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SOME GEOMETRICAL ASPECTS OF BERTOTTI-ROBINSON LIKE SPACETIME

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Using the fundamental principle of Einstein space and following Gödel's cosmological model [1]. Radojević [2] derived solutions for the Bertotti-Robinson like spacetime described by the metric

$$ds^2 = a^2\left[m\,dx^2 + \tfrac{1}{2}n^2\,dy^2 + m\,dz^2 - (dt + n\,dy)^2\right] \tag{1}$$

where $a =$ constant, $m = m(x)$ and $n = n(t)$.

Recently Mohanty and Pattanaik [3, 4] studied the role of cosmological constant and cosmological mass model in the presence of source free electromagnetic field in this spacetime. Besides the work of Radojević the important geometrical aspects of the spacetime (1) needed for physical study are not yet studied in detail. Therefore in this paper we took an attempt to derive solutions to Einstein's vacuum field equations. Again also we studied the nature of the spacetime (1) as regards to the space of constant curvature. However it does not provide any solution for which the spacetime (1) represents a space of constant curvature. Further we derived solution for the symmetric space and established the recurrent vector for recurrent space.

Einstein's vacuum solution. Out of twenty independent components of Riemannian curvature tensor for the metric (1), there are anly two distinct non-vanishing components which are

$$R_{1313} = (a^2/2m)(m'^2 - m''m) \tag{2}$$

and

$$R_{2424} = (a^2/2)n\ddot{n}. \tag{3}$$

Hereafterwards the prime and dot represent exact differentiations with respect to x and t respectively.

The Ricci tensor for the metric (1) can be calculated from (2) and (3) and obtained in the following form:

$$R_{ij} = \begin{bmatrix} m''/(2m) - m'^2/(2m^2) & 0 & 0 & 0 \\ 0 & -(1/2)n\ddot{n} & 0 & -\ddot{n} \\ 0 & 0 & m''/(2m) - m'^2/(2m^2) & 0 \\ 0 & -\ddot{n} & 0 & -\ddot{n}/n \end{bmatrix} \tag{4}$$

The Einstein vacuum field equations can be written as

$$R_{ij} = 0 \tag{5}$$

with the help of which (4) yields

$$\ddot{n} = 0 \tag{6}$$

$$\text{and} \quad mm'' - m'^2 = 0 \tag{7}$$

Equations (6) and (7) imply that n depends on t linearly and m depends on x exponentially in the following form:

$$n = At + B \tag{8}$$

$$\text{and} \quad m = De^{Cx} \tag{9}$$

where A, B, C and D are arbitrary constants.

For this solution, we find as $t \to \infty$, $n \to \pm\infty$ depending on the sign of constant A and as $x \to \infty$, $m \to \pm\infty$ or 0, according as $C > 0$ or $C < 0$. However, the solution is not asymptotically flat at spatial infinity as in the case of other symmetrical spacetimes. But it corresponds to singularity at both spatial and temporal infinities.

Space of constant curvature. A space for which

$$R_{hijk} = K(g_{hj}g_{ik} - g_{hk}g_{ij}) \tag{10}$$

(K is the curvature of the Riemannian manifold) is called a space of constant curvature [5]. The following study in the sense of Radojević [2] has the inference that the spacetime (1) is not of constant curvature in general curved space (i.e. for $K \neq 0$).

The calculation of R_{hijk} in the light of metric (1) and equations (10) yields

$$R_{1214} = R_{3234} = -K(a^4 mn) \tag{11}$$

$$R_{1414} = R_{3434} = -K(a^4 m) \tag{12}$$

$$R_{1212} = R_{2323} = -K(a^4 n^2 m/2) \tag{13}$$

$$R_{1313} = K(a^4 m^2) \quad \text{and} \tag{14}$$

$$R_{2424} = -K(a^4 n^2/2). \tag{15}$$

It is already shown that there are only two surviving components of R_{hijk} (viz., R_{1313} and R_{2424}). Therefore, equations (11)–(15) yield inconsistency for $K \neq 0$ in support of the space described by (1) to be a space of constant curvature. Thus the method of Radojević [2] is quite irrelevant to show that the space (1) is a space of constant curvature for non zero curvature. But the spacetime (1) trivially corresponds to a space of constant curvature for $K = 0$. Hence the solution for the space of constant curvature in the trivial case can be given by (8) and (9).

Symmetric space. The symmetric spaces [5] are characterised by the equation

$$R_{hijk;l} = 0. \tag{16}$$

Henceforth semicolon denotes covariant differentiation. For the two surviving components of R_{hijk} relative to the metric (1) equation (16) reduces to

$$R_{1313;1} = \frac{d}{dx}\left[\frac{a^2}{2m}(m'^2 - m''m)\right] - \frac{m'}{m}\left[\frac{a^2}{m}(m'^2 - m''m)\right] = 0 \tag{17}$$

$$R_{2424;2} = -a^2 n\dot{n}\ddot{n} = 0 \tag{18a}$$

$$R_{2424;4} = \frac{a^2}{2}(n\dddot{n} - \dot{n}\ddot{n}) = 0. \tag{18b}$$

It may be verified that except these three equations of $R_{hijk;l} = 0$, it leads to the fact the space described by the metric (1) is a symmetric space. Thus we design the space (1) as a symmetric space by solving equations (17) and (18).

From equation (18), one can easily get solution for n either as

$$n = \text{constant} \tag{19a}$$

$$\text{or} \quad n = Et + F \tag{19b}$$

where E and F are arbitrary constants.

Now equation (17) can be easily put in the form

$$(3mm'm'' - 3m'^3)/m^3 = (mm''' - m'm'')/m^2. \tag{20}$$

Integrating equation (20) we get

$$\left(\frac{m'}{m}\right)^2 = \frac{2}{3}\left(\frac{m''}{m}\right). \tag{21}$$

As the equation is highly non-linear, here the constants of integration is taken to be zero to avoid the mathematical complexity in subsequent calculation.

Again integrating equation (21), we get

$$m = 1/(Gx + H)^2 \tag{22}$$

where G and H are arbitrary constants.

Hence with the help of (19) and (22) we can confirm that the space described by (1) is a symmetric space.

Recurrent space. The recurrent spaces are defined [5] by the condition

$$R_{hijk;l} = K_l R_{hijk} \tag{23}$$

where (K_l) is the recurrent vector.

With the help of (2) eqn. (23) yields

$$K_1 = \frac{3m'^3 + m^2 m'''}{(m''m - m'^2)m}. \tag{24}$$

Again using (3) in equation (23), we obtain

$$K_2 = -2\dot{n} \tag{25a}$$

$$\text{and} \quad K_4 = (\dddot{n}/\ddot{n} - 1). \tag{25b}$$

The other component of recurrent vector (i.e. K_3) does not survive as the covariant derivative of both existing and non-existing components of R_{hijk} are zero. Hence the recurrent vector takes the form

$$\begin{aligned}(K_l) &= (K_1, K_2, 0, K_4) \\ &= \left(\frac{3m'^3m^2m'''}{(m''m - m'^2)m}, -2\dot{n}, 0, \left(\frac{\dddot{n}}{\ddot{n}} - 1\right)\right).\end{aligned} \tag{26}$$

It may be mentioned here that the solution of symmetric space and that of Einstein's vacuum case do not describe a recurrent space because the recurrent vector does not exist. Thus the spacetime (1) describes a recurrent space iff it does not represent a symmetric space. The interacting gravitation field in this spacetime may represent a recurrent space depending on the nature of its gravitational potential.

REFERENCES

[1] Gödel, K.: *An Example of a New Type of Cosmological Solutions to Einstein's Field Equations of Gravity*, Rev. Mov. Phys. **21**, 447 (1949).

[2] Radojević, D.: *One Example of Einstein Space*, Teor. i Prim. Meh. **13**, 77–81, (1988).

[3] Mohanty, G. and Pattanaik, R. R.: *Cosmological Electrovac Model in Bertotti-Robinson Spacetime*, Astrophys. Sp. Sc. **154**, 217–227 (1989).

[4] Mohanty, G. and Pattanaik, R. R.: *Cosmological Mass Model in General Relativity*, Astrophys. Sp. Sc. **161**, 253–261 (1989).

[5] Willmore, T. J.: *An Introduction to Differential Geometry*, Oxford University Press, Oxford, pp. 236–38, (1978).

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