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ESTIMATION OF STATE VARIABLES FROM MEASUREMENTS

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1. Introduction

The use of a dynamic programming filter to identify systems has been presented by Distefano and Rath [1] for the continuous case. This was expanded by Simonian [2–3] who applied the filter to the estimation of wind forces on structures. This paper presents a discrete version of the dynamic programming filter. The advantage of the discrete version is to more easily analyze nonlinear systems, both conceptually and computationally.

Consider a dynamical process described by an n dimensional system of difference equations

$$x_{i+1} = M_i x_i + q_i + \varepsilon_i \tag{1a}$$

where x_i is an $(n \times 1)$ column vector of state variables, M_i is an $(n \times n)$ transfer matrix, q_i is an $(n \times 1)$ known forcing term and ε_i is an $(n \times 1)$ vector of dynamical errors. It is assumed that observations on the state vector are obtained at every step and related to x_i by

$$d_i = H_i x_i + \eta_i \tag{1b}$$

where d_i is an $(n \times 1)$ column vector of measurements, H_i is an $(n_d \times n)$ matrix relating the measurements to the state vector and η_i is an $(n \times 1)$ column vector of observational errors.

It is desired to optimally estimate x_i using a least squares criterion. In addition, suppose that N measurements have been taken and that all the x_i have been optimally estimated. Now another measurement d_{N+1} is taken. We would like to optimally estimate only x_{i+1} without having to recalculate all the previous optimal estimates. This is necessary in a real time constraint where only the current state is of importance.

The least squares criterion for N measurements is

$$E_N = \sum_{i=1}^{N}(d_i - H_i x_i, d_i - H_i x_i) + \\ + \sum_{i=2}^{N}(x_i - M_{i-1}x_{i-1} - q_{i-1}, K(x_i - M_{i-1}x_{i-1} - q_{i-1})) \quad (2)$$

where (x, y) denotes the inner product of two vectors and K represents an $(n \times n)$ weighting matrix.

To restate the problem we wish to minimize Eq. (2) with respect to all x_i, i.e. $(x_1, x_2, \ldots, x_N)$. Then suppose we add another data point d_{N+1} which gives

$$E_{N+1} = \sum_{i=1}^{N+1}(d_i - H_i x_i, d_i - H_i x_i) + \\ + \sum_{i=2}^{N+1}(x_i - M_{i-1}x_{i-1} - q_{i-1}, K(x_i - M_{i-1}x_{i-1} - q_{i-1})). \quad (3)$$

We now wish to minimize E_{N+1} with respect to $(x_1, x_2, \ldots, x_N)$. It is very important to note that all of the minimizing x_i's will change by adding the one data point. Also, we only want the value of x_{N+1} so that the computations are kept to a minimum. Dynamic programming can be used to provide the sequential formulas.

2. Sequential filter equations

First define $\phi_N(c)$ as the minimum of E_N conditioned on $x_N = c$. That is suppose x_N is fixed at c and we minimize E_N with respect to $x_1, x_2, \ldots, x_{N-1}$. The value of E_N is $O_N(c)$. Using a dynamic programming argument it follows that

$$\phi_{N+1}(c) = \min_{x_N}\big[(d_{N+1} - H_{N+1}c, d_{N+1} - H_{N+1}c) + \\ + (c - M_N x_N - q_N, K(c - M_N x_N - q_N)) + \phi_N(x_N)\big]. \quad (4)$$

Equation (4) will give the function $\phi_{N+1}(c)$ which is valid for all $x_{N+1} = c$. However, the optimal estimate of x_{N+1} is the one that minimizes $\phi_{N+1}(c)$. It is easy to show that all $\phi_N(c)$ will be of the form

$$\phi_N(c) = b_N - 2(c, s_N) + (c, R_N c) \quad (5)$$

where s_N is an $(n \times 1)$ recursive vector and R_N is an $(n \times n)$ Riccati matrix. Recursive formulas for s_N and R_N can be obtained. Substituting Eq. (5) into Eq. (4) yields

$$b_{N+1} - 2(c, s_{N+1}) + (c, R_{N+1}c) = \min_{X_N}\big[(d_{N+1}, d_{N+1}) - 2(d_{N+1}, H_{N+1}c) + \\ + (H_{N+1}c, H_{N+1}c) + (c, Kc) - 2(c, KM_N x_N) - 2(c, Kq_N) + (M_N x_N, KM_N x_N) + \\ + 2(M_N x_N, Kq_N) + (q_N, Kq_N) - 2(x_N, s_N) + (x_N, R_N x_N) + b_N\big]. \quad (6)$$

Minimizing Eq. (6) with respect to x_N yields

$$x_N^* = Q_N M_N^T K(c - q_N) + Q_N s_N \tag{7}$$

where

$$Q_N^{-1} = M_N^T K M_N + R_N. \tag{8}$$

Substituting Eq. (7) into Eq. (6) and equating like coefficients in c yield

$$s_{N+1} = H_{N+1}^T d_{N+1} + K q_N - K M_N Q_N M_N^T K q_N + K M_N Q_N s_N \tag{9}$$

$$R_{N+1} = H_{N+1}^T H_{N+1} + K - K M_N Q_N M_N^T K\,. \tag{10}$$

The optimal estimate of x_N is produced by minimizing $\phi_N(c)$ which is denoted by c^*. Thus

$$2R_N c_N^* - 2s_N = 0$$

or

$$c_N^* = R_N^{-1} s_N. \tag{12}$$

Equation (12) can serve as the final calculation. However, a more ambitious calculation is

$$c_{N+1}^* = R_{N+1}^{-1} s_{N+1}. \tag{13}$$

Using Eq. (8), (9), and (10) we find

$$c_{N+1}^* = M_N c_N^* + q_N + R_{N+1}^{-1} H_{N+1}^T [d_{N+1} - H_{N+1}(M_n c_N^* + q_N)] \tag{14}$$

which involves only the optimal estimates.

3. Application to nonlinear systems

In the previous discussion, the linear equation was used to represent the system

$$x_{i+1} = M_i x_i + q_i \tag{15}$$

This can easily represent a nonlinear system in the following manner: Let the original differential equation be of the form

$$\dot{x} + Kx + n(x) = f(t) \tag{16}$$

where all of the linear terms have been grouped into Kx leaving the nonlinear terms in $n(x)$. Expanding $n(x)$ into a Taylor's expansion about some state x_0 gives

$$n(x) = n(x_0) + A(x - x_0) \tag{17}$$

where A represents the Jacobian matrix of n evaluated at x_0. This gives

$$\dot{x} + (K + A)x = -n(x_0) + Ax_0 + f. \tag{18}$$

Using a difference formula for x and evaluating x at the average gives

$$[I + (K + A)h/2]x_{i+1} = [I - (K + A)h/2]x_i - hn(x_0) + hAx_0 + hf \tag{19}$$

where h is the time step. Now let $x_0 = x_i$, then the terms in Eq. (19) become

$$M_i = [I + (K + A_i)h/2]^{-1}[I - (K + A_i)h/2] \tag{20}$$

$$q_i = -[I + (K + A_i)h/2]^{-1}[h(n(x_i) - A_x - f_i)] \tag{21}$$

It is important that these integration formula remain stable. It should be noted that as $h \to 0$, Eq. (19) reduces to Eq. (16) and A_i is not involved. There is evidence that A_i can be used to stabilize the integration process since it is somewhat arbitrary. However, here A_i will be evaluated as the Jacobian matrix of $n(x)$. With these definitions of M_i and q_i the previous filter equations can now be used directly for a nonlinear equation. The Jacobian matrix and the nonlinear term must be evaluated at the previous state.

4. Illustrative numerical example

The use of the filter to determine unknown constants in a model involves an extension of the dynamic system to include the unknown constants. The following simple example will illustrate the use of the filter to determine the unknown constants. It will also show how the various terms in Eq. (19) are derived. Consider a simple spring mass system given by

$$\ddot{x} + 2\xi\omega\dot{x} + \omega^2 x = 0 \tag{22}$$

where ξ and ω are considered as constants to be determined from measurements of either velocity or displacements. For the case considered here, only the displacement measurement will be used. In order to use the filter equations, the constants must be considered as variables. Define the following vector

$$x = [x_1 \;\; x_2 \;\; x_3 \;\; x_4]^{\mathrm{T}} = [x \;\; \dot{x} \;\; \xi \;\; \omega]^{\mathrm{T}} . \tag{23}$$

The complete dynamic system is given by

$$\begin{aligned} x_1 - x_2 &= 0 \\ x_2 + 2x_2x_3x_4 + x_1x_4^2 &= 0 \\ x_3 &= 0 \\ x_4 &= 0 . \end{aligned} \qquad (24)$$

The matrices K and A are now filled in with the appropriate terms. The nonlinear vector consists of one term which is given as

$$n_2(x) = 2x_2x_3x_4 + x_1x_4^2 . \qquad (24)$$

The nonzero entries in the Jacobian matrix A are

$$A(2,1) = x_4^2, \qquad A(2,2) = 2x_3x_4,$$
$$A(2,3) = 2x_2x_4, \qquad A(2,4) = 2(x_2x_3 + x_1x_4) .$$

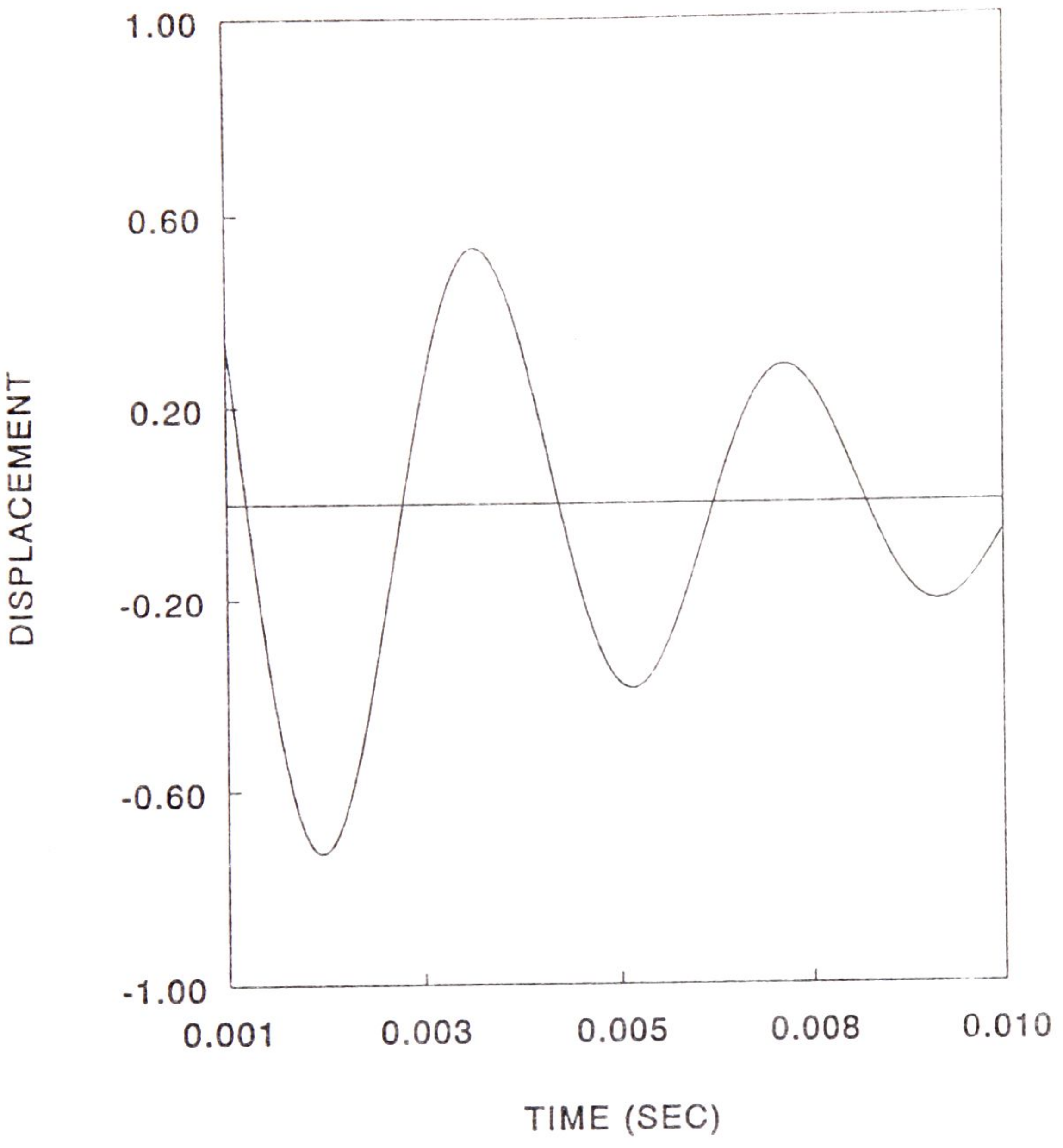

Fig. 1

An integration time step of 2.5E-5 seconds was used for a total time of 0.01 seconds. The identification problem now considers the constants as unknowns. These are to be estimated using only the generated displacement data.

The initial values used were $x_1 = 1.0$, $x_2 = 0.0$, $x_3 = .05$, $x_4 = 1000.$, the original displacement data and the results of the filter are shown in Fig. (1). The filter had no difficulty in following the data. In addition the velocity has also been reproduced and compared with the original expression (see Fig. (2)). The progress of the constants is shown in Figs. (3) and (4). Both of the constants reach the correct values of 0.1 and 1715 Hz in 0.0040 seconds.

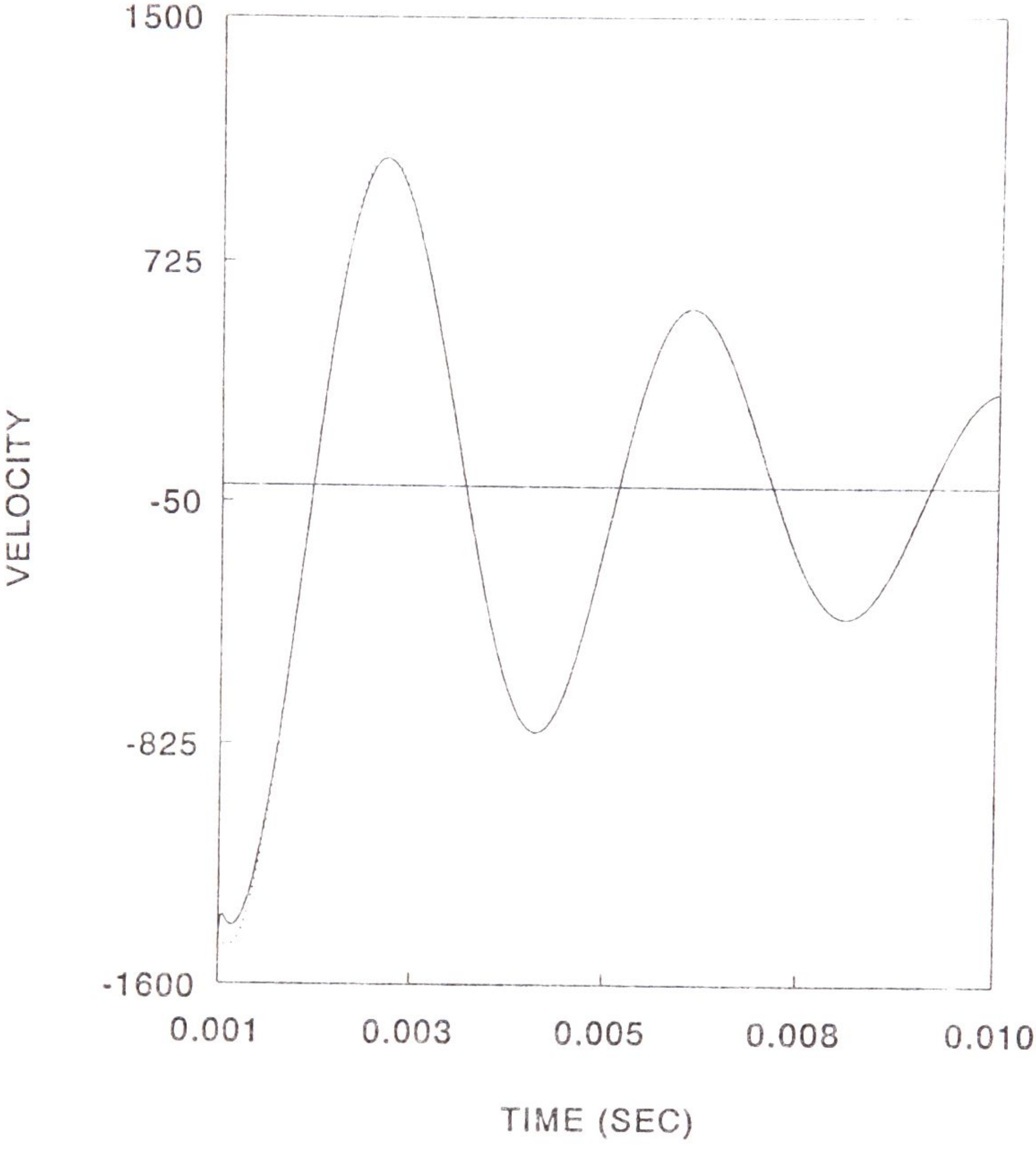

Fig. 2

5. Results and conclusions

Equations have been derived for a sequential least squares estimator using the technique of dynamic programming. A method was presented so that the filter could be used for nonlinear systems. Additional work needs to be performed to determine the overall capability of the sequential filter. The filter worked well when a simple model of a spring mass system was investigated, however the model was free of noise.

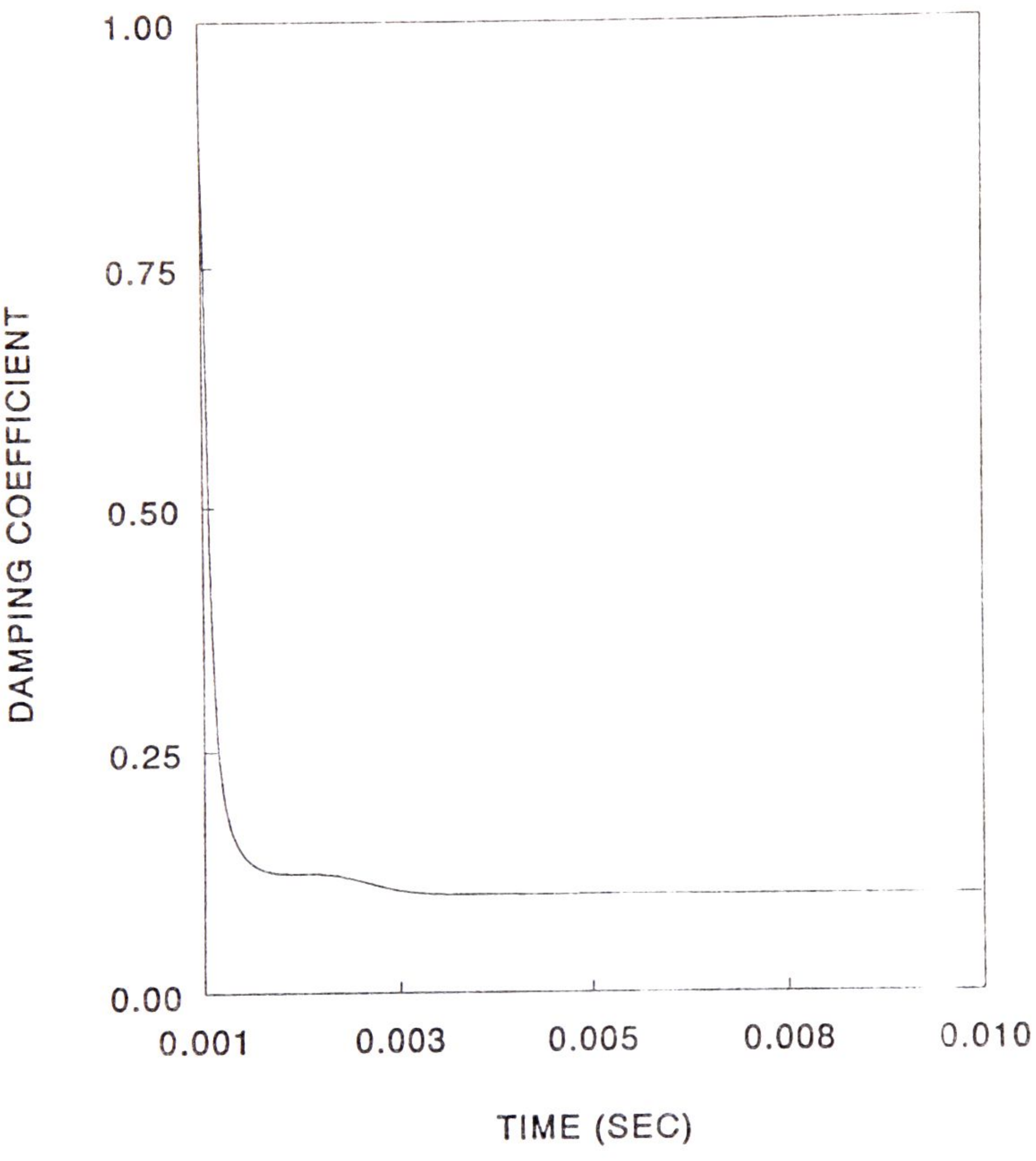

Fig. 3

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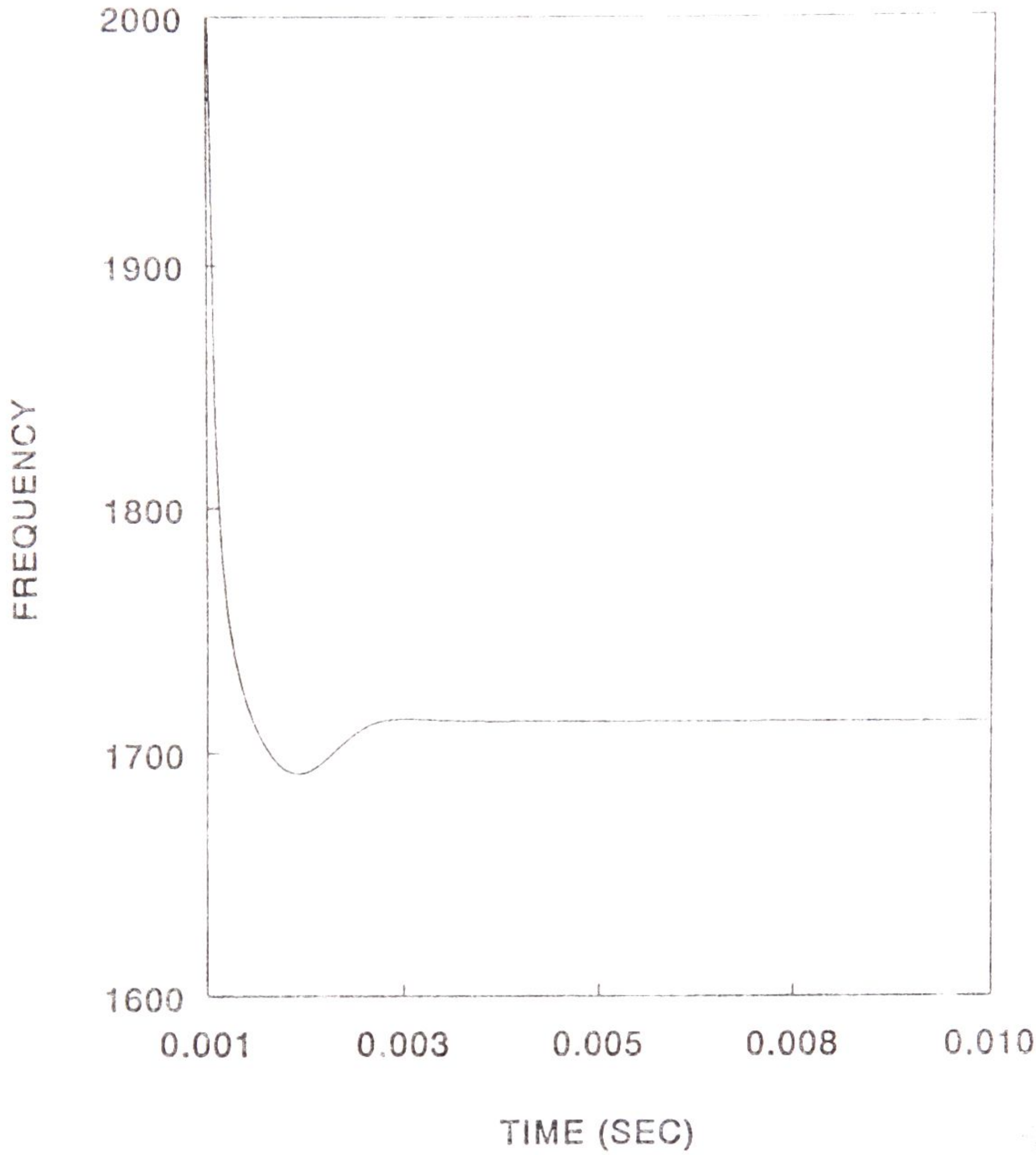

Fig. 4

ESTIMATION DES VARIABLES D'ÉTAT OBTENUES PAR MESURES

Le problème considéré est une estimation sequentielle des variables d'état susceptible d'être appliqué aux systèmes nonlineaires. Les équations sont presentées pour la mise à jour sequentielle de la solution optimale en temps discret („discrete-time") pendant que le processus continue et les observations nouvelles sont obtenues. La technique de la programmation dynamique est appliquée en vue d'obtenir les estimations.

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