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TRANSONIC FLOW SOLUTION OF THE EULER EQUATIONS USING IMPLICIT LU FACTORIZATION

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1. Introduction

An application of the explicit scheme in the Euler equation solution [1], [2] and [3] made a very accurate transonic three-dimensional inviscid flow analysis possible by elimination of drawbacks connected with potential theory results. However, basic disadvantage of the explicit approach lies in the Courant number limitation in time step size choice. LU implicit factorization scheme increases allowed Courant number and makes transonic flow calculation more acceptable, even in unsteady flow solutions. Two-pass LU implicit factorization approach, based on finite volume method with flux splitting, is described in this paper. Special attention will be paid to correct implementation of the boundary conditions, related to the flux splitting scheme.

2. Theoretical postulation

The three-dimensional unsteady Euler equations may be written in Cartesian coordinate system in conservation form as follows:

$$\partial_t q + \partial_x F + \partial_y G + \partial_z H = 0, \tag{2.1}$$

where q is the flow properties vector

$$q = (\rho, \rho u, \rho v, \rho w, e)^{\mathrm{T}}, \tag{2.2}$$

while F, G and H are flux vector projections on Cartesian axes:

$$F = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ \rho uw \\ u(e+p) \end{pmatrix}, \qquad G = \begin{pmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ \rho vw \\ v(e+p) \end{pmatrix} \qquad \text{and} \qquad H = \begin{pmatrix} \rho w \\ \rho uw \\ \rho vw \\ \rho w^2 + p \\ w(e+p) \end{pmatrix}. \tag{2.3}$$

In the equations (2.2) and (2.3) quantities ρ, u, v, w, p and e denote density, velocity vector projections, pressure and total energy per unit volume, respectively.

An application of the curvilinear coordinate system transformation, grid generated in physical space, which surfaces are related to the constant values of curvilinear coordinates (ξ, η, ζ), is mapped to the rectangular computational grid. In the transformed space the equation (2.1) becomes

$$\partial_\tau \overline{q} + \partial_\xi \overline{F} + \partial_\eta \overline{G} + \partial_\zeta \overline{H} = 0, \tag{2.1.1}$$

where $\overline{q}, \overline{F}, \overline{G}$ and $\overline{H}$ are quantities defined as follows:

$$\overline{q} = Jq \tag{2.2.1}$$

and

$$\begin{aligned} \overline{F} &= J(\xi_\tau q + \xi_x F + \xi_y G + \xi_z H), \\ \overline{G} &= J(\eta_\tau q + \eta_x F + \eta_y G + \eta_z H), \\ \overline{H} &= J(\zeta_\tau q + \zeta_x F + \zeta_y G + \zeta_z H). \end{aligned} \tag{2.3.1}$$

After transformation of the equations (2.3.1), quantities $\overline{F}$, $\overline{G}$ and $\overline{H}$ become

$$\overline{F} = J \begin{pmatrix} \rho U \\ \rho u U + \xi_x p \\ \rho v U + \xi_y p \\ \rho w U + \xi_z p \\ U(e+p) - \xi_t p \end{pmatrix}, \qquad \overline{G} = J \begin{pmatrix} \rho V \\ \rho u V + \eta_x p \\ \rho v V + \eta_y p \\ \rho w V + \eta_z p \\ V(e+p) - \eta_t p \end{pmatrix}$$

$$\text{and} \qquad \overline{H} = J \begin{pmatrix} \rho W \\ \rho u W + \zeta_x p \\ \rho v W + \zeta_y p \\ \rho w W + \zeta_z p \\ W(e+p) - \zeta_t p \end{pmatrix}, \tag{2.3.2}$$

where U, V and W are velocity vector contravariant coordinates, defined by the following transformation:

$$\begin{aligned} U &= \xi_t + \xi_x u + \xi_y v + \xi_z w, \\ V &= \eta_t + \eta_x u + \eta_y v + \eta_z w, \\ W &= \zeta_t + \zeta_x u + \zeta_y v + \zeta_z w, \end{aligned} \tag{2.3.3}$$

while ξ_t, η_t and ζ_t are evaluated by equations

$$\begin{aligned} \xi_t &= -x_\tau \xi_x - y_\tau \xi_y - z_\tau \xi_z, \\ \eta_t &= -x_\tau \eta_x - y_\tau \eta_y - z_\tau \eta_z, \\ \zeta_t &= -x_\tau \zeta_x - y_\tau \zeta_y - z_\tau \zeta_z, \end{aligned} \tag{2.3.4}$$

having in mind known the coordinate transformation law

$$\begin{aligned} \xi &= \xi(x, y, z, t), \\ \eta &= \eta(x, y, z, t), \\ \zeta &= \zeta(x, y, z, t), \\ \tau &= t. \end{aligned} \tag{2.3.5}$$

The quantities ξ_t, η_t and ζ_t in the equations (2.3.4) are equal to zero, for a stationary, fixed grid.

In the equations (2.2.1), (2.3.1) and (2.3.2) Jacobian $J = \partial(x,y,z)/\partial(\xi,\eta,\zeta)$ is evaluated from the expression

$$J = x_\xi(y_\eta z_\zeta - z_\eta y_\zeta) - y_\xi(x_\eta z_\zeta - z_\eta x_\zeta) + z_\xi(x_\eta y_\zeta - y_\eta x_\zeta). \tag{2.3.6}$$

For a perfect gas the system of equations (2.1) is completed by the definition of total fluid energy

$$e = \frac{1}{\gamma - 1} p + \frac{1}{2}\rho(u^2 + v^2 + w^2). \tag{2.4}$$

The unknown in the system of equations (2.1) and (2.1.1) is the flow property vector $\overline{q}$. The quantities $\overline{F}$, $\overline{G}$ and $\overline{H}$ are nonlinear functions of $\overline{q}$ at the time level $n+1$. These functions were linearized about time level n:

$$\begin{aligned} \overline{F}^{n+1} &= \overline{F}^n + [D\overline{F}/D\overline{q}]^n \, \Delta\overline{q}^n, \\ \overline{G}^{n+1} &= \overline{G}^n + [D\overline{G}/D\overline{q}]^n \, \Delta\overline{q}^n, \\ \overline{H}^{n+1} &= \overline{H}^n + [D\overline{H}/D\overline{q}]^n \, \Delta\overline{q}^n, \end{aligned} \tag{2.5}$$

where

$$\Delta\overline{q}^n = \overline{q}^{n+1} - \overline{q}^n. \tag{2.6}$$

In the equations (2.5) matrices $[D\overline{F}/D\overline{q}]^n$, $[D\overline{G}/D\overline{q}]^n$ and $[D\overline{H}/D\overline{q}]^n$, denoted by $\overline{\mathbf{A}}$, $\overline{\mathbf{B}}$ and $\overline{\mathbf{C}}$, respectively, are defined as follows:

$$\overline{\mathbf{A}}, \quad \overline{\mathbf{B}}, \quad \overline{\mathbf{C}} = \tag{2.7}$$

$$\begin{pmatrix} k_t & k_x & k_y & k_z & 0 \\ k_x\phi^2 - u\theta & k_t + \theta - k_x(\gamma-2)u & k_y u - k_x(\gamma-1)v & k_z u - k_x(\gamma-1)w & k_x(\gamma-1) \\ k_y\phi^2 - v\theta & k_x v - k_y(\gamma-1)u & k_t + \theta - k_y(\gamma-2)v & k_z v - k_y(\gamma-1)w & k_y(\gamma-1) \\ k_z\phi^2 - w\theta & k_x w - k_z(\gamma-1)u & k_y w - k_z(\gamma-1)v & k_t + \theta - k_z(\gamma-2)w & k_z(\gamma-1) \\ \theta(\phi^2 - \omega) & k_x\omega - (\gamma-1)u\theta & k_y\omega - (\gamma-1)v\theta & k_z\omega - (\gamma-1)w\theta & k_t + \gamma\theta \end{pmatrix}.$$

where $k = (\xi, \eta, \zeta)$ for matrices $\overline{\mathbf{A}}$, $\overline{\mathbf{B}}$ and $\overline{\mathbf{C}}$, respectively. The quantities ϕ^2, θ and ω in the equations (2.7) are evaluated in the following way:

$$\begin{aligned} \phi^2 &= \tfrac{1}{2}(\gamma-1)(u^2+v^2+w^2), \\ \theta &= k_x u + k_y v + k_z w, \\ \omega &= \gamma e/\rho - \phi^2. \end{aligned} \tag{2.8}$$

In the numerical approach of the Euler equation solution with finite volume method (FVM), the computational domain is discretised by dividing into hexahedral cells and then the system of equations (2.1)–(2.4) is approximated. It is assumed that the value of the dependent variable $\overline{q}$ is known at the point (i,j,k), where each such point is the center of one of the cells, then the approximative system of equations (2.1.1) may be written in the following form:

$$\left[\mathbf{I} + \beta\Delta t(\delta_\xi \overline{\mathbf{A}}^n + \delta_\eta \overline{\mathbf{B}}^n + \delta_\zeta \overline{\mathbf{C}}^n)\right] \Delta\overline{q}^n + \Delta t \, \overline{R}^n = 0, \tag{2.9}$$

where the residual $\overline{R}^n$ is

$$\overline{R}^n = \delta_\xi \overline{F}(\overline{q}^n) + \delta_\eta \overline{G}(\overline{q}^n) + \delta_\zeta \overline{H}(\overline{q}^n). \tag{2.9.1}$$

In the equations (2.9) and (2.9.1) δ_ξ, δ_η and δ_ζ denote the central difference operators $\partial/\partial\xi$, $\partial/\partial\eta$ and $\partial/\partial\zeta$.

The parameter β in the relation (2.9) defines time accuracy of the applied scheme. If $\beta = 1/2$, the scheme remains second-order accurate in time; for other values of β the time accuracy drops to first order. The unfactored implicit scheme, defined by the equation (2.9), requires a huge storage for a very large block-banded matrix, which is very costly to invert especially for a 3D flow case. An unconditionally stable implicit scheme that has error terms at most of the order $(\Delta t)^2$ in any number of space dimensions can be derived by LU factorization [4], [5] and [6]

$$\left[\mathbf{I} + \beta\Delta t(\delta_\xi^- \overline{\mathbf{A}}^+ + \delta_\eta^- \overline{\mathbf{B}}^+ + \delta_\zeta^- \overline{\mathbf{C}}^+)\right]^n \times$$
$$\times \left[\mathbf{I} + \beta\Delta t(\delta_\xi^+ \overline{\mathbf{A}}^- + \delta_\eta^+ \overline{\mathbf{B}}^- + \delta_\zeta^+ \overline{\mathbf{C}}^-)\right]^n \Delta\overline{q}^n + \Delta t\,\overline{R}^n = 0, \tag{2.10}$$

where δ_ξ^-, δ_η^- and δ_ζ^- are backward difference operators and δ_ξ^+, δ_η^+ and δ_ζ^+ are forward difference operators

$$\begin{aligned}
\delta_\xi^-(\overline{\mathbf{A}}^+ \Delta\overline{q}^n)_{i,j,k} &= \overline{\mathbf{A}}^+_{i+1/2,j,k}\Delta\overline{q}^n_{i,j,k} - \overline{\mathbf{A}}^+_{i-1/2,j,k}\Delta\overline{q}^n_{i-1,j,k},\\
\delta_\xi^+(\overline{\mathbf{A}}^- \Delta\overline{q}^n)_{i,j,k} &= \overline{\mathbf{A}}^-_{i+1/2,j,k}\Delta\overline{q}^n_{i+1,j,k} - \overline{\mathbf{A}}^-_{i-1/2,j,k}\Delta\overline{q}^n_{i,j,k}.
\end{aligned} \tag{2.10.1}$$

The values of the matrix elements with indices $(i+1/2, j, k)$ and $(i-1/2, j, k)$ are evaluated by averaging between the cell points (i, j, k) and $(i+1, j, k)$, or $(i-1, j, k)$ and (i, j, k), respectively. The operators for two remaining coordinate directions can be calculated in a similar manner.

The reason for splitting in the equation (2.10) is to ensure the diagonal dominance of lower and upper factors as well as to make possible usage of the built-in implicit dissipation. Flux matrices $\overline{\mathbf{A}}^+$, $\overline{\mathbf{B}}^+$, $\overline{\mathbf{C}}^+$, $\overline{\mathbf{A}}^-$, $\overline{\mathbf{B}}^-$ and $\overline{\mathbf{C}}^-$ in the equation (2.10) are constructed so that the eigenvalues of "+" matrices are nonnegative and those of "−" matrices are nonpositive

$$\begin{aligned}
\overline{\mathbf{A}}^+ &= \tfrac{1}{2}(\overline{\mathbf{A}} + r_A\mathbf{I}), & \overline{\mathbf{A}}^- &= \tfrac{1}{2}(\overline{\mathbf{A}} - r_A\mathbf{I}),\\
\overline{\mathbf{B}}^+ &= \tfrac{1}{2}(\overline{\mathbf{B}} + r_B\mathbf{I}), & \overline{\mathbf{B}}^- &= \tfrac{1}{2}(\overline{\mathbf{B}} - r_B\mathbf{I}),\\
\overline{\mathbf{C}}^+ &= \tfrac{1}{2}(\overline{\mathbf{C}} + r_C\mathbf{I}), & \overline{\mathbf{C}}^- &= \tfrac{1}{2}(\overline{\mathbf{C}} - r_C\mathbf{I}),
\end{aligned} \tag{2.10.2}$$

where $\mathbf{I}$ is the unit matrix and factors r_A, r_B and r_C are defined as follows:

$$\begin{aligned}
r_A &\geq \max(|\lambda_A|),\\
r_B &\geq \max(|\lambda_B|),\\
r_C &\geq \max(|\lambda_C|).
\end{aligned} \tag{2.10.3}$$

In the equations (2.10.3) λ_A, λ_B and λ_C represent the eigenvalues of the Jacobian matrices $\overline{\mathbf{A}}$, $\overline{\mathbf{B}}$ and $\overline{\mathbf{C}}$, respectively:

$$\begin{aligned}
\lambda_A &= (U, U, U, U + c(\xi_x^2 + \xi_y^2 + \xi_z^2)^{1/2}, U - c(\xi_x^2 + \xi_y^2 + \xi_z^2)^{1/2}),\\
\lambda_B &= (V, V, V, V + c(\eta_x^2 + \eta_y^2 + \eta_z^2)^{1/2}, V - c(\eta_x^2 + \eta_y^2 + \eta_z^2)^{1/2}),\\
\lambda_C &= (W, W, W, W + c(\zeta_x^2 + \zeta_y^2 + \zeta_z^2)^{1/2}, W - c(\zeta_x^2 + \zeta_y^2 + \zeta_z^2)^{1/2}).
\end{aligned} \tag{2.10.4}$$

In the expressions (2.10.4) the quantities U, V and W are contravariant velocity coordinates as defined by the equations (2.3.3), while c represents local speed of sound.

The equations (2.10) can be inverted in two steps:

$$\begin{aligned} &\left[\mathrm{I} + \beta\Delta t(\delta_\xi^- \overline{\mathrm{A}}^+ + \delta_\eta^- \overline{\mathrm{B}}^+ + \delta_\zeta^- \overline{\mathrm{C}}^+)\right]^n \Delta\overline{q}^{*n} = -\Delta t\, \overline{R}^n, \\ &\left[\mathrm{I} + \beta\Delta t(\delta_\xi^+ \overline{\mathrm{A}}^- + \delta_\eta^+ \overline{\mathrm{B}}^- + \delta_\zeta^+ \overline{\mathrm{C}}^-)\right]^n \Delta\overline{q}^{n} = \Delta\overline{q}^{*n}. \end{aligned} \tag{2.10.5}$$

The solution of the first system of equations (2.10.5) is done by a simple forward substitution and solution of the second system by a simple back substitution.

Two-pass LU implicit factorization scheme requires a correct introduction of the boundary condition on the left side of the equation (2.10.5). An approach, based on the characteristic variable boundary condition [7], is used in this paper. For a boundary across which there is no flow, the following conditions are specified:

$$\begin{aligned} p_b &= p_1 + \rho_1 c_1 (\overline{\zeta}_t + \overline{\zeta}_x u_1 + \overline{\zeta}_y v_1 + \overline{\zeta}_z w_1), \\ \rho_b &= \rho_1 + (p_b - p_1)/c_1^2, \\ u_b &= u_1 - \overline{\zeta}_x (\overline{\zeta}_t + \overline{\zeta}_x u_1 + \overline{\zeta}_y v_1 + \overline{\zeta}_z w_1), \\ v_b &= v_1 - \overline{\zeta}_y (\overline{\zeta}_t + \overline{\zeta}_x u_1 + \overline{\zeta}_y v_1 + \overline{\zeta}_z w_1), \\ w_b &= w_1 - \overline{\zeta}_z (\overline{\zeta}_t + \overline{\zeta}_x u_1 + \overline{\zeta}_y v_1 + \overline{\zeta}_z w_1), \end{aligned} \tag{2.10.6}$$

where $\overline{\zeta}_t$, $\overline{\zeta}_x$, $\overline{\zeta}_y$ and $\overline{\zeta}_z$ are

$$\begin{aligned} \overline{\zeta}_t &= \frac{\zeta_t}{(\zeta_x^2 + \zeta_y^2 + \zeta_z^2)^{1/2}}, & \overline{\zeta}_x &= \frac{\zeta_x}{(\zeta_x^2 + \zeta_y^2 + \zeta_z^2)^{1/2}}, \\ \overline{\zeta}_y &= \frac{\zeta_y}{(\zeta_x^2 + \zeta_y^2 + \zeta_z^2)^{1/2}}, & \overline{\zeta}_z &= \frac{\zeta_z}{(\zeta_x^2 + \zeta_y^2 + \zeta_z^2)^{1/2}}. \end{aligned} \tag{2.10.7}$$

In the relations (2.10.6) the index 1 denotes values of flow variables at the center of the first cell from the boundary, while the index b denotes values at the flow boundary. At the far-field boundaries (upstream, lateral and downstream), the flow was assumed to be undisturbed whenever the freestream Mach number M_∞ was less than one. If the freestream Mach number exceeded unity ($M_\infty \geq 1$), all the five flow variables were extrapolated at the outflow boundary from the nearest inside cells.

At the boundaries of the impermeable surfaces, information about the values of flow variables inside the body are needed, as can be concluded from the equations (2.10) and (2.10.1). In order to overcome the mentioned obstacle it is necessary to modify the applied scheme for cell indices $k = 1$:

$$\overline{\mathrm{C}}^+_{i,j,k-1/2} \Delta\overline{q}_{i,j,k-1} = \mathrm{E}\, \overline{\mathrm{C}}^-_{i,j,k-1/2} \Delta\overline{q}_{i,j,k}, \tag{2.10.8}$$

where the matrix $\mathbf{E}$ is defined in the following way:

$$\mathbf{E} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} . \tag{2.10.9}$$

This modification eliminates flow across the boundary. Similar can be done for cells at the plane of symmetry of aircraft. At the far-field boundaries Whitfield [8] has shown the validity of the assumption $\Delta\overline{q}^n = 0$. Now, the solution of the system of equations (2.10.5) can be efficiently solved by inversion of sparse triangular matrices without using large computer storage. Owing to the fact that there are only two factors present in this scheme, the factorization error can be reduced significantly. Although the alternating direction implicit (ADI) scheme has been valuable in two-dimensional problems, its inherent limitation in three dimensions suggests the LU approach. If the equation (2.10.5) is solved by a pass through computational space sweeping through diagonal plane, defined by $i + j + k = const$, program code can be fully vectorized and adapted for usage at supercomputers.

Central difference approximations in evaluation of the residual $\overline{R}^n$ in the equation (2.9.1) require an artificial viscosity in order to converge to a steady state of the fluid dynamic equations [9]. Additional dissipative terms, known as artificial viscosity terms, are added in order to inhibit any odd-even decoupling of the numerical solution by introduction of dissipation. On the other hand, artificial dissipation terms are added to eliminate high frequency oscillations in the neighborhood of shock waves. Also, from the mathematical theory for hyperbolic systems of inviscid conservation laws [10], the introduction of artificial dissipation is necessary to guarantee a unique weak solution. The artificial dissipation employed in this paper is the blending of second and fourth differences

$$R_1^n = (D_\xi^2 + D_\eta^2 + D_\zeta^2 - D_\xi^4 - D_\eta^4 - D_\zeta^4) q_{i,j,k}^n . \tag{2.11}$$

In the expression (2.11) term $D_\xi^2 q_{i,j,k}^n$ is defined as

$$D_\xi^2 q_{i,j,k}^n = d_{i+1/2,j,k}^{(2)} - d_{i-1/2,j,k}^{(2)} , \tag{2.11.1}$$

where

$$d_{i+1/2,j,k}^{(2)} = \epsilon_{i+1/2,j,k}^{(2)} \frac{J_{i+1/2,j,k}}{\Delta t} \delta_\xi^+ q_{i,j,k}^n , \tag{2.11.2}$$

while the term $D_\xi^4 q_{i,j,k}^n$ is evaluated from the relation

$$D_\xi^4 q_{i,j,k}^n = d_{i+1/2,j,k}^{(4)} - d_{i-1/2,j,k}^{(4)} , \tag{2.11.3}$$

where

$$d_{i+1/2,j,k}^{(4)} = \epsilon_{i+1/2,j,k}^{(4)} \frac{J_{i+1/2,j,k}}{\Delta t} \delta^{3+}_\xi q_{i,j,k}^n . \tag{2.11.4}$$

In the equations (2.11.3) and (2.11.4) the quantities $\delta_\xi^+ q_{i,j,k}^n$ and $\delta^3{}_\xi^+ q_{i,j,k}^n$ denote forward difference operators

$$\delta_\xi^+ q_{i,j,k}^n = q_{i+1,j,k}^n - q_{i,j,k}^n \tag{2.11.5}$$

and

$$\delta^3{}_\xi^+ q_{i,j,k}^n = q_{i+2,j,k}^n - 3q_{i+1,j,k}^n + 3q_{i,j,k}^n - q_{i-1,j,k}^n. \tag{2.11.6}$$

The remainder of dissipation terms from the equation (2.11) can be evaluated in a similar manner. The quantities $\epsilon_{i+1/2,j,k}^{(2)}$ and $\epsilon_{i+1/2,j,k}^{(4)}$, present in the equations (2.11.2) and (2.11.4) are defined as follows:

$$\epsilon_{i+1/2,j,k}^{(2)} = k^{(2)} \max(\nu_{i+1,j,k}, \nu_{i,j,k}) \tag{2.11.7}$$

and

$$\epsilon_{i+1/2,j,k}^{(4)} = \max(0, k^{(4)} - \epsilon_{i+1/2,j,k}^{(2)}), \tag{2.11.8}$$

where the quantity $\nu_{i,j,k}$ is defined by

$$\nu_{i,j,k} = \frac{|p_{i+1,j,k} - 2p_{i,j,k} + p_{i-1,j,k}|}{|p_{i+1,j,k} + 2p_{i,j,k} + p_{i-1,j,k}|}. \tag{2.11.9}$$

In the equations (2.11.7) and (2.11.8) values of the constants $k^{(2)} = O(1)$ and $k^{(4)}$ determine the amount of artificial viscosity added, and make transition from fourth order accurate scheme to second order accuracy in the neighborhood of shock waves. This is obvious in the zone of an abrupt change of pressure distribution, as can be concluded from the equation (2.11.8). On the other hand, amount of artificial dissipation determines the numerical stability of a scheme [11]. By increasing the values of $k^{(2)}$ and $k^{(4)}$ the numerical stability is improved by smoothing of the solution at the expense of reduced accuracy in the vicinity of shock wave location.

The boundary dissipation operators that are applied in this paper, based on finite volume discretisation, are introduced with special care. Having in mind the equations (2.11.5) and (2.11.6) evaluating the second difference dissipation term at the boundary requires information at one neighboring cell in each coordinate direction, while in the case of the fourth difference dissipation contribution is needed from two neighboring cells. For these cells a solution is found by a combination of the given physical and numerical boundary condition, which is a way to obtain necessary information and required stability of chosen difference stencil.

After the dissipation terms are added, the equation (2.10) takes the final form of

$$[\mathrm{I} + \beta\Delta t(\delta_\xi^- \overline{\mathrm{A}}^+ + \delta_\eta^- \overline{\mathrm{B}}^+ + \delta_\zeta^- \overline{\mathrm{C}}^+)]^n \times$$
$$\times [\mathrm{I} + \beta\Delta t(\delta_\xi^+ \overline{\mathrm{A}}^- + \delta_\eta^+ \overline{\mathrm{B}}^- + \delta_\zeta^+ \overline{\mathrm{C}}^-)]^n \Delta\overline{q}^n + \Delta t[\overline{R}^n - R_1^n] = 0. \tag{2.12}$$

Also, it is very important to determine correctly the time step size Δt, having in mind that the highest acceptable value is determined by a time interval of perturbation propagation from one side of a cell to another. The local time step Δt for the cell with indices (i,j,k) is evaluated in the following way:

$$\Delta t_{i,j,k} = \left[\frac{1}{(\Delta t_\xi)_{i,j,k}} + \frac{1}{(\Delta t_\eta)_{i,j,k}} + \frac{1}{(\Delta t_\zeta)_{i,j,k}} \right]^{-1}, \tag{2.13}$$

where $(\Delta t_\xi)_{i,j,k}$, $(\Delta t_\eta)_{i,j,k}$ and $(\Delta t_\zeta)_{i,j,k}$ are time intervals of perturbation propagation inside a cell in given coordinate directions. Time intervals $(\Delta t_\xi)_{i,j,k}$, $(\Delta t_\eta)_{i,j,k}$ and $(\Delta t_\zeta)_{i,j,k}$ can be evaluated in physical space by the following expressions:

$$\begin{aligned} (\Delta t_\xi)_{i,j,k} &= \left(|U| + c\sqrt{\xi_x^2 + \xi_y^2 + \xi_z^2} \right)^{-1}_{i,j,k}, \\ (\Delta t_\eta)_{i,j,k} &= \left(|V| + c\sqrt{\eta_x^2 + \eta_y^2 + \eta_z^2} \right)^{-1}_{i,j,k}, \\ (\Delta t_\zeta)_{i,j,k} &= \left(|W| + c\sqrt{\zeta_x^2 + \zeta_y^2 + \zeta_z^2} \right)^{-1}_{i,j,k}, \end{aligned} \tag{2.13.1}$$

In the equations (2.13.1) U, V and W are contravariant velocity coordinates, while c is the local speed of sound.

The time step size Δt, evaluated in the equation (2.13), is to be scaled with constant, known as the Courant number. Stability analysis of the applied two-pass implicit LU scheme [12] has shown insensitivity to relatively high values of the Courant number. In order to accelerate convergence to a steady state usage of local time stepping is highly recommended, having in mind that cell sizes may differ very drastically. Since flow properties do not vary rapidly inside one iteration cycle it is not necessary to repeat time step calculation after one single iteration.

It is obvious that shown discussion is not valid for unsteady flows, since constant time step $\Delta t = \min(\Delta t_{i,j,k})$ should be used.

3. Results

Computer results are presented for (1) steady transonic flow past a rectangular wing and (2) unsteady flow past a rectangular wing in plunge motion. Three-dimensional algebraically generated non-orthogonal "C–H" grids are used in all cases. In the calculations reported in this paper the convergence was considered to have been achieved when the value of residual was reduced by four orders of magnitude.

Computation for the first case is performed for a freestream Mach number $M_\infty = 0.8$ and $\alpha = 1.25°$ angle of attack. A coarse grid $(65 \times 11 \times 15)$ is employed and pressure distribution is presented for the section of the plane of symmetry. As shown in Fig. 1, the present coarse-grid solution for the wall-pressure distribution agrees quite well with the results of Swanson and Turkel [9], based on explicit scheme approach.

As can be seen from Fig. 1 agreement is very good around the leading edge of the airfoil and slightly worse in the neighborhood of the shock wave. The reason for this discrepancy lies in the application of very coarse grid. Grid refinement will obviously improve the pressure distribution in the vicinity of steep pressure gradients.

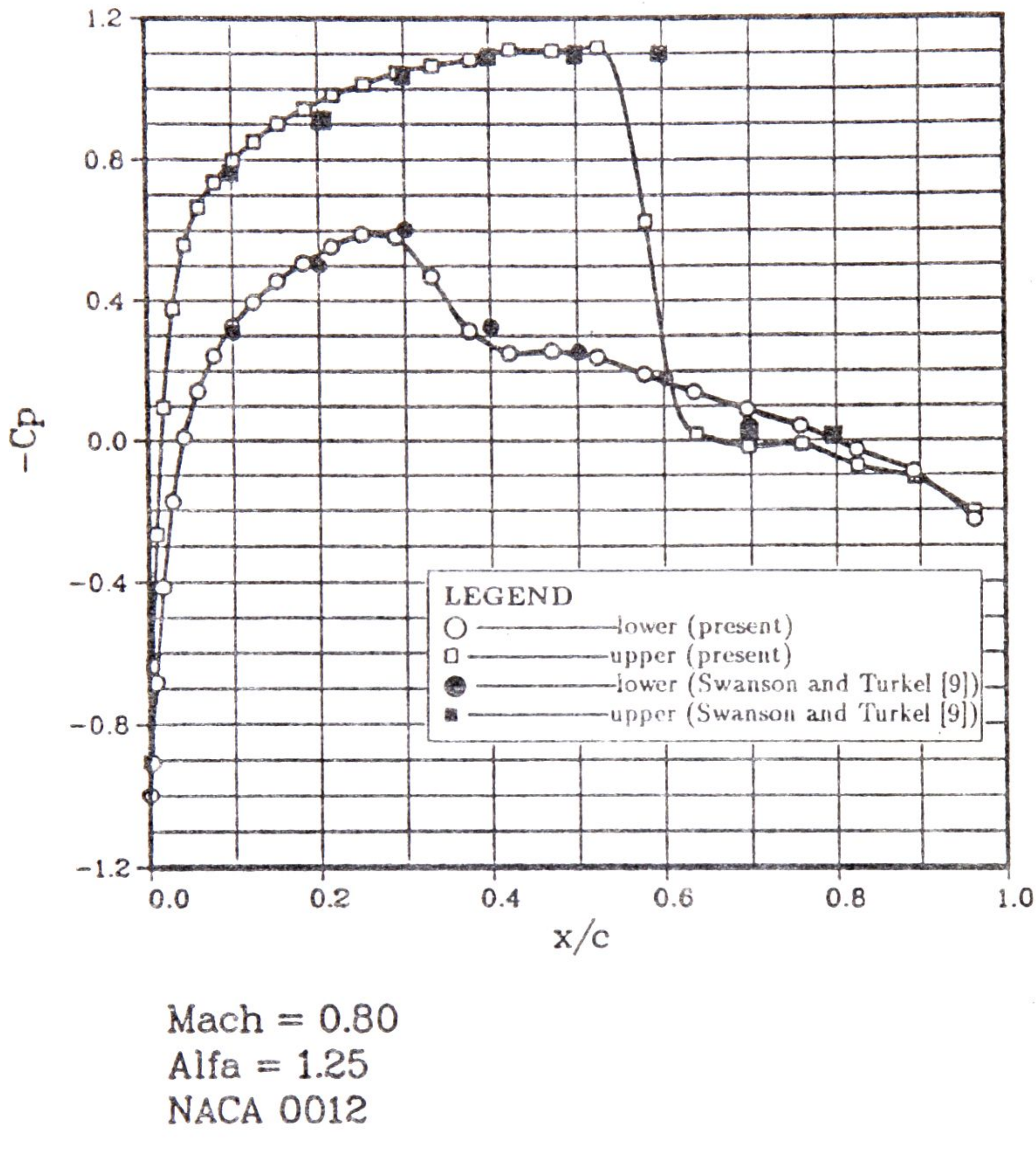

Fig. 1. Calculation of inviscid flow over a rectangular wing with NACA 0012 airfoil

In order to verify the validity of the present approach for unsteady flow calculation numerical experiment for plunge motion of rectangular NACA 65A010 wing is done.

The comparison of the present coarse-grid solution with Steger results, presented in Fig. 2, shows very good agreement of C_L distribution. The representative

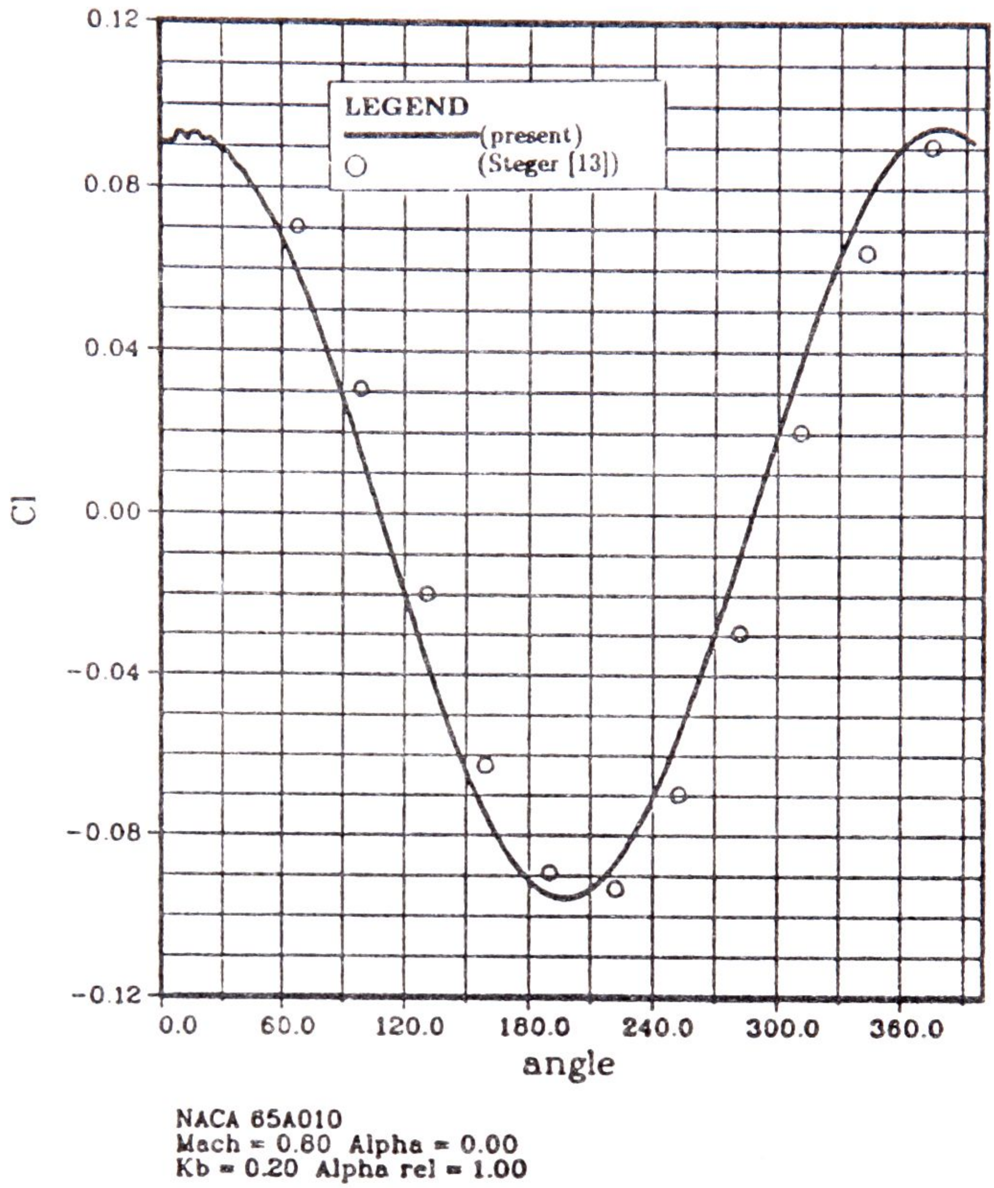

Fig. 2. C_L versus plunge angle in unsteady calculations of flow over rectangular NACA 65A010 wing

wing section is taken to lie in the plane of wing symmetry.

4. Conclusion

The applied approach in inviscid transonic flow analysis provides accurate aerodynamic load calculation especially in presence of very strong shock waves, when usage of the potential theory is unacceptable. The numerical stability and fast convergence in differential equation solution make a significant improvement in comparison with explicit scheme employment. Flux splitting LU implicit factorization scheme ensures the numerical stability even for very large time step sizes, i.e.

for $CFL \geq 20$. In highly time step dependent flow calculation, such as in the case of 3D unsteady motion, application of the present approach is quite acceptable. The presence of only two factors, even in the case of 3D flow reduces the factorization error. In comparison with classical ADI schemes LU implicit factorization decreases an amount of CPU time required for calculation. The chosen approach does not require large storage in solution of system of equations and employs only inversion of fifth order matrices. Correct program coding supports easy vectorization and usage at supercomputers. High accuracy of a solution is obtained by the modification of the scheme at physical boundaries, improving precise boundary condition definition.

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РАСЧЕТ ТРАНСЗВУКОВОГО ТЕЧЕНИЯ С ПОМОЩЮ НЕЯВНОЙ ФАКТОРИЗАЦИИ СИСТЕМ УРАВНЕНИЙ ЭЙЛЕРА

В работе разработан метод численного решения уравнений трехмерного безвязкостного трансзвукового течения, заснован на методе конечних объемов. Устойчивость и быстрая сходимость обеспечена введением членов искуственной вязкости второго и четвертого порядков. Система дифференциальных уравнений решена с помощью аппроксимативной „ЛУ“ неявной факторизации с разделением потока Джеймсон-Юна.

PRORAČUN TRANSONIČNOG STRUJNOG POLJA PRIMENOM LU IMPLICITNE FAKTORIZACIJE NA SISTEM JEDNAČINA EULER-A

U radu je izložen postupak numeričkog rešavanja jednačina trodimenzionalnog bezviskoznog transoničnog strujanja, baziran na metodi konačnih zapremina. Stabilnost i brza konvergencija postupka obezbeđuje se uvođenjem članova veštačke viskoznosti drugog i četvrtog reda. Sistem diferencijalnih jednačina rešavan je primenom aproksimativne LU implicitne faktorizacije sa „razdvajanjem" fluksa metodom Jameson-Yoon-a.

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