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PLASTICITY IN A DISK WITH TAPER

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1. Introduction

In thin disks (assuming $\sigma_z = 0$) of slightly variable thickness t, neglecting body forces (gravity, centrifugal, thermal), the equilibrium is given by the following stress balance between radial σ_R and circumferential σ_ϕ (hoop) components w.r.t. radii r [1]

$$\frac{d}{dr}(tr\sigma_R) = t\sigma_\phi. \tag{1}$$

This for hyperbolic shape $t = t_0(a/r)^n$, where $n \geq 0$, $a \leq r \leq b$, becomes (if $' = d/dr$)

$$\sigma_R(1-n) + r\sigma_R' = \sigma_\phi. \tag{2}$$

In the elastic domain ($c \leq r \leq b$, $c \geq a$) Hookean constitutive law applies, leading to the known 2nd order differential equation [2]

$$\sigma_R'' + \frac{3-n}{r}\sigma_R' - \frac{n(1+\nu)}{r^2}\sigma_R = 0 \tag{3}$$

with its general solution

$$\sigma_R = C_1 r^p + C_2 r^q \tag{4}$$

where p and q are roots of the characteristic polynomial

$$w^2 + (2-n)w - n(1+\nu) = 0 \tag{5}$$

i.e.

$$w = \frac{n}{2} - 1 \pm \sqrt{\left(\tfrac{1}{2}n\right)^2 + \nu n + 1} = (p, q). \tag{6}$$

Using boundary conditions ($\sigma_R(r = a) = -p_a$, $\sigma_R(r = b) = -p_b$) integration constants become

$$C_1 = \frac{p_b a^q - p_a b^q}{a^p b^q - a^q b^p}, \tag{7}$$

$$C_2 = \frac{p_b a^p - p_a b^p}{a^p b^q - a^q b^p}. \tag{8}$$

Taking into account internal loading only ($p_b = 0$) the stresses in elastic part are given as

$$\sigma_R = p_a \frac{b^p r^q - b^q r^p}{a^p b^q - a^q b^p}, \tag{9}$$

$$\sigma_\phi = p_a \frac{(1-n+q)b^p r^q - (1-n+p)b^q r^p}{a^p b^q - a^q b^p}. \tag{10}$$

If the internal pressure increases, plastic zone will develop inside of the disk body. Yield criterion has to be considered, which is given by Tresca or Mises formulae respectively [3]

$$\sigma_\phi - \sigma_R = 2K, \tag{11}$$

$$\sigma_\phi^2 - \sigma_\phi \sigma_R + \sigma_R^2 = 3K^2 \tag{12}$$

where K is shear yield, which is 15% higher for Mises than by Tresca.

2. Tresca case

Using eqs. (2) and (11) the following 1st order differential eq. is obtained

$$r\sigma_R' - n\sigma_R = 2K. \tag{13}$$

Its homogeneous left hand side may be integrated by separation of variables, giving

$$\sigma_R = Ar^n \tag{14}$$

and by variation of the constant A, the complete solution becomes

$$\sigma_R = Br^n - 2K/n \tag{15}$$

where B is an integration constant to be determined from boundary conditions.

With $n = 0$ the corresponding solution has to be modified into

$$\sigma_R = B + 2K\ln(r). \tag{16}$$

If boundary conditions are used ($\sigma_R(a) = -p_a$, $\sigma_R(b) = 0$) for $n = 0$ if follows that

$$\sigma_R = -p_a + 2K\ln(r/a) \tag{17}$$

and with ($n \neq 0$)

$$\sigma_R = -p_a\left(\frac{r}{a}\right)^n + \frac{2K}{n}\left(\left(\frac{r}{a}\right)^n - 1\right) = \left(-p_a + \frac{2K}{n}\right)\left(\frac{r}{a}\right)^n - \frac{2K}{n}. \tag{18}$$

On the plastic-elastic interface at $r = c$, using Tresca criterion (11) the pressure is

$$p_c = 2K\frac{c^p b^q - c^q b^p}{(q-n)b^p c^q - (p-n)b^q c^p} \tag{19}$$

so that stresses in the external elastic zone are given by

$$\sigma_R = 2K\frac{b^p r^q - b^q r^p}{(q-n)b^p c^q - (p-n)b^q c^p}, \tag{20}$$

$$\sigma_\phi = 2K\frac{(1-n+q)b^p r^q - (1-n+p)b^q r^p}{(q-n)b^p c^q - (p-n)b^q c^p}. \tag{21}$$

Continuity of σ_R at $r = c$ leads to the result, using eqs. (18) and (20)

$$p_a = \frac{2K}{n}\left(1 + a^n \frac{pb^q c^{p-n} - qb^p c^{q-n}}{(q-n)b^p c^q - (p-n)b^q c^p}\right) \tag{22}$$

and at the first yield ($c = a$) the internal pressure required is

$$p_a = 2K \frac{b^q a^p - b^p a^q}{(q-n)b^p a^q - (p-n)b^q a^p} \tag{23}$$

while the ultimate pressure, causing the whole disk to be plastic, ($c = b$)

$$p_a = \frac{2K}{n}\left(1 - \left(\frac{a}{b}\right)^n\right). \tag{24}$$

For $n = 0$ stresses in the external elastic zone are dependent upon the interface pressure, which in this case is

$$p_c = K\left(1 - \left(\frac{c}{b}\right)^2\right) \tag{25}$$

so that

$$\sigma_R = -K\left(\left(\frac{c}{r}\right)^2 - \left(\frac{c}{b}\right)^2\right), \tag{26}$$

$$\sigma_\phi = K\left(\left(\frac{c}{r}\right)^2 + \left(\frac{c}{b}\right)^2\right). \tag{27}$$

Continuity of σ_R at $r = c$ leads now to the known result [2] for internal pressure required

$$p_a = K\left(1 - \left(\frac{c}{b}\right)^2 + 2\ln\left(\frac{c}{a}\right)\right) \tag{28}$$

and at the first yield ($c = a$)

$$p_a = K\left(1 - \left(\frac{a}{b}\right)^2\right) \tag{29}$$

while the ultimate pressure ($c = b$) is

$$p_a = 2K\ln\left(\frac{b}{a}\right). \tag{30}$$

3. Mises case

Eq. (12) may be solved for hoop stress

$$\sigma_\phi = \frac{\sigma_R}{2} + \frac{\sqrt{3}}{2}\sqrt{4K^2 - \sigma_R^2}. \tag{31}$$

When this expression is substituted in eq. (2), the differential equation is separable [3].

Defining a new variable

$$s = -\frac{\sigma_R}{2K} \tag{32}$$

it follows

$$\frac{dr}{r} = \frac{-ds}{(\sqrt{3}/2)\sqrt{1-s^2} + (1/2)s(1-2n)} \tag{33}$$

which can be integrated by means of a substitution

$$s = \sin(\theta + \delta) \tag{34}$$

where

$$\delta = \arctan\left(\frac{1-2n}{\sqrt{3}}\right). \tag{35}$$

So the eq. (33) turns to become

$$\frac{dr}{r} = \frac{1}{2(1-n+n^2)}((1-2n)\tan\theta - \sqrt{3})\, d\theta \tag{36}$$

the solution of which is

$$\ln(r) = C - \frac{1}{2(1-n+n^2)}((1-2n)\ln(\cos(\theta)) + \sqrt{3}\theta) \tag{37}$$

where C is the constant of integration.

For $n = 0$ the corresponding result is

$$\ln(r) = C - \tfrac{1}{2}\left(\ln(\cos(\theta)) + \sqrt{3}\theta\right). \tag{38}$$

Consider now a fully plastic plane stress solution. In this case internal radial pressure is $p = 2K$, so that $r = a$ corresponds to (since $\sigma_R = -p = -2K$, and $\sigma_\phi = -K$) $s = 1$ or $\theta = \pi/2 - \delta$, while $r = b$ gives $s = 0$ or $\theta = -\delta$. It follows that b/a is given by

$$\ln\left(\frac{a}{b}\right) = \frac{1}{2(1-n+n^2)}\left[(1-2n)\ln\left(\frac{\cos(\pi/2-\delta)}{\cos(\delta)}\right) + \frac{\pi\sqrt{3}}{2}\right]. \tag{39}$$

For $0 < n \leq 1/2$ (since δ must be positive), the limit value (if $n = 1/2$) is

$$\ln\left(\frac{a}{b}\right) = \frac{\pi}{\sqrt{3}} = 1.8138\,.$$

With $n = 0$: $\delta = \arctan(1/\sqrt{3}) = \pi/6$ it follows that

$$\ln\left(\frac{a}{b}\right) = \frac{\pi\sqrt{3} - \ln 3}{4} = 1.0857\,.$$

The fully plastic solution is thus possible only if $b/a \leq e^{1.0857}$, $(n = 0)$, and $b/a \leq e^{1.8138}$, $(n = 1/2)$.

To determine the greatest extent of the plastic domain in the limit as $b/a \to \infty$, let's again set $\theta = \pi/2 - \delta$ at $r = a$, and $\theta = \pi/6 - \delta$ at $r = c$ since $\sigma_R = -K$ and $\sigma_\phi = K$ there, thus obtaining

$$\ln\left(\frac{c}{a}\right) = \frac{1}{2(1-n+n^2)}\left[(1-2n)\ln\left(\frac{\cos(\pi/2-\delta)}{\cos(\pi/6-\delta)}\right) + \frac{\pi\sqrt{3}}{3}\right]. \tag{40}$$

For $n = 1/2$ its value is

$$\ln\left(\frac{c}{a}\right) = \frac{2\pi}{3\sqrt{3}} = 1.2092$$

while with $n = 0$

$$\ln\left(\frac{c}{a}\right) = \frac{\pi/\sqrt{3} - \ln 2}{2} = 0.5603\,.$$

For example, with $n = 1/4$, $\ln(b/a) = 1.2920$ and $\ln(c/a) = 0.7307$.

4. Work hardening

With Tresca yield criterion (11), equilibrium (13) and flow rule

$$H = \frac{d\sigma_Y}{d\varepsilon^p_\phi} \tag{41}$$

the following differential equation for the plastic hoop strain in the hyperbolic disk is obtained at plane stress ($\sigma_z = 0$)

$$2\varepsilon^p_\phi + r\left(1 + \frac{H}{E}\right)\frac{d\varepsilon^p_\phi}{dr} + \frac{1}{E}(2\sigma_Y + n\sigma_R(1-\nu)) = 0\,. \tag{42}$$

Bearing in mind the radial stress distribution (15) renders the form

$$2\varepsilon^p_\phi + r\left(1 + \frac{H}{E}\right)\frac{d\varepsilon^p_\phi}{dr} + \frac{1}{E}\left((1+\nu)\sigma_Y + (1-\nu)(\sigma_Y - np_a)\left(\frac{r}{a}\right)^n\right) = 0 \tag{43}$$

the solution of which is

$$\varepsilon^p_\phi = \frac{C}{r^{2/(1+H/E)}} - \frac{1}{E}\left((1+\nu)\frac{\sigma_Y}{2} + \frac{(1-\nu)(\sigma_Y - np_a)(r/a)^n}{2 + n(1+H/E)}\right). \tag{44}$$

The integration constant C is to be determined subject to boundary condition $\varepsilon^p_\phi(r = c) = 0$, while $\sigma_Y(\varepsilon^p_\phi = 0) = 2K$, producing

$$\varepsilon^p_\phi = \frac{1}{E}\left[\frac{1+\nu}{2}\left(2K\left(\frac{c}{r}\right)^{2/(1+H/E)} - \sigma_Y\right) + \frac{1-\nu}{2+n(1+H/E)}\times\right.$$
$$\left.\times\left((2K - np_a)\left(\frac{c}{a}\right)^n\left(\frac{c}{r}\right)^{2/(1+H/E)} - (\sigma_Y - np_a)\left(\frac{r}{a}\right)^n\right)\right] \tag{45}$$

where Poisson's ratio $\nu = 1/2$ applies if noncompressible (isohoric) behaviour is assumed. For $n = 0$ the last formula reduces to

$$\varepsilon^p_\phi = \frac{1}{E}\left(2K\left(\frac{c}{r}\right)^{2/(1+H/E)} - \sigma_Y\right) \tag{46}$$

which may also be used for long tubes, just multiplied by $(1-\nu^2)$ in order to switch from the plane stress to plane strain case ($\varepsilon_z = 0$).

As shown by [3], the equation (46) may also be approximated to

$$\varepsilon^p_\phi = \frac{1}{E}\left(2K\left(\frac{c}{r}\right)^2 - \sigma_Y\right). \tag{47}$$

If $\varepsilon^p_\phi(r = a) = \varepsilon^p_0$, this means

$$c^2 = a^2 \frac{\sigma_Y + E\varepsilon^p_0}{2K} . \tag{48}$$

With boundary conditions

$$\sigma_R(r = a) = -p_a , \qquad \sigma_R(r = c) = K\left(1 - \left(\frac{c}{b}\right)^2\right) \tag{49}$$

the required pressure is given as

$$p_a = K\left(1 - \left(\frac{c}{b}\right)^2\right) + \frac{1}{2}\int_0^{\varepsilon^p_0} \frac{E + H}{E\varepsilon^p_\phi + \sigma_Y}\sigma_Y \, d\varepsilon^p_\phi . \tag{50}$$

The last two equations (49) and (50) provide the relation between p_a and c trough the parameter ε^p_0. The integration has in general to be carried out numerically, but an explicit relation is easily obtained if the hardening is linear, that is if H modulus is invariable.

Attempts to use Mises criterion (12) in place to Tresca (11) while determining work hardening in considered cylinders, appear infeasible.

5. Conclusion

Plasticity in a hyperbolic disk under internal pressure has been investigated. Closed form solutions are given subject to Tresca and Mises alternative criteria.

However, if the disk profile is assumed parabolic ($f \geq b$, $n > 0$)

$$t = t_0 \frac{f^n - r^n}{f^n - a^n} = t_0 \frac{1 - (r/f)^n}{1 - (a/f)^n}$$

$$\frac{t'}{t} = \frac{nr^{-1}}{1 - (f/r)^n} , \qquad \frac{t''}{t} = \frac{n(n-1)r^{-2}}{1 - (f/r)^n}$$

the solution in a closed form cannot be found, and numerical means have to be employed in order to sort out given differential equation of elastic and plastic behaviour respectively.

REFERENCES

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PLASTIZITÄT IN SCHEIBEN VERÄNDERLICHER DICKE

Im Beitrag sind Spannungen in dünnen Scheiben betrachtet im Falle wo die Belastung am inneren Rande die Fliessgrenze übersteigt. Resultate sind mit bekannten Werten für Scheiben mit konstanter Dicke ($n = 0$) verglichen, wobei Tresca

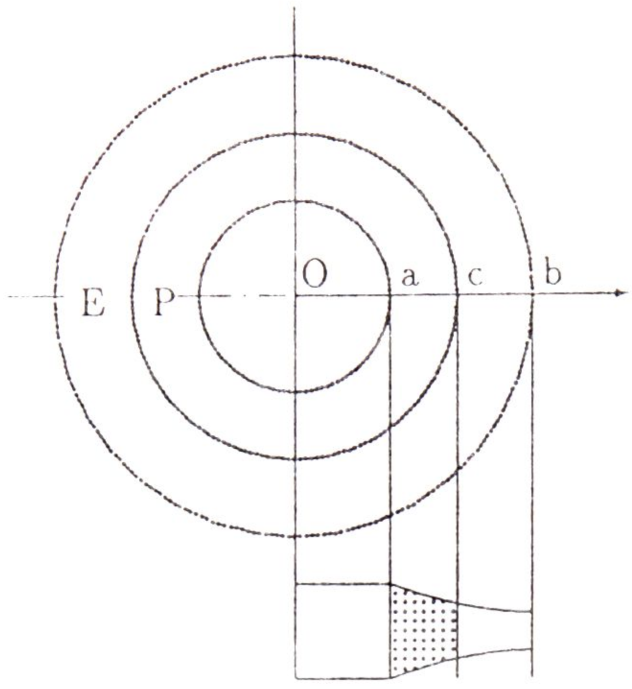

Fig. 1: Hyperbolic disk plastification

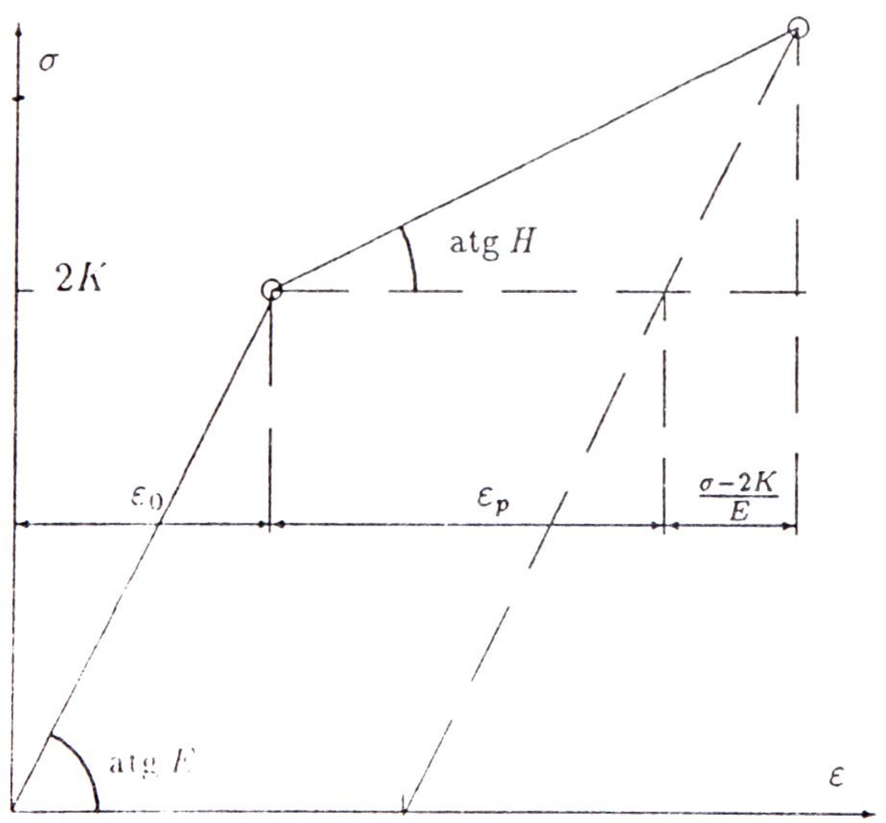

Fig. 2: Linear hardening

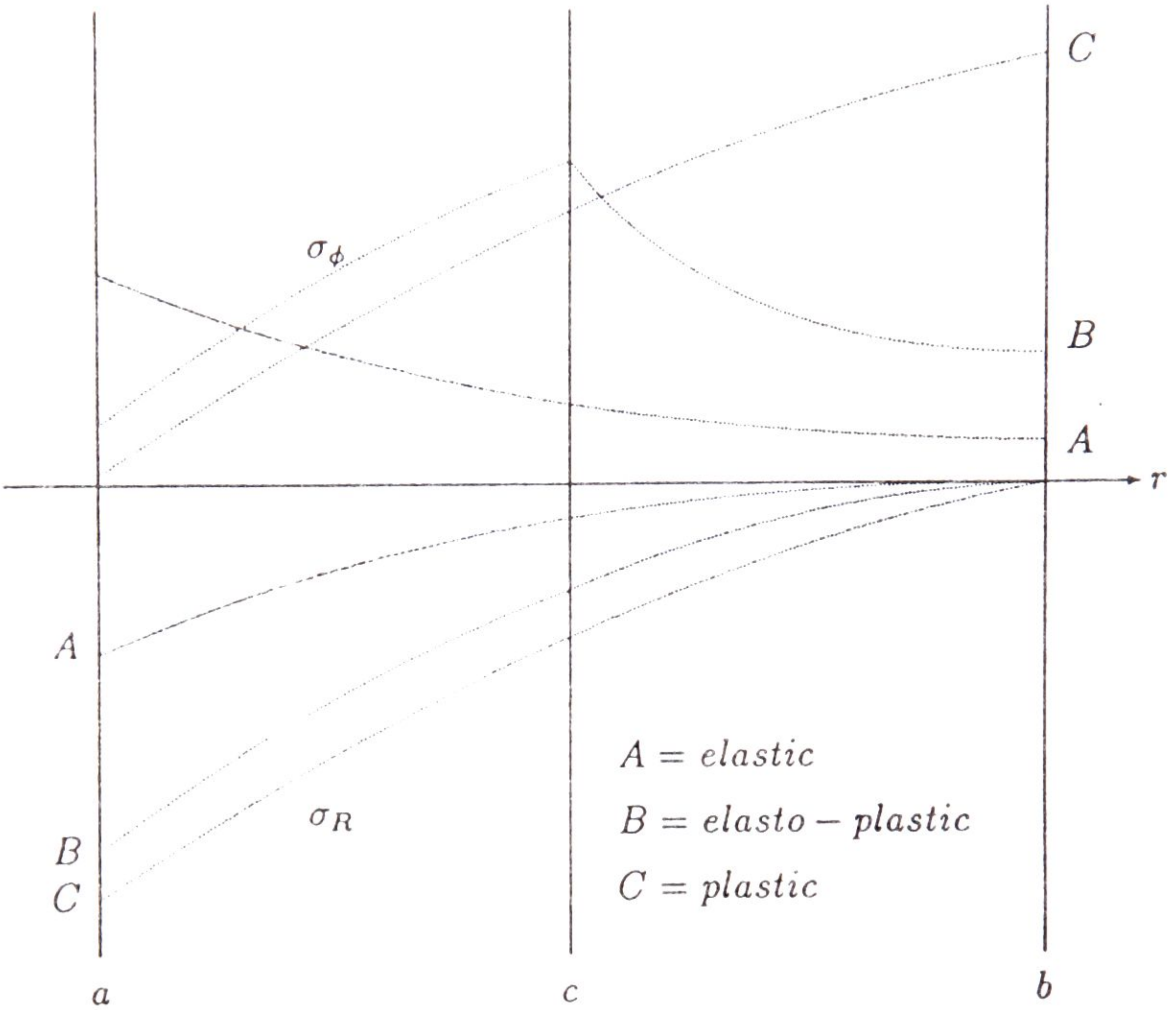

Fig. 3: Comparison of stress distributions due to internal pressure variation

oder von Mises Hypothesen und plastische Werkstoffsfliessen mit bzw. ohne Verfestigung angewandt sind.

PLASTIFIKACIJA DISKA SPREMENLJIVE DEBELINE

V prispevku obravnavamo razmere v hiperboličnem disku, ko zaradi visokega tlaka na notranjem polmeru pride do plastičnosti. Rezultati predstavljajo razširitev znanih obrazcev za nespremenljivo debelino ($n = 0$), pri čemer upoštevamo Tresca oziroma Misesovo hipotezo in plastično utrjevanje snovi.

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