mathematical procedure. The problem of bifurcation of the equilibrium is formulated as an eigenvalue problem of the homogeneous integral equation, Eq (9). The known solving technique of the elastic bifurcation problem is expanded to the case of aging linear viscoelastic problem using the linear integral operator procedure.

It is shown that every aging viscoelastic eigenvalue linearly depends on the nondimensional relaxation function and that the coefficient of proportionality is the corresponding elastic eigenvalue. Their boundaries in the time inteval $t = \tau = t_0$ and $t \to \infty$ are determined.

The first eigenvalue determines the aging viscoelastic critical force as a time function representing a product of the corresponding Euler critical force and the nondimensional relaxation function. Its limit value when $t \to \infty$ represents the safe load limit. Using Aroutiounian's law the critical force as a time function is examined for different times of the load application and for different values of the parameters.

The theoretical results obtained are valid in the special case of the hereditary creep, too.

Appendix 1 — Notation

$E(t)$	modulus of elasticity at time t
E_0	modulus of elasticity at time t_0; elastic modulus
$F^* = F^*(t,\tau)$	nondimensional creep function
I	minimum moment of inertia of the cross section of the column
$g = g(t)$	any given time function
$K^e_{\lambda\mu}(x,\xi)$	influence function of a elastic beam for the lateral deflection ($\lambda = v$), the deflection slope ($\lambda = \varphi$), at point x due to the unit lateral force ($\mu = v$), the unit moment ($\mu = m$), at point ξ.
$K^*_{\lambda\mu}(x,\xi,t,\tau)$	influence function of a viscoelastic beam for the lateral deflection ($\lambda = v$), the deflection slope ($\lambda = \varphi$), at time t at point x due to the unit lateral force ($\mu = v$), the unit moment ($\mu = m$), at point ξ subjected at time τ.
l	length of the column
$m^* = m^*(x,t,\tau)$	additional load by distributed moments
$N^* = N^*(t,\tau)$	axial force of an undeformated viscoelastic column
P	multiplied by the load parameter represents the axial compressive force subjected to column ends
$P^* = P^*(t,\tau)$	viscoelastic critical force
P_E	Euler's elastic critical force
P_S	safe load limit
$R^* = R^*(t,\tau)$	nondimensional relaxation function
t, θ, ω, τ	time
t_0	time of the load application
$v = v(x)$	elastic lateral deflection
$v^* = v^*(x,t,\tau)$	viscoelastic lateral deflection
x, ξ	coordinate along the column axis
α	elastic load parameter
$\alpha^* = \alpha^*(t,\tau)$	viscoelastic load parameter
α^e_k	k-th eigenvalue of the kernel of Eq. (24)
$\alpha^*_k = \alpha^*_k(t,\tau)$	k-th eigenvalue of the kernel of Eq. (23)
$\varphi(x)$	elastic deflection slope
$\varphi^* = \varphi^*(x,t,\tau)$	viscoelastic deflection slope
$\varphi^e_k(x)$	k-th eigenfunction of Eq. (24)
$\varepsilon = \varepsilon(t,\tau)$	strain
$\sigma = \sigma(t,\tau)$	stress
$1' = 1'(t,\tau)$	Dirac function $\delta(t-\tau) = \delta(\tau - t)$
$1^* = 1^*(t,\tau)$	Heaviside step function $H(t-\tau)$.

Appendix 2 — Some mathematical explanations

The uniaxial stress-strain relationship describing the mechanical properties of an aging lineal viscoelastic material can be symbolically written in the operator form:

$$\varepsilon(t,\tau) = (1/E_0)\widetilde{F}'(t,\theta)\sigma(\theta,\tau), \qquad \tau \geq t_0\,, \tag{A.1}$$

expressing the Boltzmann-Volterra principle, where:

$$F' = F'(t,\theta) = -\frac{\partial F^*(t,\theta)}{\partial\theta}, \qquad \text{for } t > \theta, \tag{A.2}$$

and:

$$\widetilde{F}'\sigma = \widetilde{F}'(t,\theta)\sigma(\theta,\tau) = -\int_\tau^t \frac{\partial F^*(t,\theta)}{\partial\theta}\sigma(\theta,\tau)\,d\theta, \quad \tau \geq t_0, \tag{A.3}$$

$\widetilde{F}'$ being the linear integral operator, associated to the function F'.

$$F^* = F^*(t,\tau) = \widetilde{F}'(t,\theta)\,1^*(\theta,\tau), \qquad \tau \geq t_0, \tag{A.4}$$

F^* being the integral of the function F'.

Considering $\widetilde{F}'$ as a linear integral operator, associated to the function F', its definition is given by Eq (A.3). Depending on the function the sign "−" or "+" can appear. Analogous to Eq (A.4) $\widetilde{R}'1^* = R^*$, $\widetilde{\alpha}'1^* = \alpha^*$, etc., represent the integrals of the corresponding functions R', α', etc.

The unity operator $\tilde{1}'$ is associated to the Dirac function $1'$. Each linear integral operator multiplied by the unity operator gives this operator, keeping in mind that the commutative law is valid as well as $\tilde{1}'1^* = 1^*$.

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БИФУРКАЦИЯ СТЕРЖНЕЙ ИЗ СТАРЕЮЩЕГО ЛИНЕЙНОГО ВЯЗКОУПРУГОГО МАТЕРИАЛА

Представлена теоретическая студия бифуркации равновесного состояния прямых витких стержней из стареющего линейного вязкоупругого материала. Проблема сформулирована как проблема собственных значений однородного интегрального уравнения. Точное решение относится к стержням с различными типами укреплений на концах и к любой даной функции ползучести. Определяется критическая сила как функция времени. В случае функции ползучести со старением Арутюняна, рассматриваются влияние момента нагружения как и влияние значений величины параметров являющихся в функции ползучести.

BIFURKACIJA ŠTAPOVA OD LINEARNOG VISKOELASTIČNOG MATERIJALA SA OSOBINOM STARENJA

U radu je razmatran problem bifurkacije ravnoteže pravih štapova od linearnog viskoelastičnog materijala sa osobinom starenja. Problem je formulisan preko svojstvenih vrednosti homogene integralne jednačine. Tačno rešenje se odnosi na proizvoljnu funkciju puzanja ovih materijala i na štapove različitih vrsta oslanjanja. Određena je kritična sila izvijanja kao funkcija vremena. Posebno, za Arutunjanovu funkciju puzanja ispitani su uticaji vrednosti parametara ove funkcije i starosti materijala prilikom opterećavanja na veličine kritičnih sila izvijanja tokom vremena.

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