TEORIJSKA I PRIMENJENA MEHANIKA 16, pp 79–83, 1990, UDK 531

ON A CONFORMALLY FLAT SPACE-TIME

Dragi Radojević

(Received 16.03.1990.)

In this paper we propose a metric based on Gödel cosmological model [1]. The cosmological model of K. Gödel was published in 1949. The model is stationary, axisymmetric and with cosmological constant. The cosmological constant was introoduced by Einstein in 1917 [2], to explain some global properties of space-time treated as a Riemannian space.

We propose a metric of the following form

$$ds^2 = a^2\left\{\xi(dx^1)^2 + \frac{1}{2}f^2(dx^2)^2 + \xi(dx^3)^2 - (f\,dx^2 + dx^4)^2\right\}.$$

The metric tensor is represented by two functions f and ξ both depending on time x^4 only. First, we calculate the Christoffel symbols of the second kind and write down only those which are different from zero. The overhead dot denotes differentiation with respect to time x^4

$$\left\{ {2 \atop 1\,1} \right\} = \frac{\dot\xi}{f} \qquad \left\{ {4 \atop 1\,1} \right\} = -\frac{1}{2}\dot\xi$$

$$\left\{ {1 \atop 1\,4} \right\} = \frac{1}{2}\frac{\dot\xi}{\xi}$$

$$\left\{ {2 \atop 2\,2} \right\} = -\dot f \qquad \left\{ {4 \atop 2\,2} \right\} = \frac{1}{2}f\dot f$$

$$\left\{ {2 \atop 2\,4} \right\} = -\frac{\dot f}{f} \qquad \left\{ {4 \atop 2\,4} \right\} = \dot f$$

$$\left\{ {2 \atop 3\,3} \right\} = \frac{1}{2}\frac{\dot\xi}{f} \qquad \left\{ {4 \atop 3\,3} \right\} = -\frac{1}{2}\dot\xi$$

$$\left\{ {3 \atop 3\,4} \right\} = \frac{1}{2}\frac{\dot\xi}{\xi}$$

$$\left\{ {2 \atop 4\,4} \right\} = -2\frac{\dot f}{f^2} \qquad \left\{ {4 \atop 4\,4} \right\} = 2\frac{\dot f}{f}.$$

Next we calculate the Ricci tensor. The result can be presented as the following matrix

$$[R_{\alpha\beta}] = \begin{bmatrix} \frac{1}{2}\ddot{\xi} + \frac{1}{2}\dot{\xi}\frac{\dot{f}}{f} & 0 & 0 & 0 \\ 0 & -\frac{1}{2}f\ddot{f} - \frac{1}{2}f\dot{f}\frac{\dot{\xi}}{f} & 0 & -\ddot{f} - \dot{f}\frac{\dot{\xi}}{\xi} \\ 0 & 0 & \frac{1}{2}\ddot{\xi} + \frac{1}{2}\dot{\xi}\frac{\dot{f}}{f} & 0 \\ 0 & -\ddot{f} - \dot{f}\frac{\dot{\xi}}{\xi} & 0 & \frac{\ddot{\xi}}{\xi} - \frac{1}{2}\frac{\dot{\xi}^2}{\xi^2} - \frac{\ddot{f}}{f} - 2\frac{\dot{\xi}}{\xi}\frac{\dot{f}}{f} \end{bmatrix}$$

we note the scalar curvature also

$$R = R^\alpha_\alpha = \frac{1}{a^2}\left\{2\frac{\ddot{\xi}}{\xi} - \frac{1}{2}\frac{\dot{\xi}^2}{\xi^2} + 2\frac{\ddot{f}}{f} + 2\frac{\dot{\xi}}{\xi}\frac{\dot{f}}{f}\right\}.$$

The form of the Ricci tensor allows us to put the condition

$$R_{\alpha\beta} = \kappa g_{\alpha\beta}.$$

If the solutions f and ξ satisfying this condition exist, the scalar curvature will be constant as in [1]. When the Ricci tensor is proportional to the metric tensor the space is called Einstein space [3].

The equations obtained from the conditions

$$R_{11} = \kappa g_{11} \qquad R_{33} = \kappa g_{33}$$

are identical, and the equations obtained from the conditions

$$R_{22} = \kappa g_{22} \qquad R_{24} = \kappa g_{24}$$

are equivalent. The equation

$$R_{44} = \kappa g_{44}$$

is independent. We get the following system of three equations containing only two unknown functions

$$\frac{1}{2}\ddot{\xi} + \dot{\xi}\frac{\dot{f}}{f} = \kappa a^2 \xi$$

$$-\ddot{f} - \dot{f}\frac{\dot{\xi}}{\xi} = -\kappa a^2 f$$

$$\frac{\ddot{\xi}}{\xi} - \frac{1}{2}\frac{\dot{\xi}^2}{\xi^2} - \frac{\ddot{f}}{f} - 2\frac{\dot{\xi}}{\xi}\frac{\dot{f}}{f} = -\kappa a^2.$$

We shall try to find some special solutions by supposing ξ and f to be of form

$$\xi = e^{\nu x^4} \qquad f = e^{n x^4}.$$

Substituting these expressions in the two first equations, we determine the constants ν and n and we get

$$\xi = e^{2a\sqrt{\kappa/3}x^4} \qquad f = e^{a\sqrt{\kappa/3}x^4}, \qquad \kappa > 0.$$

These solutions satisfy the third equation also. Function $\sqrt{\xi}$ is positive and so we have a correct signauture.

This possibility enables us to interpret the proposed metric, with ξ and f we obtained, as an example of Einstein space.

However, we have no physical interpretation of the coordinates. We shall try to find out some coordinate transformations that would put the metric into a diagonal form. The time part of the metric could be transformed in order to get the exact differential

$$f\,dx^2 + dx^4 = f\left(dx^2 + \frac{dx^4}{f}\right) = fd\left(x^2 + \int \frac{dx^4}{f}\right)$$
$$= fd\left(x^2 - \frac{\sqrt{3}}{a\sqrt{\kappa}} e^{-a\sqrt{\kappa/3}x^4}\right).$$

Next, we introduce a conformal metric

$$d\tilde{s}^2 = \frac{ds^2}{f} = a^2\left\{\frac{\xi}{f^2}(dx^1)^2 + \frac{1}{2}(dx^2)^2 + \frac{\xi}{f^2}(dx^3)^2 - \left(dx^2 + \frac{dx^4}{f}\right)^2\right\}.$$

This metric becomes simpler after substituting ξ and f

$$d\tilde{s}^2 = a^2\left\{(dx^1)^2 + \frac{1}{2}(dx^2)^2 + (dx^3)^2 - \left(dx^2 + \frac{dx^4}{f}\right)^2\right\}.$$

Now we have to propose a coordinate transformation. We chose

$$\bar{x}^1 = x^1$$
$$\bar{x}^2 = \nu x^2 + \psi$$
$$\bar{x}^3 = x^3$$
$$\bar{x}^4 = x^2 + \mu(x^4), \qquad \dot{\mu} = \frac{1}{f}.$$

Using the law of transformations of the metric tensor

$$g_{\alpha\beta} = \bar{g}_{\rho\nu}\frac{\partial \bar{x}^\rho}{\partial x^\alpha}\frac{\partial \bar{x}^\nu}{\partial x^\beta}$$

we form the system of equations for determining the components of the metric tensor, expressed in the new system of coordinates $\bar{x}^\alpha$.

$$\bar{g}_{11} = g_{11} \qquad \bar{g}_{13} = 0 \qquad \bar{g}_{33} = g_{33}$$
$$\nu\bar{g}_{12} + \bar{g}_{14} = 0 \qquad \nu\bar{g}_{23} + \bar{g}_{43} = 0$$
$$\left(\dot{\nu}x^2 + \dot{\psi}\right)\bar{g}_{12} + (1/f)\bar{g}_{14} = 0 \qquad \left(\dot{\nu}x^2 + \dot{\psi}\right)\bar{g}_{23} + (1/f)\bar{g}_{43} = 0$$
$$\nu^2\bar{g}_{22} + 2\nu\bar{g}_{24} + \bar{g}_{44} = -a^2/2$$
$$\nu\left(\dot{\nu}x^2 + \dot{\psi}\right)\bar{g}_{22} + \left((\nu/f) + \dot{\nu}x^2 + \dot{\psi}\right)\bar{g}_{24} + (1/f)\bar{g}_{44} = -a^2/f$$
$$\left(\dot{\nu}x^2 + \dot{\psi}\right)^2\bar{g}_{22} + (2/f)\left(\dot{\nu}x^2 + \dot{\psi}\right)\bar{g}_{24} + (1/f^2)\bar{g}_{44} = -(a^2/f^2)$$

This is a system of ten equations containing ten unknown components of the metric tensor and two unknown functions ν and ψ. We shall express the components of the metric tensor by the functions ν and ψ. By a suitable choice of these functions we shall try to give this conformal metric a diagonal form. From the threee last equations we calculate

$$\bar{g}_{24} = -\frac{a^2}{2}\frac{\dot{\nu}x^2 + \dot{\psi}}{f(\dot{\nu} + \dot{\psi} - (\nu/f))^2}.$$

In order to make this component null we choose

$$\nu = \nu_0 = \text{const} \qquad \psi = \psi_0 = \text{const}.$$

By substituting this solution in the system, we finq that all nondiagonal components become equal to zero and we get a diagonal metric

$$d\bar{s}^2 = a^2\left\{(d\bar{x}^1)^2 + \frac{1}{2\nu_0{}^2}(d\bar{x}^2)^2 + (d\bar{x}^3)^2 - (d\bar{x}^4)^2\right\}.$$

The components of the metric tensor are constants, the conformal metric represents a flat, Minkowski space.

Furthermore, we shall specify the constants

$$\nu_0 = -1 \qquad \psi_0 = 0$$

in order to make $\bar{x}^4$ the rising function of x^2 and x^4.

Now, we could express the metric we proposed in the beginning of the paper by new coordinates

$$ds^2 = \frac{3}{a^2\sqrt{\kappa}(\bar{x}^2 + \bar{x}^4)^2}d\bar{s}^2.$$

The modified Gödel metric, modified in the way we proposed, represents a conformally flat space.

REFERENCES

[1] Gödel K.: *An example of a new type of cosmological solution of Einstein field equation of gravitation*, Rev. Mod. Phys. **21**, 447 (1949).

[2] Einstein A.: *Cosmological considerations on the general theory of relativity*, Dover (1952).

[3] Petrov A. Z.: *Einstein Spaces*, Pergamon Press, (1969).

SUR UNE ESPACE-TEMPS CONFORMEMENT PLANE

Dans ce article on propose une métrique fondée sur le modèle cosmologique de Gödel. Le tenseur métrique ne depend que de temps. Deux fonctions de la métrique proposée, sont determinées de la condition que le tenseur de Ricci est proportionel à le tenseur métrique. En cette manière on obtient une example des espaces d'Einstein. Ensuite, on trouve des transformations de coordonées qui transforment la métrique proposée dans une métrique diagonale. De cette manière on demontre que la métrique proposée represente l'espace conforme à celui de Minkowski.

O JEDNOM KONFORMNO RAVNOM PROSTORU-VREMENU

Predlaže se metrika koja je bazirana na idejama vezanim za Gedelov kosmološki model. Metrički tenzor predstavljen je preko dve funkcije koje zavise samo od vremena. Te funkcije su određene iz uslova da Ričijev tenzor krivine bude proporcionalan metričkom tenzoru. Tako je dobijen jedan primer Ajnštajnovog prostora. Zatim se uvode koordinatne transformacije koje razmatranu metriku prevode u dijagonalni oblik. Korišćenjem tog, jednostavnijeg, oblika dokazano je da je predložena metrika konformno ravna.

Dragi Radojević
Matematički institut
Kneza Mihaila 35
11000 Beograd, Yugoslavia