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SOME ESTIMATIONS OF THE NOLINEAR OSCILLATOR AMPLITUDE SUBJECTED TO RANDOM PARAMETRIC EXITATION

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Introduction

In engineering practice an important role is played by the equations of the form:

$$\ddot{y} + g(y)\dot{y} + f(y,t)y = 0$$

which represent mathematical models of the elastic systems motion with one degree of freedom or discretization of the dynamic model of an elastic body in the basic form of the dynamic equilibrium. Concerning these equations the researcher's interest is concentrated on exploring the bifurcational behaviour of the solutions for the sake of exploring possible stable and unstable elastic equilibrium forms of the system and their possible stochasticity under deterministic conditions; it is also concentrated upon behaviours of certain solutions at the emergence of random influences and parameters, that is, of random forces.

Many authors, for instance, Ariartanam, Wei Chan Xie, [1], [2], and others, have studied various examples in Mechanics using the mathematical models as special cases of the previously given equation. For example, they have studied the problems of the elastic systems dynamic stability, of elastic form of beams, plates and shells under the action of the random axial stresses, that is, compressive loads in the middle surface of plates and shells or random excitations of elastic beam ends or plate contours whose behaviour can be described by the previously-given equation by the discretization of the mathematical model.

Having in view the previously mentioned examples which are important for the exploration of the elastic systems stability the aim of this paper is to enrich the known research methods by another one based on the mathematical theory of the stochastic differential equations. The method itself will be illustrated on the example of the nonlinear oscillator subjected to the random excitation of the Gaussian white noise type.

This paper started from the Monograph [7], and the information from reference [14] in which the influence of some coefficients on the mean square value of the non-linear oscillator amplitude is considered when subjected to the random excitation of the type of the Gaussian white noise, which it is mathematically described as the Gaussian stationary wideband random process of small intensity and correlation time with mathematical expectation equal to zero.

In order to carry out a desired analysis we shall describe in short some results of the quoted references.

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a fixed complete probability space on which the Gaussian random process $f(t,\omega)$ is defined, whose correlation function and spectral density function are, respectively:

$$\mathsf{K}(\tau) \stackrel{\text{def}}{=} \mathsf{E}\{f(t,s), f(t+\tau,\omega)\} = \sigma^2\delta(\tau)$$

$$\mathsf{S}(\omega) \stackrel{\text{def}}{=} \int_{\infty}^{+\infty} \mathsf{K}(\tau)\cos\omega\tau\, d\tau.$$

Note that all the random processes and random variables will be considered on the given probability space.

The nonlinear oscillator motion is mathematically described as a stochastic differential equation:

$$\ddot{y} + (\alpha + \beta y^2)\dot{y} + [\omega_0^2 + \gamma y^2 + f(t,\omega)]y = 0 \tag{1}$$

in which α, β, γ are positive constants, small comparing to one, of the same intensity order as the spectral density of the random process $f(t,\omega)$, and ω_0 is an natural frequency of the undisturbed (unperturbed) system oscillation.

By introducing new variables $y_1 = y$, $y_2 = \dot{y}$, the differential equation (1) is transformed into the system of the random differential equations:

$$\frac{dy_1}{dt} = y_2$$

$$\frac{dy_2}{dt} = -\alpha y_2 - \beta y_1^2 y_2 - \omega_0^2 y_1 - \gamma y_1^3 - y_1 f(t,\omega).$$

By substituting the variables $y_1(t)$ and $y_2(t)$ and by their representation in the standard form:

$$y_1(t) = a(t)\cos\Phi(t) \qquad \Phi(t) = \omega_0 t + \theta(t)$$

$$y_2(t) = -a(t)\,\omega_0 \sin\Phi(t)$$

in which $a(t)$ is the solution amplitude for elongation, whereas $\theta(t)$ is a phase and $\Phi(t)$ is a phase angle, the previous system is transformed into:

$$\frac{da(t)}{dt} = -(\alpha + \beta a^2\cos^2\Phi)a\sin^2\Phi + \frac{\gamma}{\omega_0^2}a^3\cos^3\Phi\sin\Phi + \frac{a}{\omega_0}\sin\Phi\cos\Phi f(t)$$

$$\frac{d\theta(t)}{dt} = -(\alpha + \beta a^2\cos^2\Phi)a\sin\Phi\cos\Phi + \frac{\gamma}{\omega_0^2}a^2\cos^4\Phi + \frac{1}{\omega_0}\cos^2\Phi f(t).$$

Considering the assumptions about the parameters α, β, γ of the dynamic system and characteristics of the Gaussian process $f(t)$, the Khasminsky averaging method can be applied to the last system (see [12]), the method which is based upon the idea of the Krilov-Bogoliubov-Mitropolsky [4], [6], [16] averaging method. Without going into details, as a result of the application of this method, random differential equations of the Itô (later marked as SDE) with respect to the averaged amplitude $\overline{a}(t)$ and the averaged phase $\overline{\theta}(t)$ in "the first approximation" are given. SDE will be of interest to us with respect to the averaged amplitude:

$$d\overline{a}(t) = \left[\frac{3}{16}\,\frac{S(2\omega_0)}{\omega_0^2}\,\overline{a}(t) - \frac{1}{2}\left(\alpha + \frac{\beta}{4}\overline{a}^2(t)\right)\overline{a}(t)\right] dt + \sqrt{\frac{S(2\omega_0)}{8\omega_0^2}}\, dW(t), \quad t > 0 \quad (2)$$

$\overline{a}(0) = \eta$ with the probability 1, on which $W = [(W_t, \mathcal{F}_t), t \geq 0]$ is the standard Wiener process defined on the given probability space $(\Omega, \mathcal{F}, \mathcal{P})$ adapted to a filtration satisfying the usual conditions. The solution of this SDE, if it exists, is almost surely a continuous homogeneous Markovian process measurable with respect to $(\mathcal{F}_t, t \geq 0)$ (compatible in the sense that for each t a random occurrence ω; $a(s,\omega)$, $s \leq t \in \mathcal{F}_t$).

Since the SDE (2) cannot be effectively solved, (see [7], [14], [1]) the usual procedure for determining the stationary probability density $p(a)$ is used from the corresponding Kolmogorov-Fokker-Planck equation for a conditional probability density, wherefrom mean square value of the stationary averaged amplitude is obtained:

$$\mathsf{E}\overline{a}^2 = \int_0^{+\infty} a^2 p(a)\, da = \frac{S(2\omega_0) - 4\omega_0^2\alpha}{\beta\omega_0^2}\,.$$

Main results

The idea of this paper is to estimate the moment of the second order of the averaged amplitude by means of solving a series of linear random differential equations of Itô type.

For the sake of simpler writing, we are introducing the following notations:

$$\frac{3}{16}\,\frac{S(2\omega_0)}{\omega_0^2} - \frac{\alpha}{2} = A, \qquad \frac{\beta}{8} = B, \qquad \sqrt{\frac{S(2\omega_0)}{8\omega_0^2}} = C$$

Therefore, SDE (2) becomes:

$$d\overline{a}(t) = [A\overline{a}(t) - B\overline{a}^3(t)]\, dt + c\overline{a}(t)\, dW(t), \qquad t > 0 \qquad (3)$$

$\overline{a}(0) = \eta$ with probability 1.

Obviously, one solution is $\overline{a}(t) = 0$, $\forall t \geq 0$, with probability 1. Since the mean square value of the averaged amplitude is to be estimated, that is $\mathsf{E}\overline{a}^2(t)$, we are introducing the substitution:

$$\mathsf{Y}(t) = \overline{a}^2(t) \qquad \forall t \geq 0 \qquad \text{with probability 1.}$$

Therefore,

$$\mathsf{Y}(t) \geq 0 \qquad \forall t \geq 0 \qquad \text{with probability 1.}$$

By applying the Itô formula for stochastic differentiation of the complex function $\mathsf{F}(x) = x^2$ (see, for instance, Gihman [5]), the SDE with respect to an unknown random process $\mathsf{Y}(t)$ [3] is obtained:

$$d\mathsf{Y}(t) = [(2A + C^2)\mathsf{Y}(t) - B\mathsf{Y}^2(t)]\,dt + 2C\overline{\mathsf{Y}}(t)\,dW(t), \qquad t > 0 \tag{4}$$
$$\mathsf{Y}(0) = \eta^2 \qquad \text{with probability 1.}$$

The solution of this SDE is also a Markovian homogeneous process compatible with respected to the same family of the σ-algebras $(\mathcal{F}_t, t \geq 0))$.

Since the SDE (4) cannot be effectively solved, we shall compare its solution with the one of some solvable Itô SDE, for instance, of the linear Itô SDE. In this sense we shall present one comparison theorem of solutions of two Itô SDE.

COMPARISON THEOREM. *If the coefficients of at least one Itô SDE*

$$d\mathsf{X}_i(t) = f_1(t, \mathsf{X}_i(t))\,dt + g(t, \mathsf{X}_i(t))\,dW(t), \qquad t > 0$$
$$\mathsf{X}_i(0) = \eta \qquad \textit{with probability } 1,\ i = 1, 2,$$

satisfy the uniform Lipschitz condition and the condition of the restriction of growth with respect to the second argument in its own space of definition $[0, \alpha) \times \mathrm{R}$, *and if*

$$f_1(t, x) \leq f_2(t, x) \qquad \forall (t, x) \in [0, \alpha) \times \mathrm{R},$$

then

$$\mathsf{X}_1(t) \leq \mathsf{X}_2(t) \qquad \textit{with probability } 1.$$

Since in the equation (4) the drift coefficient

$$f_1(x) = (2A + C^2)x - Bx^2$$

satisfies the Lipschitz condition and the condition of the restriction of growth only for a limited x, we shall perform section of the random process $\mathsf{Y}(t)$ at the distance n, $n \in \mathrm{N}$, from the axis t, namely, we shall define the stopping time (see Fig. 1) with respect to the family of the σ-algebras $(\mathcal{F}_t, t \geq 0)$:

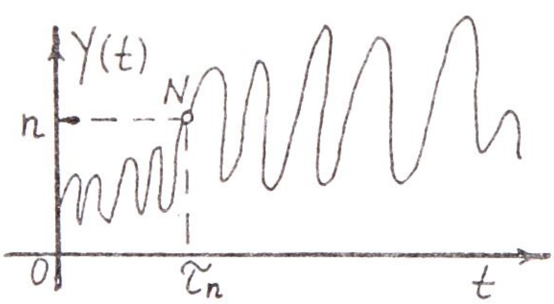

Fig. 1

$$\tau_n = \begin{cases} \inf\{t : \mathsf{Y}(t) \geq n\} & \\ \infty & \mathsf{Y}(t) < n\,. \end{cases}$$

This is, therefore, the first time moment in which the random process $\mathsf{Y}(t)$ achieves value n. Because of the almost sure continuity of the random process $\mathsf{Y}(t)$, τ_n is a random variable adapted to the σ-algebras $(\mathcal{F}_t, t \geq 0)$.

Let us look at the SDE

$$d\mathsf{Y}_n(t) = [(2A + C^2)\mathsf{Y}_n(t) - B\mathsf{Y}_n^2(t)]\,dt + 2C\mathsf{Y}_n(t)\,dW(t), \quad t \in [0, \tau_n]$$
$$\mathsf{Y}_n(0) = \eta^2 \qquad \text{with probability 1.} \tag{5}$$

The drift coefficient $f(x) = (2A + C^2)x - Bx^2$ and the diffusion coefficient $g(x) = x$ satisfy the Lipschitz condition as well as the one of the limited growth at $[0, n]$; therefore, according to the basic existence and uniqueness theorem of the solution of the Itô SDE (see [8], [5], [3]) it follows that the SDE (5) has a unique solution $\mathsf{Y}_n(t)$ defined on the random interval $[0, \tau_n]$, almost surely continuous and measurable with respect to $((\mathcal{F}_t),\ t \in [0, \tau_n])$. According to the local uniqueness theorem (see [5]), it follows that it is:

$$\mathsf{Y}(t) = \mathsf{Y}_n(t), \qquad t \in [0, \tau_n], \quad \text{with probability 1.}$$

The solution of the SDE (5) can be compared with the one of the linear homogeneous Itô SDE:

$$d\mathsf{Z}(t) = (2A + C^2)\mathsf{Z}(t)\,dt + 2C\mathsf{Z}(t)\,dW(t), \qquad t \geq 0,$$
$$\mathsf{Z}(0) = \eta^2 \qquad \text{with probability 1.}$$

According to the comparison theorem and since $(2A+C^2)x - Bx^2 \leq (2A+C^2)x$, it follows that:

$$\mathsf{Y}_n(t) \leq \mathsf{Z}(t), \qquad t \in [0, \tau_n], \quad \text{with probability 1.}$$

It can be proved (see, for instance [5], [9]) that it is:

$$\lim_{n\to\infty} \tau_n = \infty \qquad \text{with probability 1,}$$

as well as

$$\lim_{n\to\infty} \mathsf{Y}_n(t) = \mathsf{Y}(t), \qquad t \in [0, \infty), \quad \text{with probability 1.}$$

Therefore,

$$\mathsf{Y}(t) \leq \mathsf{Z}(t), \qquad t \in [0, \infty), \quad \text{with probability 1.}$$

It is known (see [3]) that the linear homogeneous Itô SDE with the constant coefficients

$$dX(t) = \alpha X(t)\,dt + \beta X(t)\,dW(t), \qquad X(0) = \gamma \ \text{ with probability 1}$$

has a solution

$$X(t) = \gamma \exp[(\alpha - \beta^2/2)t + \beta W(t)]$$

and the moment of the order k

$$\mathsf{E}|X(t)|^k = \mathsf{E}|\gamma|^k \exp[k(\alpha - \beta^2/2)t + k^2\beta^2 t/2].$$

In our case:

$$\mathsf{Z}(t) = \eta^2 \exp[(2A - C^2)t + 2CW(t)], \qquad t \geq 0$$

$$\mathsf{E}|\mathsf{Z}(t)|^k = \eta^{2k} \exp\{[k(2A - C^2) + 2k^2C^2]t\}, \qquad t \geq 0$$

Therefore,

$$\mathsf{Z}(t) = \eta^2 \exp\left(\frac{S(2\omega_0) - 4\omega_0^2\alpha}{4\omega_0^2}t + 2\sqrt{\frac{S(2\omega_0)}{8\omega_0^2}}\,W(t)\right), \qquad t \geq 0$$

$$\mathsf{E}|\mathsf{Z}(t)|^k = \eta^{2k} \exp\left(\frac{k[S(2\omega_0)(1+k) - 4\alpha\omega_0^2]}{4\omega_0^2}t\right), \qquad t \geq 0,$$

namely,

$$\overline{a}^2(t) = \mathsf{Y}(t) \leq \mathsf{Z}(t), \qquad t \geq 0, \quad \text{with probability 1}$$

and

$$\mathsf{E}\overline{a}^2(t) = \mathsf{E}\mathsf{Y}(t) \leq \mathsf{E}\mathsf{Z}(t) = \eta^2 \exp\left(\frac{S(2\omega_0) - 4\omega_0^2\alpha}{2\omega_0^2}t\right), \qquad t \geq 0.$$

Since $\mathsf{Z}(t) \to 0$ when $t \to \infty$ with probability 1 if and only if it is:

$$\frac{S(2\omega_0) - 4\omega_0^2\alpha}{4\omega_0^2} < 0,$$

then $\overline{a}^2(t) \to 0$ when $t \to \infty$ with probability 1 if it is:

$$\alpha > \frac{S(2\omega_0)}{4\omega_0^2}\,. \tag{7}$$

Analogously, $\mathsf{E}\overline{a}^2(t) \to 0$ when $t \to 0$ if

$$\alpha > \frac{S(2\omega_0)}{2\omega_0^2}. \tag{8}$$

Therefore, the condition (7) represents the sufficient condition for the uniform asymptotic almost sure stability of the averaged amplitude, whereas (8) is a sufficient condition of the uniform asymptotic mean square stability of the averaged amplitude.

Remark. In the general case the solution of the SDE (3) can be of arbitrary sign so the drift coefficient cannot be majorated by the linear function which is another reason for transfer to the SDE (4).

Conclusions

This relatively simple method of estimation of the averaged amplitude is illustrated on the example which allows for the application of others, mostly classical estimation methods and it can be inspiring while being used in dealing with the problems which cannot be solved in the traditional way such as, for instance, the case when the stationary probability density cannot be determined from the Kolmogorov-Fokker-Planck equation. It should also be noted that the comparison cannot be necessarily done with the linear Itô SDE; it can be done some other well-chosen effectively solvable Itô SDE.

Since after applying the Khasminsky averaging method many problems of estimation of amplitudes of various types of nonlinear oscillators subjected to one or more independent random excitations of the Gaussian white noise type, are reduced to the estimation of the solution of an autonomous Itô SDE, the authors' intention is to proceed in their further work with searching for a way an effective estimation of the mathematical expectation of the different order of solution of the general autonomous Itô SDE:

$$dX(t) = f(X(t))\,dt + g(X(t))\,dW(t), \qquad t \in [t_0, +\infty)$$
$$X(t_0) = \eta \qquad \text{with probability } 1.$$

One of the ways would be a search for an appropriate substitution by which this SDE will transform into the SDE with a linear diffusion coefficient which represents a necessary condition for comparison with liner SDE.

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REFERENCES

[1] Ariaratnam, S.T., (1972), *Stability of mechanical systems under stochastic parametric exitation*, Proc. IUTAM Symposium on Stability of Stochastic Dynamical Systems, Lecture Notes in Mathematics **294**, Springer-Verlag, 291.

[2] Ariaratnam, S.T. and Wei Ch. Xie, (1989), *Lyapunov exponent and rotation number of a two-dimensional nilpotent stochastic system*, in: *Dynamic and Stability of Systems*, Oxford University Press.

[3] Arnold, L., (1973), *Stochastic Differential Equations*, Theory and Applications, John Wiley & Sons, New York.

[4] Bogoliubov, N.N., and Mitropolsky, Y.A., (1961), *Asymptotic Methods in the Theory of Nonlinear Oscillations*, Gordon and Breach, New York.

[5] Gihman, I.I., Skorohod, A.V., (1982), *Stochastic Differential Equations and Applications*, Naukova dumka, Kiev, (in Russian).

[6] Hedrih, K., (1975), *Izabrana poglavlja teorije nelinearnih oscilacija*, Univerzitet u Nišu, Niš.

[7] Ibrahim, A.R., (1985), *Parametric Random Vibration*, John Wiley & Sons, Inc.

[8] Ikeda, N., and Watanabe, S., (1981), *Stochastic Differential Equations and Diffusion Processes*, North-Holland.

[9] Janković, S., (1987), *Iterative procedure for solving stochastic differential equations*, Mathematica Balcanica, New Series, **1** (2), 64–71.

[10] Janković, S., (1987), *Neki iterativni postupci i granične teoreme u teoriji slučajnih diferencijalnih jednačina*, Doktorska disertacija, Prirodno-matematički fakultet, Beograd.

[11] Khasminski, R.Z., (1966), *Predeljnaja teorema dla rešenij uravnjenij so slučajnoj pravoj častju*, Teorija verovatn. i jejo primenjenija, 444–462 (in Russian).

[12] Khasminski, R.Z., (1966), *A limit theorem for the solution of differential equations with random right-hand sides*, Theory Probab. Appl. **11**, 390.

[13] Rašković, D., (1965), *Teorija oscilacija*, Naučna knjiga, Beograd.

[14] *Stohastički procesi u dinamičkim sistemima sa primenama na mašinske sisteme*, studija (1989), Scientific project, K. (Stevanović) Hedrih, supervisor, and P. Kozić, R. Pavlović, Sl. Mitić, V. Nikolić and al., Mechanical Engineering Faculty, University of Niš.

[15] Stratonovich, R.L., (1976), *Topics in the Random Noise, Vol. II*, New York, Gordon and Breach.

[16] Vujičić, V., (1967), *Teorija oscilacija*, Savremena administracija, Beograd.

НЕКОТОРЫЕ ОЦЕНКИ УСТОЙЧИВОСТИ АМПЛИТУДЫ НЕЛИНЕЙНОГО ОСЦИЛЛЯТОРА ПРИ ВОЗДЕЙСТВИИ СЛУЧАЙНОГО ПАРАМЕТРИЧЕСКОГО ВОЗБУЖДЕНИЯ

В этой работе рассматриваются некоторые средне квадратические и почти известные устойчивости нелинейного осциллятора при воздействии случайного параметрического возбуждения в форме широкополосного гауссовского белого шума малого интенситета и корреляционого времени. Известно, что применением метода усреднения Хасминьского на стохастическое дифференциальное уравнение осциллятора получается стохастическое дифференциальное уравнение Ито. Усредненная амплитуда, как решение этого дифференциального уравнения, оценивается с помощью метода сравнения с решением некоторого линейного стохастического дифференциального уравнения Ито.

NEKE OCENE STABILNOSTI AMPLITUDE NELINEARNOG OSCILATORA PRI DEJSTVU SLUČAJNE PARAMETARSKE POBUDE

U ovom radu razmatrane su neke ocene srednje kvadratne i skoro izvesne stabilnosti nelinearnog oscilatora pri dejstvu slučajne pobude tipa širokopojasnog (wideband random processes) Gauss-ovog belog šuma (the white noise-limited white noise) malog intenzieta i korelacionog vremena (correlation time). Poznato je da se primenom metode usrednjenja Khasminskog na stohastičku diferencijalnu jednačinu oscilatora dobija stohastička diferencijalna jednačina Itô-a. Usrednjena amplituda, kao rešenje ove jednačine ocenjuje se metodom upoređivanja sa rešenjem neke linearne slučajne diferencijalne jednačine Itô-a.

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