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ON THE DERIVATION OF E. CESÀRO'S FORMULA IN CURVILINEAR COORDINATES[1]

Z. V. Drašković

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1. Introduction

This paper is the result of an attempt to use the idea of an absolute or covariant integral for determining the displacement vector coordinates from infinitesimal strain tensor coordinates, prescribed in an arbitrary curvilinear coordinate system.

2. An absolute or covariant tensor field integral in Euclidean space

It has been shown in [8] that, for an absolute integral[2] (or a covariant one) of the absolute differential of a sufficiently smooth vector function $\boldsymbol{v}$, from the point P_0 to the point P on an arbitrary curve in Euclidean space, one can write:

$$(1)\qquad \int_{P_0P}^{\nabla} g_{i}^{\cdot p}(M,P)\,Dv^{i}(M) = v^{p}(P) - v^{i}(P_0)g_{i}^{\cdot p}(P_0,P),$$

where M is the "current" point of integration, and $g_{i}^{\cdot p}$ is the shifting operator ("Euclidean shifter"; [1, p. 806]); Einstein's summation convention for diagonally repeated indices is used, and all Latin indices have the range $1,2,3$.

It is clear that relation (1) can also be extended on any tensor field; e.g. for a second order tensor $\boldsymbol{t}$ we shall have:

$$(2)\qquad \int_{P_0P}^{\nabla} g^{i}_{\cdot m}(M,P)g^{j}_{\cdot n}(M,P)\,Dt_{ij}(M) = t_{mn}(P) - t_{ij}(P_0)g^{i}_{\cdot m}(P_0,P)g^{j}_{\cdot n}(P_0,P)\,.$$

The last member in this expression (as well as the last member in (1)), being obtained by a parallel displacement of a tensor $t_{ij}(P_0)$, is a covariantly constant tensor field.

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[2]For the first time this notion was introduced in [3].

3. Determination of the displacement vector coordinates corresponding to the given strain tensor coordinates in an arbitrary curvilinear coordinate system

Let us concern ourselves with the proof that the three-dimensional compatibility conditions:

$$e_{ij,kl} - e_{ik,jl} - e_{lj,ki} + e_{kl,ij} = 0 \tag{3}$$

(where e is the Eulerian infinitesimal strain tensor, while the comma denotes covariant differentiation with respect to the three-dimensional metric tensor) are necessary and sufficient conditions for the existence (in a simply-connected region) of a displacement field u such that:

$$u_{i,j} + u_{j,i} = 2e_{ij} \,. \tag{4}$$

We shall proceed similarly as in [9, pp. 56–57], but without supposing that rectangular Cartesian coordinates are in question. Let us start from the relation (cf. e.g. with (8.1) in [9]):

$$u_{i,j} = e_{ij} - \omega_{ij} \,, \tag{5}$$

where $u_{i,j}$ are the displacement gradients, and:

$$\omega_{ij} = \tfrac{1}{2}(u_{j,i} - u_{i,j}) \tag{6}$$

are the linear Eulerian rotation tensor coordinates. From (5) follows that a displacement vector absolute differential is:

$$Du_i = u_{i,j}\, dx^j = u_{i,j}\, Dr^j = (e_{ij} - \omega_{ij})\, Dr^j \,, \tag{7}$$

where r^j are the coordinates, in curvilinear coordinates x^j, of the position vector r. If we perform, according to (1), an absolute integration of the relation (7), we shall have:

$$\begin{aligned} u_m(P) - u_i(P_0)g^{i}_{.m}(P_0, P) &= \int_{P_0P}^{\nabla} g^{i}_{.m}(M,P)\, Du_i(M) \\ &= \int_{P_0P}^{\nabla} g^{i}_{.m}(M,P)[e_{ij}(M) - \omega_{ij}(M)]\, Dr^j(M) \\ &= \int_{P_0P}^{\nabla} g^{i}_{.m}(M,P)e_{ij}(M)\, Dr^j(M) - \\ &\quad - \int_{P_0P}^{\nabla} g^{i}_{.m}(M,P)\, D[\omega_{ij}(M)r^j(M)] + \\ &\quad + \int_{P_0P}^{\nabla} g^{i}_{.m}(M,P)r^j(M)\, D\omega_{ij}(M) \,. \end{aligned} \tag{8}$$

From (6) it follows:

$$\begin{aligned} \omega_{ij,k} &= \tfrac{1}{2}(u_{j,ik} - u_{i,jk}) = \tfrac{1}{2}(u_{j,ki} + u_{k,ji}) - \tfrac{1}{2}(u_{k,ij} + u_{i,kj}) \\ &= e_{jk,i} - e_{ki,j} \,, \end{aligned} \tag{9}$$

and:

$$D\omega_{ij} = \omega_{ij,k}\,dx^k = (e_{jk,i} - e_{ki,j})\,Dr^k\,, \tag{10}$$

so (according to (2)) an absolute integration of this relation gives:

$$\begin{aligned}\omega_{ml}(P) - \omega_{ij}(P_0)g^{i}_{\cdot m}(P_0,P)g^{j}_{\cdot l}(P_0,P) &= \int_{P_0P}^{\nabla} g^{i}_{\cdot m}(M,P)g^{j}_{\cdot l}(M,P)\,D\omega_{ij}(M)\\ &= \int_{P_0P}^{\nabla} g^{i}_{\cdot m}(M,P)g^{j}_{\cdot l}(M,P)[e_{jk,i}(M) - e_{ki,j}(M)]\,Dr^k(M)\,.\end{aligned} \tag{11}$$

Using (10) and (11), and after some indices exchange, we can rewrite (8) in the following way:

$$\begin{aligned}u_m(P) - u_i(P_0)g^{i}_{\cdot m}(P_0,P) &= \int_{P_0P}^{\nabla} g^{i}_{\cdot m}(M,P)e_{ij}(M)\,Dr^j(M) -\\ &- \omega_{mj}(P)r^j(P) + \omega_{ij}(P_0)r^j(P_0)g^{i}_{\cdot m}(P_0,P) +\\ &+ \int_{P_0P}^{\nabla} g^{i}_{\cdot m}(M,P)r^j(M)[e_{jk,i}(M) - e_{ki,j}(M)]\,Dr^k(M)\\ &= \int_{P_0P}^{\nabla} g^{i}_{\cdot m}(M,P)\{e_{ik}(M) + r^j(M)[e_{jk,i}(M) - e_{ki,j}(M)]\}\,Dr^k(M) -\\ &- \omega_{ij}(P_0)g^{i}_{\cdot m}(P_0,P)g^{j}_{\cdot l}(P_0,P)r^l(P) + \omega_{ij}(P_0)r^j(P_0)g^{i}_{\cdot m}(P_0,P) -\\ &- \int_{P_0P}^{\nabla} g^{i}_{\cdot m}(M,P)g^{j}_{\cdot l}(M,P)[e_{jk,i}(M) - e_{ki,j}(M)]r^l(P)\,Dr^k(M)\,,\end{aligned} \tag{12}$$

finally obtaining:

$$\begin{aligned}u_m(P) = u_i(P_0)&g^{i}_{\cdot m}(P_0,P) -\\ &- \omega_{ij}(P_0)g^{i}_{\cdot m}(P_0,P)[g^{j}_{\cdot l}(P_0,P)r^l(P) - r^j(P_0)] +\\ &+ \int_{P_0P}^{\nabla} g^{i}_{\cdot m}(M,P)\{e_{ik}(M) - [g^{j}_{\cdot l}(M,P)r^l(P) - r^j(M)]\times\\ &\qquad\times [e_{jk,i}(M) - e_{ki,j}(M)]\}\,Dr^k(M)\end{aligned} \tag{13}$$

and that is the coordinate form, in arbitrary curvilinear coordinates, of E. Cesàro's formula for determining the displacement field from a prescribed infinitesimal deformation field. Concerning the integrability of expression (13), it can be proved by checking the path independence conditions for the corresponding line integral. These conditions read:

$$\begin{aligned}&\{e_{ik}(M) - [g^{j}_{\cdot m}(M,P)r^m(P) - r^j(M)][e_{jk,i}(M) - e_{ki,j}(M)]\}_{,l} =\\ &= \{e_{il}(M) - [g^{j}_{\cdot m}(M,P)r^m(P) - r^j(M)][e_{jl,i}(M) - e_{li,j}(M)]\}_{,k}\,.\end{aligned} \tag{14}$$

That they are satisfied follows from the fact that, having in mind the performing of covariant differentiation at the point M, we can (similarly as in [9, p. 57]) show their equivalence to the compatibility conditions (3).

4. Example

Let us determine the displacement filed for an infinitesimal relative strain tensor prescribed in cylindrical polar coordinates $\{x^1, x^2, x^3\} = \{\rho, \varphi, z\}$:

$$\{e_{ij}\} = k \left\{ \begin{matrix} \sin(2\varphi) & \rho\cos(2\varphi) & 0 \\ \rho\cos(2\varphi) & -\rho^2\sin(2\varphi) & 0 \\ 0 & 0 & 0 \end{matrix} \right\} \tag{15}$$

(k is an infinitesimal constant), if the displacement $u_i(P_0)$ and rotation $\omega_{ij}(P_0)$ at the point $P_0(\rho = 1, \varphi = 0, z = 0)$ are equal to zero.

Taking into account that the only three Christoffel symbol coordinates different from zero, in the cylindrical polar system, are:

$$\Gamma^1_{22} = -\rho, \qquad \Gamma^2_{12} = \Gamma^2_{21} = 1/\rho, \tag{16}$$

it is easy to show (s. e.g. [7, p. 45]) the covariant constancy of the prescribed infinitesimal strain tensor:

$$e_{ij,k} = 0. \tag{17}$$

From the relation (17) it immediately follows that the compatibility conditions (3) are satisfied, so we can use the formula (13), which, because of (17) and the assumptions $u_i(P_0) = 0$ and $\omega_{ij}(P_0) = 0$, reduces to:

$$u_m(P) = \int_{P_0P}^{\nabla} g^{i}_{.m}(M,P) e_{ik}(M)\, Dr^k(M). \tag{18}$$

Using the equality $\{Dr^k\} = \{dx^k\} = \{d\rho, d\varphi, dz\}$ as well as the fact that $e_{i3} = e_{3i} = 0$ (s. (15)), we can present (18) in the following form:

$$\begin{aligned} u_1(P) = \int_{P_0P} \{ & [g^1_{.1}(M,P)e_{11}(M) + g^2_{.1}(M,P)e_{21}(M)]\, d\rho(M) + \\ & [g^1_{.1}(M,P)e_{12}(M) + g^2_{.1}(M,P)e_{22}(M)]\, d\varphi(M)\} \\ u_2(P) = \int_{P_0P} \{ & [g^1_{.2}(M,P)e_{11}(M) + g^2_{.2}(M,P)e_{21}(M)]\, d\rho(M) + \\ & [g^1_{.2}(M,P)e_{12}(M) + g^2_{.2}(M,P)e_{22}(M)]\, d\varphi(M)\} \\ u_3(P) = 0. & \end{aligned} \tag{19}$$

However, the coordinates of the shifter which relates to the points $M(\rho, \varphi, z)$ and $P(R, \Phi, Z)$, in the case of cylindrical polar coordinates, are equal (s. e.g. (17.2) in [1], (3.A.23) in [6] or p. 11 in [7]):

$$\{g^{i}_{.m}(M,P)\} = \left\{ \begin{matrix} \cos(\varphi - \Phi) & R\sin(\varphi - \Phi) & 0 \\ -1/\rho\sin(\varphi - \Phi) & R/\rho\cos(\varphi - \Phi) & 0 \\ 0 & 0 & 1 \end{matrix} \right\}. \tag{20}$$

Having in mind (15) and suitably choosing an integration path[3] from P_0 to P, e.g. over the points $(R, 0, 0)$ and $(R, \Phi, 0)$, we can reduce the curvilinear integrals in (19) to ordinary ones:

$$\begin{aligned}
(21)\qquad u_1(P) = k\Big\{ \int_1^R & [\cos(\varphi - \Phi)\sin(2\varphi) - \sin(\varphi - \Phi)\cos(2\varphi)]|_{\varphi=0}\, d\rho + \\
&+ \int_0^\Phi [\rho\cos(\varphi - \Phi)\cos(2\varphi) + \rho\sin(\varphi - \Phi)\sin(2\varphi)]|_{\rho=R}\, d\varphi \Big\} \\
u_2(P) = kR\Big\{ \int_1^R & [\sin(\varphi - \Phi)\sin(2\varphi) + \cos(\varphi - \Phi)\cos(2\varphi)]|_{\varphi=0}\, d\rho + \\
&+ \int_0^\Phi [\rho\sin(\varphi - \Phi)\cos(2\varphi) - \rho\cos(\varphi - \Phi)\sin(2\varphi)]|_{\rho=R}\, d\varphi \Big\}\,.
\end{aligned}$$

Now we immediately obtain that the first and second displacement field coordinates are:

$$(22)\qquad \begin{aligned} u_1(P) &= k[R\sin(2\Phi) - \sin(\Phi)] \\ u_2(P) &= kR[R\cos(2\Phi) - \cos(\Phi)] \end{aligned}$$

and these are exactly the expressions obtained in [7] (p. 47), in the same example, by solving a system of partial differential equations which follows from the starting system (4) after explicitly expressing the covariant derivatives in the cylindrical polar system.

5. Concluding remarks

It has been explained in [8] how the idea of an absolute or covariant integral (which was postulated in [3]) imposes itself naturally in Euclidean space. Namely, following J. L. Ericksen's concept of integration in curvilinear coordinates (s. p. 808 in [1]), an invariant integral of the absolute differential of a tensor field has been established in the form (1), i.e. (2).

The notion of such an integral here is used in order to carry out E. Cesàro's formula entirely in coordinate form in an arbitrary system of curvilinear coordinates. Of course, in the case of Cartesian coordinates (when the Euclidean shifters are the Kronecker delta), the formula (13) reduces to the usual one (s. e.g. p. 41 in [4] or p. 57 in [9]).

It should be noted that the derivation of E. Cesàro's formula in direct notation (i.e. without indices introducing in the corresponding vector or tensor field kernel, thus without pointing out to the coordinate system in question) can be found in [2], p. 63. On the basis of formula (2.2.2) derived there, one can obtain its coordinate form (13) in arbitrary curvilinear coordinates by consistent use of Euclidean shifters; in that case, integrals of form (1) or (2) (obtained in [8] by arbitrary curvilinear

[3]Path independence of the curvilinear integrals in (19) is provided by the above mentioned satisfaction of the compatibility conditions.

coordinates introducing in integral sums of the corresponding limit process) should arise.

Finally, we emphasize that this paper shows, on the example of E. Cesàro's formula, that the derivation in the coordinate form of various integral relations in Euclidean space should not be limited to Cartesian coordinates, which is usually motivated by procedural simplicity and the wish to avoid "some formal difficulties" in using curvilinear coordinates.

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SUR LA DÉRIVATION DE LA FORMULE D'E. CESÀRO EN COORDONNÉES CURVILIGNES

Dans cet article, en suivant le concept d'intégrale absolue ou covariante dans l'espace euclidien, on propose une procedure d'intégration des relations entre les composantes d'un tenseur de déformation et d'un vecteur déplacement en coordonnées curvilignes quelconques.

O IZVOĐENJU FORMULE E. ČEZARA U SISTEMU KRIVOLINIJSKIH KOORDINATA

U radu je, u skladu sa konceptom tzv. apsolutnog ili kovarijantnog integraljenja u euklidskom prostoru ([3], [5], [8]), izvršeno integraljenje veze pomeranja i deformacije u proizvoljnim krivolinijskim koordinatama.

Zoran Drašković
Matematički institut
Odeljenje za mehaniku
Kneza Mihaila 35
11000 Beograd, Jugoslavija