

## ON VISCOPLASTICITY OF TRANSVERSELY ISOTROPIC MATERIALS

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**Introduction.** The objective of this paper is to present a rational thermodynamic approach to the so-called as-received anisotropic materials i.e. materials which are anisotropic even without anisotropy induced by a previous plastic deformation. The latter is usually referred to as Bauschinger effect.

In this paper like in [1-6] an associativity of flow rule i.e. the normality of the plastic strain rate tensor onto a yield surface has not been taken as granted even if such an approach is accepted in the majority of the papers dealing with the subject (compare with [7-10] and references mentioned in these papers). Such an approach is seriously questioned not only by the theoretical but by experimental results as well (see [11] for a comparison between tension and torsion and [12, 13] for experiments dealing with cruciform specimens.). For these reasons the normality is abandoned and instead of such an assumption evolution equations are naturally based on the appropriate geometry of deformation and the extended irreversible thermodynamics. This geometry is founded on the continuum theory of dislocations (compare with [14-19]) and is shortly reviewed in the next section. A very attractive approach to the extended thermodynamics has been proposed in [20] with a rational analysis of thermodynamic processes leading to the desired thermodynamic restrictions on general constitutive equations. This approach with the Liu's theorem [21] was applied to viscoplastic materials in [1] and to inelastic composite materials in [2, 3]. However despite of its beauty an inherent coldness function (which is not quite clear from the experimental point of view) is inevitable. In this paper an alternative approach [22, 23] (already applied to thermoplasticity of irradiated materials in [6]) is chosen instead.

In the first two sections geometry of the deformation process is considered and general constitutive as well as evolution equations are given and analyzed. Based on such a general background a special illustrative case of small thermoelastic and finite plastic strains is presented in the subsequent section in order to make an explicit distinction between the original and the acquired anisotropy. Such an assumption is realistic in the case of metal matrix composites for instance.

**Geometrical preliminaries.** As a prerequisite, a correct geometric description of an inelastic deformation process analyzed is needed. Consider a crystalline

body in a real configuration  $(\psi_t)$  with dislocations and an inhomogeneous temperature field  $\theta(X, t)$  (where  $t$  stands for time and  $X$  for the considered particle of the body) subject to surface tractions. Corresponding to  $(\psi_t)$  there exists, usually, an initial reference configuration  $(\kappa_0)$  with (differently distributed) dislocations at a homogeneous temperature  $\theta_0$  without surface tractions. Due to these defects such a configuration is not stressfree but contains an equilibrated residual stress (sometimes named ad "back-stress"). It is generally accepted that linear mapping function  $\mathbf{F}(\cdot, t) : (\kappa_0) \rightarrow (\psi_t)$  is compatible second order *total deformation gradient tensor*. Here  $t$  as scalar parameter allows for time developing family of deformed configurations  $(\psi_t)$ . In the papers dealing with continuum representations of dislocation distributions  $(\psi_t)$ -configuration is imagined to be cut into small elements denoted by  $(\nu_t)$ , these being subsequently brought to  $(\theta_0)$  free of neighbors. The deformation tensor  $\Delta_E(\cdot, t) : (\nu_t) \rightarrow (\psi_t)$  obtained in such a way is incompatible and should be called the *thermoelastic distortion tensor* whereas  $(\nu_t)$ -elements are commonly named as *natural state local reference configurations* (cf. eg. [14–19]). Moreover often as a plastic distortion tensor  $\Pi(\cdot, t) : (\kappa_\Pi) \rightarrow (\psi_t)$  is defined where  $(\kappa_\Pi)$  is a global ideal crystal having the same intrinsic crystalline structure as  $(\nu_t)$ -elements themselves. However, such a distortion is not unique since there are many indistinguishable configurations  $(\kappa_\Pi)$  with various shapes but the same intrinsic structure. Their indistinguishability is promoted to be a *replacement invariance principle* in [24]. The announced difficulty is overcome by the following definition of the *plastic distortion tensor*

$$\Delta_P(\cdot, t) := \Delta_E(\cdot, t)^{-1} \cdot \mathbf{F}(\cdot, t), \quad (1)$$

where  $\mathbf{F}(\cdot, t)$  is found by comparison of material fibres in  $(\kappa_0)$  and  $(\psi_t)$  whereas  $\Delta_E(\cdot, t)$  is determined by crystallographic vectors in  $(\nu_t)$  and  $(\psi_t)$ . Multiplying (1) from the left hand side by  $\Delta_E(\cdot, t)$  we reach at Kröner's decomposition rule which is often wrongly named as Lee's decomposition formula. The above two definitions of plastic distortion are easily connected by  $\Pi(\cdot, t) = \Delta_P(\cdot, t) \cdot \Delta_0(\cdot)$ . It is worthy of note that  $\text{curl } \Delta_E(\cdot, t)^{-1} \neq \mathbf{0}$  and this *incompatibility* is commonly connected to an asymmetric second rank tensor of dislocation density.

Additionally to the above description essentially already given elsewhere (compare for instance [18]) it is supposed that material of the considered body is anisotropic in  $(\kappa_0)$ -configuration with a direction of anisotropy given by the unit vector  $\vec{A}$ . If the cause of anisotropy is a continuous distribution of fibres (aligned in such a way to form one family of curvilinear lines) then  $\vec{A}$  is the unit tangent vector of fibre lines. If the corresponding unit tangent vectors in  $(\psi_t)$  and  $(\nu_t)$  are designated respectively by  $\vec{a}$  and  $\vec{A}_\nu$ , then they are related by means of the following formulae [3]:

$$\vec{a} = \frac{1}{\lambda_E} \vec{\Delta}_E \cdot \vec{A}_\nu = \frac{1}{\lambda} \mathbf{F} \cdot \vec{A}, \quad \vec{A}_\nu = \frac{1}{\lambda_P} \Delta_P \cdot \vec{A}, \quad (2)$$

where by  $\lambda_E$ ,  $\lambda_P$  and  $\lambda$  elastic, plastic and total fibre extension ratios are respectively denoted.

**General constitutive and evolution equations.** Accepting internal variables approach the following set of state variables is introduced

$$\Gamma := \{ \mathbf{E}_E, \theta, \text{grad } \theta, \vec{q}, \Delta_P, \mathbf{A} \}, \quad \Gamma \in \mathfrak{C}_\Gamma, \quad (3)$$

where

$\mathbf{E}_E = \frac{1}{2}(\Delta_E^T \cdot \Delta_E - 1)$  is the thermoelastic strain tensor,

$\text{grad } \theta = \Delta_E^{-1} \cdot \text{grad}_\psi \theta$  — temperature gradient with respect to natural state  $(\nu_t)$ -configuration,

$\vec{q} = J_E \Delta_E^{-1} \cdot \vec{q}_\psi$  — the  $(\nu_t)$ -defined heat flux vector,

$\mathbf{A} := \vec{A}_\nu \otimes \vec{A}_\nu$ ,  $J_E \equiv \det(1 + 2\mathbf{E}_E)^{1/2}$ ,

while  $\text{grad}_\psi \theta$  and  $\vec{q}_\psi$  are defined in  $(\psi_t)$ -configuration. The set (3) might be otherwise termed as temperature-deformation point belonging to the extended deformation space  $\mathfrak{C}_\Gamma$ .

To a temperature-deformation configuration point there corresponds reaction point represented by the response set

$$\Sigma_1 := \{\mathbf{S}, u, s, \vec{\phi}\}, \quad \Sigma_1 \in \mathfrak{C}_\Sigma \quad (4)$$

whose elements are:

$\mathbf{S} = J_E \Delta_E^{-1} \cdot \mathbf{S}_\psi \cdot \Delta_E^{-T}$  — the symmetric Piola-Kirchhoff stress tensor of the second kind related to  $(\nu_t)$ ,

$\mathbf{S}_\psi \equiv \mathbf{T}$  — the Cauchy stress in  $(\psi_t)$ -configuration,

$\vec{\phi} = J_E \Delta_E^{-1} \cdot \vec{\phi}_\psi$  — the entropy flux vector related to  $(\nu_t)$ ,

$u$  — the internal energy density, and

$s$  — the entropy density,

whereas  $\mathfrak{C}_\Sigma$  is the extended stress space.

At this place the constitutive equations are simply stated by the bijective mapping:

$$\Sigma_1 = \mathfrak{R}(\Gamma) = \hat{\Sigma}_1(\Gamma) \quad \text{or} \quad \mathfrak{R} : \mathfrak{C}_\Gamma \rightarrow \mathfrak{C}_\Sigma, \quad (5)$$

which is too general so that the thermodynamic analysis presented henceforth is aimed to introduce restrictions concordant with the second law of thermodynamics. It should be noted that the proposed constitutive equations are already frame indifferent i.e. objective with regard to a rigid body motion of an observer of  $(\psi_t)$ -configuration.

The evolution functions compatible with (3-5) are now collected into the set

$$\Sigma_2 := \{\vec{Q}, \mathbf{L}\}, \quad (6)$$

so that (objective) evolution equations simply read:

$$D\vec{q} = \vec{Q}(\Gamma), \quad (7)$$

$$\mathbf{L}_P = \mathbf{L}(\Gamma) \quad \text{where} \quad \mathbf{L}_P := D\Delta_P \cdot \Delta_P^{-1} \quad (8)$$

and the material time derivative is designated by  $D$ . The simplicity of left hand sides of (7-8) due to the absence of corotational (or convective) time derivatives has the origin in the isoclinicity of crystallographic vectors in all  $(\nu_t)$ -configurations during time elapsing [1]. Decomposition of the plastic "velocity gradient" tensor into its symmetric and antisymmetric parts i.e.

$$2\mathbf{D}_P := \mathbf{L}_P + \mathbf{L}_P^T, \quad 2\mathbf{W}_P := \mathbf{L}_P - \mathbf{L}_P^T, \quad (9)$$

gives respectively *plastic stretching* and *plastic spin tensor*.

A thermodynamic process in the considered body is described by the following balance laws ( $\rho_0$  and  $\rho$  are mass densities in  $(\kappa_0)$  and  $(\psi_t)$ -configurations): the internal energy balance equation

$$\rho_0 Du - \mathbf{S} : [D\mathbf{E}_E + (1 + 2\mathbf{E}_E) \cdot \mathbf{L}_P] + \text{div } \vec{q} = 0, \quad (10)$$

mass conservation law

$$\rho_0 - \rho J_E = 0, \quad (11)$$

and the equation of balance of momentum ( $\vec{v}$  is the velocity of the particle considered)

$$\rho \vec{v} - \text{div}_\psi (J_E^{-1} \Delta_E \cdot \mathbf{S} \cdot \Delta_E^T) = \vec{0}, \quad (12)$$

as well as by the evolution equations. These imply constraints on the elements of the set  $\{\Gamma, D\Gamma\}$  causing breaking of their independence which is the essence of the Liu's theorem [21]. In the case of an inelastic deformation process however the notion of the yield surface becomes essential dividing sharply the two regions of material behavior. Let us define dynamic and static scalar yield functions in the following way:

$$f = \hat{f}(\mathbf{S}, \theta, \Delta_P) \equiv h(\Gamma), \quad (13)$$

$$f_0 = \hat{f}(\mathbf{S}^\#, \theta, \Delta_P) \equiv h_0(\Gamma), \quad (14)$$

where  $\mathbf{S}^\#$  is static stress corresponding to the dynamic viscoplastic stress  $\mathbf{S}$  whereas their difference is usually called the *overstress tensor* which may be represented by a function of  $\mathbf{L}_P$  as follows:

$$\Delta \mathbf{S} := \mathbf{S} - \mathbf{S}^\# = \mathcal{I}(\Gamma) : \mathbf{L}_P(\Gamma), \quad (15)$$

with  $\mathcal{I}(\Gamma)$  being *fourth rank tensor of plastic viscosity coefficients*. Introducing the *plastic strain rate intensity* by

$$Dp := (\mathbf{L}_P^T : \mathbf{L}_P)^{1/2} \equiv \|\mathbf{L}_P\| \geq 0, \quad (16)$$

the classification:

$$f > 0, f_0 = 0, Dp > 0 \text{ — viscoplastic behavior;}$$

$$f = f_0 = 0, Dp = 0 \text{ — elastoplastic frontier;}$$

$$f = f_0 < 0, Dp = 0 \text{ — elastic behavior;}$$

and the kinematic constraint:

$$\langle f \rangle Df_0 = 0 \quad (17)$$

may be formulated in a straightforward way. In the above equation the function  $\langle x \rangle = 1$  for  $x$  positive and  $\langle x \rangle = 0$  otherwise.

All thermodynamic processes must obey the master law of nature i.e. the second law of thermodynamics which in our case reads:

$$\rho_0 Ds + \text{div } \vec{\phi} - \rho_0 \frac{r}{\theta} \geq 0, \quad (18)$$

where  $r/\theta$  is the entropy source. Precisely defined a thermodynamic process is a solution of evolution and balance equations which must satisfy (18). The analysis

of (18) by the Liu's theorem may be described as follows. Replacing the response functions  $\hat{s}(\Gamma)$  and  $\vec{\phi}(\Gamma)$  into the inequality (18) this becomes a differential inequality linear with respect to the elements of the set  $\{D\Gamma, \text{grad } \Gamma\}$  namely:

$$\begin{aligned} & \rho_0 Ds + \text{div } \vec{\phi} - \rho_0 \frac{r}{\theta} - \Lambda^u \{ \rho_0 Du - \mathbf{S} : [D\mathbf{E}_E + (\mathbf{1} + 2\mathbf{E}_E) \cdot \mathbf{L}_P] + \text{div } \vec{q} \} \\ & - \vec{\Lambda}^v \cdot [J_E^{-1} \rho_0 D\vec{v} - \text{div}_\psi (J_E^{-1} \cdot \Delta_E \cdot \mathbf{S} \cdot \Delta_E^T)] \\ & - \vec{\Lambda}^q \cdot [D\vec{q} - \vec{Q}(\Gamma)] - \Lambda^l : [\mathbf{L}_P - \mathbf{L}(\Gamma)] - \Lambda^f \langle f \rangle Df_0 \geq 0. \end{aligned} \quad (19)$$

By introducing Lagrange multipliers all the elements of the set  $\{D\Gamma, \text{grad } \Gamma\}$  except  $\text{grad } \theta$  (which is already included into  $\Gamma$ ) become independent. Therefore, in thus extended inequality all the coefficients escorting the elements of the set  $\{D\Gamma, \text{grad } \Gamma\}$  must vanish. This gives rise to the following constitutive restrictions

$$\mathbf{S} = \rho_0 \partial_{\mathbf{E}_E} F + \langle f \rangle \Lambda^f \partial_{\mathbf{E}_E} f_0, \quad (20)$$

$$s = \partial_\theta F + \langle f \rangle \rho_0^{-1} \Lambda^f \partial_\theta f_0, \quad (21)$$

$$0 = \partial_{\text{grad } \theta} F + \langle f \rangle \rho_0^{-1} \Lambda^f \partial_{\text{grad } \theta} f_0, \quad (22)$$

$$0 = \partial_{\mathbf{E}_E} \vec{\phi}, \quad 0 = \partial_{\text{grad } \theta} \vec{\phi}, \quad 0 = \partial_{\Delta_P} \vec{\phi}, \quad (23)$$

$$\frac{1}{\theta} \mathbf{1} = \partial_{\vec{q}} \vec{\phi} \quad (24)$$

and the residual dissipation inequality

$$\partial_\theta \vec{\phi} \cdot \text{grad } \theta + \vec{\Lambda}^q \cdot \vec{Q}(\Gamma) + \Lambda^l : \mathbf{L}(\Gamma) \geq 0, \quad (25)$$

where

$$F := u - s(\Lambda^u)^{-1} \equiv u - \theta s \quad (26)$$

is the free energy density. The Lagrange multipliers appearing in (20–24) are explicitly given by the following formulae:

$$\begin{aligned} \vec{\Lambda}^q &= -\theta^{-1} \partial_{\vec{q}} F - (\rho_0 \theta)^{-1} \langle f \rangle \Lambda^f \partial_{\vec{q}} f_0, & \vec{\Lambda}^v &= 0, \\ \Lambda^l &= -\theta^{-1} \mathbf{S} \cdot (\mathbf{1} + 2\mathbf{E}) - \theta^{-1} \Delta_P \cdot (\rho_0 \partial_{\Delta_P} F - \langle f \rangle \Lambda^f \partial_{\Delta_P} f_0). \end{aligned}$$

Integrating (23–24) we acquire additionally

$$\vec{\phi} = \theta^{-1} \vec{q} + \vec{\phi}_1(\theta), \quad (27)$$

where  $\vec{\phi}_1(\theta)$  is some function of temperature.

**Small thermoelastic strains.** Suppose in the sequel that:

(A1) thermoelastic strain, temperature gradient and heat flux are small but plastic strain itself is finite;

(A2) stress vanishes when pure elastic strain equals to zero.

Under such assumptions and making use of tensor representations theory (given in [25–28]) the stress tensor may be represented by the following formula:

$$\mathbf{S} = \mathcal{D} : [\mathbf{E}_E - \alpha \mathbf{1}(\theta - \theta_0)] \equiv \mathcal{D} : \mathbf{E}_e = \sum_{\alpha} a_{\alpha} \mathbf{G}_{\alpha}, \quad (28)$$

with the generators — symmetric second rank tensors given below:

$$\begin{aligned}
 \mathbf{G}_1 &= \mathbf{1}, & \mathbf{G}_2 &= \boldsymbol{\varepsilon}_P, & \mathbf{G}_3 &= \boldsymbol{\varepsilon}_P^2, & \mathbf{G}_4 &= \mathbf{A}, \\
 \mathbf{G}_5 &= \mathbf{A} \cdot \boldsymbol{\varepsilon}_P + \boldsymbol{\varepsilon}_P \cdot \mathbf{A}, & \mathbf{G}_6 &= \mathbf{A} \cdot \boldsymbol{\varepsilon}_P^2 + \boldsymbol{\varepsilon}_P^2 \cdot \mathbf{A}, \\
 \mathbf{G}_7 &= \mathbf{E}_e, & \mathbf{G}_8 &= \mathbf{E}_e \cdot \boldsymbol{\varepsilon}_P + \boldsymbol{\varepsilon}_P \cdot \mathbf{E}_e, & \mathbf{G}_9 &= \mathbf{E}_e \cdot \mathbf{A} + \mathbf{A} \cdot \mathbf{E}_e, \\
 \mathbf{G}_{10} &= \mathbf{E}_e \cdot \boldsymbol{\varepsilon}_P^2 + \boldsymbol{\varepsilon}_P^2 \cdot \mathbf{E}_e, & \mathbf{G}_{11} &= \mathbf{E}_e \cdot \mathbf{A} \cdot \boldsymbol{\varepsilon}_P + \boldsymbol{\varepsilon}_P \cdot \mathbf{A} \cdot \mathbf{E}_e, \\
 \mathbf{G}_{12} &= \mathbf{A} \cdot \mathbf{E}_e \cdot \boldsymbol{\varepsilon}_P + \boldsymbol{\varepsilon}_P \cdot \mathbf{E}_e \cdot \mathbf{A}, & \mathbf{G}_{13} &= \mathbf{A} \cdot \boldsymbol{\varepsilon}_P \cdot \mathbf{E}_e + \mathbf{E}_e \cdot \boldsymbol{\varepsilon}_P \cdot \mathbf{A}, \\
 \mathbf{G}_{14} &= \mathbf{E}_e \cdot \mathbf{A} \cdot \boldsymbol{\varepsilon}_P^2 + \boldsymbol{\varepsilon}_P^2 \cdot \mathbf{A} \cdot \mathbf{E}_e, & \mathbf{G}_{15} &= \mathbf{A} \cdot \mathbf{E}_e \cdot \boldsymbol{\varepsilon}_P^2 + \boldsymbol{\varepsilon}_P^2 \cdot \mathbf{E}_e \cdot \mathbf{A}, \\
 \mathbf{G}_{16} &= \mathbf{A} \cdot \boldsymbol{\varepsilon}_P^2 \cdot \mathbf{E}_e + \mathbf{E}_e \cdot \boldsymbol{\varepsilon}_P^2 \cdot \mathbf{A}
 \end{aligned} \tag{29}$$

whereas in terms of the following scalar invariants

$$\begin{aligned}
 i_1 &= \mathbf{1} : \mathbf{E}_e \equiv \text{tr } \mathbf{E}_e, & i_2 &= \mathbf{E}_e : \boldsymbol{\varepsilon}_P, & i_3 &= \mathbf{E}_e : \boldsymbol{\varepsilon}_P^2, & i_4 &= \mathbf{E}_e : \mathbf{A}, \\
 i_5 &= \vec{A}_\nu \cdot \mathbf{E}_e \cdot \boldsymbol{\varepsilon}_P \cdot \vec{A}_\nu, & i_6 &= \vec{A}_\nu \cdot \mathbf{E}_e \cdot \boldsymbol{\varepsilon}_P^2 \cdot \vec{A}_\nu, \\
 \pi_1 &= \mathbf{1} : \boldsymbol{\varepsilon}_P, & \pi_2 &= \mathbf{1} : \boldsymbol{\varepsilon}_P^2, & \pi_3 &= \mathbf{1} : \boldsymbol{\varepsilon}_P^3, & \pi_4 &= \mathbf{A} : \boldsymbol{\varepsilon}_P, & \pi_5 &= \mathbf{A} : \boldsymbol{\varepsilon}_P^2, \\
 \gamma_\pi &:= \{\pi_1, \pi_2, \pi_3, \pi_4, \pi_5, \theta\},
 \end{aligned} \tag{30}$$

the scalar coefficients  $a_\alpha$ ,  $\alpha \in \{1, 16\}$  appearing in (28) may be expressed in the following way:

$$a_k = \sum_1^6 a_{k\alpha}(\gamma_\pi) i_\alpha, \quad k \in \{1, 6\} \quad \text{and} \quad a_k = a_k(\gamma_\pi), \quad k \in \{7, 16\}. \tag{31}$$

In the above equations the Eulerian plastic strain tensor connected to the vectorial base in  $(\nu_t)$ -configuration is given by

$$\boldsymbol{\varepsilon}_P = \frac{1}{2}(\mathbf{1} - \Delta_P^T \cdot \Delta_P^{-1}). \tag{32}$$

Other equations being of exceptional importance for us are evolution equations for the plastic stretching tensor and for the plastic spin tensor originating from (8) under assumptions (A1–A2). It is customary to connect the plastic strain rate with the stress tensor. If this is accepted then the expression for the plastic stretching reads:

$$\mathbf{D}_P = \sum_\alpha b_\alpha \mathbf{H}_\alpha, \quad \alpha \in \{1, 37\} \tag{33}$$

with the inherent generators — symmetric second rank tensors

$$\begin{aligned}
 \mathbf{H}_1 &= \mathbf{1} & \mathbf{H}_2 &= \mathbf{S}, & \mathbf{H}_3 &= \boldsymbol{\varepsilon}_P, & \mathbf{H}_4 &= \mathbf{A}, & \mathbf{H}_5 &= \mathbf{S}^2, & \mathbf{H}_6 &= \boldsymbol{\varepsilon}_P^2, \\
 \mathbf{H}_7 &= \mathbf{S} \cdot \boldsymbol{\varepsilon}_P + \boldsymbol{\varepsilon}_P \cdot \mathbf{S}, & \mathbf{H}_8 &= \mathbf{A} \cdot \mathbf{S} + \mathbf{S} \cdot \mathbf{A}, & \mathbf{H}_9 &= \mathbf{A} \cdot \boldsymbol{\varepsilon}_P + \boldsymbol{\varepsilon}_P \cdot \mathbf{A}, \\
 \mathbf{H}_{10} &= \mathbf{S}^2 \cdot \boldsymbol{\varepsilon}_P + \boldsymbol{\varepsilon}_P \cdot \mathbf{S}^2, & \mathbf{H}_{11} &= \mathbf{A} \cdot \mathbf{S}^2 + \mathbf{S}^2 \cdot \mathbf{A}, \\
 \mathbf{H}_{12} &= \mathbf{S} \cdot \boldsymbol{\varepsilon}_P^2 + \boldsymbol{\varepsilon}_P^2 \cdot \mathbf{S}, & \mathbf{H}_{13} &= \mathbf{A} \cdot \boldsymbol{\varepsilon}_P^2 + \boldsymbol{\varepsilon}_P^2 \cdot \mathbf{A}, \\
 \mathbf{H}_{14} &= \mathbf{S} \cdot \boldsymbol{\varepsilon}_P \cdot \mathbf{A} + \mathbf{A} \cdot \boldsymbol{\varepsilon}_P \cdot \mathbf{S}, & \mathbf{H}_{15} &= \mathbf{A} \cdot \mathbf{S} \cdot \boldsymbol{\varepsilon}_P + \boldsymbol{\varepsilon}_P \cdot \mathbf{S} \cdot \mathbf{A},
 \end{aligned}$$

$$\begin{aligned}
H_{16} &= \mathbf{S} \cdot \mathbf{A} \cdot \boldsymbol{\varepsilon}_P + \boldsymbol{\varepsilon}_P \cdot \mathbf{A} \cdot \mathbf{S}, \\
H_{17} &= \mathbf{S}^2 \cdot \mathbf{A} \cdot \boldsymbol{\varepsilon}_P + \boldsymbol{\varepsilon}_P \cdot \mathbf{A} \cdot \mathbf{S}^2, & H_{18} &= \mathbf{S}^2 \cdot \boldsymbol{\varepsilon}_P \cdot \mathbf{A} + \mathbf{A} \cdot \boldsymbol{\varepsilon}_P \cdot \mathbf{S}^2, \\
H_{19} &= \mathbf{A} \cdot \mathbf{S}^2 \cdot \boldsymbol{\varepsilon}_P + \boldsymbol{\varepsilon}_P \cdot \mathbf{S}^2 \cdot \mathbf{A}, & H_{20} &= \mathbf{S} \cdot \boldsymbol{\varepsilon}_P^2 \cdot \mathbf{A} + \mathbf{A} \cdot \boldsymbol{\varepsilon}_P^2 \cdot \mathbf{S}, \\
H_{21} &= \mathbf{A} \cdot \mathbf{S} \cdot \boldsymbol{\varepsilon}_P^2 + \boldsymbol{\varepsilon}_P^2 \cdot \mathbf{S} \cdot \mathbf{A}, & H_{22} &= \mathbf{S} \cdot \mathbf{A} \cdot \boldsymbol{\varepsilon}_P^2 + \boldsymbol{\varepsilon}_P^2 \cdot \mathbf{A} \cdot \mathbf{S}, \\
H_{23} &= \mathbf{S}^2 \cdot \boldsymbol{\varepsilon}_P^2 \cdot \mathbf{A} + \mathbf{A} \cdot \boldsymbol{\varepsilon}_P^2 \cdot \mathbf{S}^2, & H_{24} &= \mathbf{A} \cdot \mathbf{S}^2 \cdot \boldsymbol{\varepsilon}_P^2 + \boldsymbol{\varepsilon}_P^2 \cdot \mathbf{S}^2 \cdot \mathbf{A}, \\
H_{25} &= \mathbf{S}^2 \cdot \mathbf{A} \cdot \boldsymbol{\varepsilon}_P^2 + \boldsymbol{\varepsilon}_P^2 \cdot \mathbf{A} \cdot \mathbf{S}^2, \\
H_{26} &= \mathbf{S}^2 \cdot \boldsymbol{\varepsilon}_P \cdot \mathbf{S} \cdot \mathbf{A} + \mathbf{A} \cdot \mathbf{S} \cdot \boldsymbol{\varepsilon}_P \cdot \mathbf{S}^2, & H_{27} &= \mathbf{S}^2 \cdot \boldsymbol{\varepsilon}_P \cdot \mathbf{A} \cdot \mathbf{S} + \mathbf{S} \cdot \mathbf{A} \cdot \boldsymbol{\varepsilon}_P \cdot \mathbf{S}^2, \\
H_{28} &= \mathbf{S}^2 \cdot \mathbf{A} \cdot \boldsymbol{\varepsilon}_P \cdot \mathbf{S} + \mathbf{S} \cdot \boldsymbol{\varepsilon}_P \cdot \mathbf{A} \cdot \mathbf{S}^2, & H_{29} &= \mathbf{S}^2 \cdot \mathbf{A} \cdot \mathbf{S} \cdot \boldsymbol{\varepsilon}_P + \boldsymbol{\varepsilon}_P \cdot \mathbf{S} \cdot \mathbf{A} \cdot \mathbf{S}^2, \\
H_{30} &= \boldsymbol{\varepsilon}_P \cdot \mathbf{S}^2 \cdot \mathbf{A} \cdot \mathbf{S} + \mathbf{S} \cdot \mathbf{A} \cdot \mathbf{S}^2 \cdot \boldsymbol{\varepsilon}_P, & H_{31} &= \mathbf{S} \cdot \boldsymbol{\varepsilon}_P \cdot \mathbf{S}^2 \cdot \mathbf{A} + \mathbf{A} \cdot \mathbf{S}^2 \cdot \boldsymbol{\varepsilon}_P \cdot \mathbf{S}, \\
H_{32} &= \boldsymbol{\varepsilon}_P^2 \cdot \mathbf{S} \cdot \boldsymbol{\varepsilon}_P \cdot \mathbf{A} + \mathbf{A} \cdot \boldsymbol{\varepsilon}_P \cdot \mathbf{S} \cdot \boldsymbol{\varepsilon}_P^2, & H_{33} &= \boldsymbol{\varepsilon}_P^2 \cdot \mathbf{A} \cdot \boldsymbol{\varepsilon}_P \cdot \mathbf{S} + \mathbf{S} \cdot \boldsymbol{\varepsilon}_P \cdot \mathbf{A} \cdot \boldsymbol{\varepsilon}_P^2, \\
H_{34} &= \boldsymbol{\varepsilon}_P^2 \cdot \mathbf{S} \cdot \mathbf{A} \cdot \boldsymbol{\varepsilon}_P + \boldsymbol{\varepsilon}_P \cdot \mathbf{A} \cdot \mathbf{S} \cdot \boldsymbol{\varepsilon}_P^2, & H_{35} &= \boldsymbol{\varepsilon}_P^2 \cdot \mathbf{A} \cdot \mathbf{S} \cdot \boldsymbol{\varepsilon}_P + \boldsymbol{\varepsilon}_P \cdot \mathbf{S} \cdot \mathbf{A} \cdot \boldsymbol{\varepsilon}_P^2, \\
H_{36} &= \mathbf{S} \cdot \boldsymbol{\varepsilon}_P^2 \cdot \mathbf{A} \cdot \boldsymbol{\varepsilon}_P + \boldsymbol{\varepsilon}_P \cdot \mathbf{A} \cdot \boldsymbol{\varepsilon}_P^2 \cdot \mathbf{S}, & H_{37} &= \mathbf{A} \cdot \boldsymbol{\varepsilon}_P^2 \cdot \mathbf{S} \cdot \boldsymbol{\varepsilon}_P + \boldsymbol{\varepsilon}_P \cdot \mathbf{S} \cdot \boldsymbol{\varepsilon}_P^2 \cdot \mathbf{A}.
\end{aligned} \tag{34}$$

Similarly, the plastic spin tensor is written in the following way:

$$W_\alpha = \sum_{\alpha} b_\alpha(\gamma_\pi) \mathbf{H}_\alpha, \quad \alpha \in \{38, 72\}, \tag{35}$$

with the next antisymmetric second rank tensors as generators

$$\begin{aligned}
H_{38} &= \mathbf{S} \cdot \mathbf{A} - \mathbf{A} \cdot \mathbf{S}, & H_{39} &= \mathbf{S} \cdot \boldsymbol{\varepsilon}_P - \boldsymbol{\varepsilon}_P \cdot \mathbf{S}, & H_{40} &= \boldsymbol{\varepsilon}_P \cdot \mathbf{A} - \mathbf{A} \cdot \boldsymbol{\varepsilon}_P, \\
H_{41} &= \mathbf{S}^2 \cdot \mathbf{A} - \mathbf{A} \cdot \mathbf{S}^2, & H_{42} &= \mathbf{S}^2 \cdot \boldsymbol{\varepsilon}_P - \boldsymbol{\varepsilon}_P \cdot \mathbf{S}^2, \\
H_{43} &= \boldsymbol{\varepsilon}_P^2 \cdot \mathbf{A} - \mathbf{A} \cdot \boldsymbol{\varepsilon}_P^2, & H_{44} &= \boldsymbol{\varepsilon}_P^2 \cdot \mathbf{S} - \mathbf{S} \cdot \boldsymbol{\varepsilon}_P^2, \\
H_{45} &= \boldsymbol{\varepsilon}_P^2 \cdot \mathbf{S}^2 - \mathbf{S}^2 \cdot \boldsymbol{\varepsilon}_P^2, \\
H_{46} &= \mathbf{S}^2 \cdot \mathbf{A} \cdot \mathbf{S} - \mathbf{S} \cdot \mathbf{A} \cdot \mathbf{S}^2, & H_{47} &= \mathbf{S}^2 \cdot \boldsymbol{\varepsilon}_P \cdot \mathbf{S} - \mathbf{S} \cdot \boldsymbol{\varepsilon}_P \cdot \mathbf{S}^2, \\
H_{48} &= \boldsymbol{\varepsilon}_P^2 \cdot \mathbf{A} \cdot \boldsymbol{\varepsilon}_P - \boldsymbol{\varepsilon}_P \cdot \mathbf{A} \cdot \boldsymbol{\varepsilon}_P^2, & H_{49} &= \boldsymbol{\varepsilon}_P^2 \cdot \mathbf{S} \cdot \boldsymbol{\varepsilon}_P - \boldsymbol{\varepsilon}_P \cdot \mathbf{S} \cdot \boldsymbol{\varepsilon}_P^2, \\
H_{50} &= \mathbf{S}^2 \cdot \boldsymbol{\varepsilon}_P^2 \cdot \mathbf{S} - \mathbf{S} \cdot \boldsymbol{\varepsilon}_P^2 \cdot \mathbf{S}^2, & H_{51} &= \boldsymbol{\varepsilon}_P^2 \cdot \mathbf{S}^2 \cdot \boldsymbol{\varepsilon}_P - \boldsymbol{\varepsilon}_P \cdot \mathbf{S}^2 \cdot \boldsymbol{\varepsilon}_P^2, \\
H_{52} &= \mathbf{S} \cdot \mathbf{A} \cdot \boldsymbol{\varepsilon}_P - \boldsymbol{\varepsilon}_P \cdot \mathbf{A} \cdot \mathbf{S}, & H_{53} &= \mathbf{S} \cdot \boldsymbol{\varepsilon}_P \cdot \mathbf{A} - \mathbf{A} \cdot \boldsymbol{\varepsilon}_P \cdot \mathbf{S}, \\
H_{54} &= \mathbf{A} \cdot \mathbf{S} \cdot \boldsymbol{\varepsilon}_P - \boldsymbol{\varepsilon}_P \cdot \mathbf{S} \cdot \mathbf{A}, \\
H_{55} &= \mathbf{S}^2 \cdot \mathbf{A} \cdot \boldsymbol{\varepsilon}_P - \boldsymbol{\varepsilon}_P \cdot \mathbf{A} \cdot \mathbf{S}^2, & H_{56} &= \mathbf{S}^2 \cdot \boldsymbol{\varepsilon}_P \cdot \mathbf{A} - \mathbf{A} \cdot \boldsymbol{\varepsilon}_P \cdot \mathbf{S}^2, \\
H_{57} &= \boldsymbol{\varepsilon}_P \cdot \mathbf{S}^2 \cdot \mathbf{A} - \mathbf{A} \cdot \mathbf{S}^2 \cdot \boldsymbol{\varepsilon}_P, & H_{58} &= \boldsymbol{\varepsilon}_P^2 \cdot \mathbf{S} \cdot \mathbf{A} - \mathbf{A} \cdot \mathbf{S} \cdot \boldsymbol{\varepsilon}_P^2, \\
H_{59} &= \mathbf{S} \cdot \mathbf{A} \cdot \boldsymbol{\varepsilon}_P^2 - \boldsymbol{\varepsilon}_P^2 \cdot \mathbf{A} \cdot \mathbf{S}, & H_{60} &= \mathbf{S} \cdot \boldsymbol{\varepsilon}_P^2 \cdot \mathbf{A} - \mathbf{A} \cdot \boldsymbol{\varepsilon}_P^2 \cdot \mathbf{S}, \\
H_{61} &= \mathbf{S}^2 \cdot \mathbf{A} \cdot \boldsymbol{\varepsilon}_P \cdot \mathbf{S} - \mathbf{S} \cdot \boldsymbol{\varepsilon}_P \cdot \mathbf{A} \cdot \mathbf{S}^2, & H_{62} &= \mathbf{S}^2 \cdot \mathbf{A} \cdot \boldsymbol{\varepsilon}_P \cdot \mathbf{S} - \boldsymbol{\varepsilon}_P \cdot \mathbf{S} \cdot \mathbf{A} \cdot \mathbf{S}^2, \\
H_{63} &= \mathbf{S}^2 \cdot \boldsymbol{\varepsilon}_P \cdot \mathbf{A} \cdot \mathbf{S} - \mathbf{S} \cdot \mathbf{A} \cdot \boldsymbol{\varepsilon}_P \cdot \mathbf{S}^2, & H_{64} &= \mathbf{S}^2 \cdot \boldsymbol{\varepsilon}_P \cdot \mathbf{S} \cdot \mathbf{A} - \mathbf{A} \cdot \mathbf{S} \cdot \boldsymbol{\varepsilon}_P \cdot \mathbf{S}^2, \\
H_{65} &= \boldsymbol{\varepsilon}_P^2 \cdot \mathbf{S} \cdot \mathbf{A} \cdot \boldsymbol{\varepsilon}_P - \boldsymbol{\varepsilon}_P \cdot \mathbf{A} \cdot \mathbf{S} \cdot \boldsymbol{\varepsilon}_P^2, & H_{66} &= \boldsymbol{\varepsilon}_P^2 \cdot \mathbf{S} \cdot \boldsymbol{\varepsilon}_P \cdot \mathbf{A} - \mathbf{A} \cdot \boldsymbol{\varepsilon}_P \cdot \mathbf{S} \cdot \boldsymbol{\varepsilon}_P^2, \\
H_{67} &= \boldsymbol{\varepsilon}_P^2 \cdot \mathbf{A} \cdot \mathbf{S} \cdot \boldsymbol{\varepsilon}_P - \boldsymbol{\varepsilon}_P \cdot \mathbf{S} \cdot \mathbf{A} \cdot \boldsymbol{\varepsilon}_P^2, & H_{68} &= \boldsymbol{\varepsilon}_P^2 \cdot \mathbf{S} \cdot \mathbf{A} \cdot \boldsymbol{\varepsilon}_P - \boldsymbol{\varepsilon}_P \cdot \mathbf{A} \cdot \mathbf{S} \cdot \boldsymbol{\varepsilon}_P^2,
\end{aligned}$$

$$\begin{aligned} \mathbf{H}_{69} &= \boldsymbol{\varepsilon}_P^2 \cdot \mathbf{S}^2 \cdot \mathbf{A} - \mathbf{A} \cdot \mathbf{S}^2 \cdot \boldsymbol{\varepsilon}_P^2, & \mathbf{H}_{70} &= \boldsymbol{\varepsilon}_P^2 \cdot \mathbf{A} \cdot \mathbf{S}^2 - \mathbf{S}^2 \cdot \mathbf{A} \cdot \boldsymbol{\varepsilon}_P^2, \\ \mathbf{H}_{71} &= \mathbf{S}^2 \cdot \boldsymbol{\varepsilon}_P^2 \cdot \mathbf{A} - \mathbf{A} \cdot \boldsymbol{\varepsilon}_P^2 \cdot \mathbf{S}^2, & \mathbf{H}_{72} &= \mathbf{S}^2 \cdot \mathbf{A} \cdot \boldsymbol{\varepsilon}_P^2 - \boldsymbol{\varepsilon}_P^2 \cdot \mathbf{A} \cdot \mathbf{S}^2. \end{aligned}$$

In the above two formulae the scalar coefficients  $b_\alpha$ ,  $\alpha \in \{1, 72\}$  depend on the following invariants:

$$\begin{aligned} j_1 &= \mathbf{S} : \mathbf{1} \equiv \text{tr } \mathbf{S}, & j_2 &= \mathbf{S}^2 : \mathbf{1}, & j_3 &= \mathbf{S}^3 : \mathbf{1}, & \pi_1, & \pi_2, & \pi_3, & \pi_4, & \pi_5, \\ j_4 &= \mathbf{S} : \mathbf{A}, & j_5 &= \mathbf{S}^2 : \mathbf{A}, & j_6 &= \mathbf{S} : \boldsymbol{\varepsilon}_P, & j_7 &= \mathbf{S}^2 : \boldsymbol{\varepsilon}_P, & j_8 &= \mathbf{S} : \boldsymbol{\varepsilon}_P^2, \\ j_9 &= \mathbf{S}^2 : \boldsymbol{\varepsilon}_P^2, & j_{10} &= \vec{A}_\nu \cdot \mathbf{S} \cdot \boldsymbol{\varepsilon}_P \cdot \vec{A}_\nu, & j_{11} &= \vec{A}_\nu \cdot \mathbf{S}^2 \cdot \boldsymbol{\varepsilon}_P \cdot \vec{A}_\nu, \\ j_{12} &= \vec{A}_\nu \cdot \mathbf{S} \cdot \boldsymbol{\varepsilon}_P^2 \cdot \vec{A}_\nu, & j_{13} &= \vec{A}_\nu \cdot \mathbf{S}^2 \cdot \boldsymbol{\varepsilon}_P^2 \cdot \vec{A}_\nu. \end{aligned}$$

It should be noted that the above two evolution equations maybe simplified if instead of the stress  $\mathbf{S}$  the elastic strain  $\mathbf{E}_e$  is adopted as a state variable. In such a case the equation for the plastic stretching simplifies into:

$$\mathbf{D}_P = \sum_{\alpha} b_{\alpha}^* \mathbf{H}_{\alpha}^*, \quad \alpha \in \{1, 22\}, \quad (33a)$$

with the new generators — symmetric second rank tensors:

$$\begin{aligned} \mathbf{H}_1^* &= \mathbf{1}, & \mathbf{H}_2^* &= \boldsymbol{\varepsilon}_P, & \mathbf{H}_3^* &= \mathbf{A}, & \mathbf{H}_4^* &= \boldsymbol{\varepsilon}_P^2, \\ \mathbf{H}_5^* &= \mathbf{A} \cdot \boldsymbol{\varepsilon}_P + \boldsymbol{\varepsilon}_P \cdot \mathbf{A}, & \mathbf{H}_6^* &= \mathbf{A} \cdot \boldsymbol{\varepsilon}_P^2 + \boldsymbol{\varepsilon}_P^2 \cdot \mathbf{A}, \\ \mathbf{H}_7^* &= \mathbf{E}_e, & \mathbf{H}_8^* &= \mathbf{E}_e \cdot \boldsymbol{\varepsilon}_P + \boldsymbol{\varepsilon}_P \cdot \mathbf{E}_e, & \mathbf{H}_9^* &= \mathbf{A} \cdot \mathbf{E}_e + \mathbf{E}_e \cdot \mathbf{A}, \\ \mathbf{H}_{10}^* &= \mathbf{E}_e \cdot \boldsymbol{\varepsilon}_P^2 + \boldsymbol{\varepsilon}_P^2 \cdot \mathbf{E}_e, \\ \mathbf{H}_{11}^* &= \mathbf{E}_e \cdot \boldsymbol{\varepsilon}_P \cdot \mathbf{A} + \mathbf{A} \cdot \boldsymbol{\varepsilon}_P \cdot \mathbf{E}_e, & \mathbf{H}_{12}^* &= \mathbf{A} \cdot \mathbf{E}_e \cdot \boldsymbol{\varepsilon}_P + \boldsymbol{\varepsilon}_P \cdot \mathbf{E}_e \cdot \mathbf{A}, \\ \mathbf{H}_{13}^* &= \mathbf{E}_e \cdot \mathbf{A} \cdot \boldsymbol{\varepsilon}_P + \boldsymbol{\varepsilon}_P \cdot \mathbf{A} \cdot \mathbf{E}_e, \\ \mathbf{H}_{14}^* &= \mathbf{E}_e \cdot \boldsymbol{\varepsilon}_P^2 \cdot \mathbf{A} + \mathbf{A} \cdot \boldsymbol{\varepsilon}_P^2 \cdot \mathbf{E}_e, & \mathbf{H}_{15}^* &= \mathbf{A} \cdot \mathbf{E}_e \cdot \boldsymbol{\varepsilon}_P^2 + \boldsymbol{\varepsilon}_P^2 \cdot \mathbf{E}_e \cdot \mathbf{A}, \\ \mathbf{H}_{16}^* &= \mathbf{E}_e \cdot \mathbf{A} \cdot \boldsymbol{\varepsilon}_P^2 + \boldsymbol{\varepsilon}_P^2 \cdot \mathbf{A} \cdot \mathbf{E}_e, \\ \mathbf{H}_{17}^* &= \boldsymbol{\varepsilon}_P^2 \cdot \mathbf{E}_e \cdot \boldsymbol{\varepsilon}_P \cdot \mathbf{A} + \mathbf{A} \cdot \boldsymbol{\varepsilon}_P \cdot \mathbf{E}_e \cdot \boldsymbol{\varepsilon}_P^2, \\ \mathbf{H}_{18}^* &= \boldsymbol{\varepsilon}_P^2 \cdot \mathbf{A} \cdot \boldsymbol{\varepsilon}_P \cdot \mathbf{E}_e + \mathbf{E}_e \cdot \boldsymbol{\varepsilon}_P \cdot \mathbf{A} \cdot \boldsymbol{\varepsilon}_P^2, \\ \mathbf{H}_{19}^* &= \boldsymbol{\varepsilon}_P^2 \cdot \mathbf{E}_e \cdot \mathbf{A} \cdot \boldsymbol{\varepsilon}_P + \boldsymbol{\varepsilon}_P \cdot \mathbf{A} \cdot \mathbf{E}_e \cdot \boldsymbol{\varepsilon}_P^2, \\ \mathbf{H}_{20}^* &= \boldsymbol{\varepsilon}_P^2 \cdot \mathbf{A} \cdot \mathbf{E}_e \cdot \boldsymbol{\varepsilon}_P + \boldsymbol{\varepsilon}_P \cdot \mathbf{E}_e \cdot \mathbf{A} \cdot \boldsymbol{\varepsilon}_P^2, \\ \mathbf{H}_{21}^* &= \mathbf{E}_e \cdot \boldsymbol{\varepsilon}_P^2 \cdot \mathbf{A} \cdot \boldsymbol{\varepsilon}_P + \boldsymbol{\varepsilon}_P \cdot \mathbf{A} \cdot \boldsymbol{\varepsilon}_P^2 \cdot \mathbf{E}_e, \\ \mathbf{H}_{22}^* &= \mathbf{A} \cdot \boldsymbol{\varepsilon}_P^2 \cdot \mathbf{E}_e \cdot \boldsymbol{\varepsilon}_P + \boldsymbol{\varepsilon}_P \cdot \mathbf{E}_e \cdot \boldsymbol{\varepsilon}_P^2 \cdot \mathbf{A}. \end{aligned} \quad (34a)$$

The corresponding evolution equation for the plastic spin is

$$\mathbf{W}_P = \sum_{\alpha} b_{\alpha}^* \mathbf{H}_{\alpha}^*, \quad \alpha \in \{23, 39\}, \quad (35a)$$

whose antisymmetric generators are now

$$\mathbf{H}_{23}^* = \mathbf{A} \cdot \boldsymbol{\varepsilon}_P - \boldsymbol{\varepsilon}_P \cdot \mathbf{A}, \quad \mathbf{H}_{24}^* = \mathbf{A} \cdot \boldsymbol{\varepsilon}_P^2 - \boldsymbol{\varepsilon}_P^2 \cdot \mathbf{A},$$



$$\begin{aligned}
 \mathbf{H}_{25}^* &= \mathbf{E}_e \cdot \boldsymbol{\varepsilon}_P - \boldsymbol{\varepsilon}_P \cdot \mathbf{E}_e, & \mathbf{H}_{26}^* &= \mathbf{A} \cdot \mathbf{E}_a - \mathbf{E}_a \cdot \mathbf{A}, \\
 \mathbf{H}_{27}^* &= \mathbf{E}_e \cdot \boldsymbol{\varepsilon}_P^2 - \boldsymbol{\varepsilon}_P^2 \cdot \mathbf{E}_e, \\
 \mathbf{H}_{28}^* &= \mathbf{E}_e \cdot \boldsymbol{\varepsilon}_P \cdot \mathbf{A} - \mathbf{A} \cdot \boldsymbol{\varepsilon}_P \cdot \mathbf{E}_e, & \mathbf{H}_{29}^* &= \mathbf{A} \cdot \mathbf{E}_e \cdot \boldsymbol{\varepsilon}_P - \boldsymbol{\varepsilon}_P \cdot \mathbf{E}_e \cdot \mathbf{A}, \\
 \mathbf{H}_{30}^* &= \mathbf{E}_e \cdot \mathbf{A} \cdot \boldsymbol{\varepsilon}_P - \boldsymbol{\varepsilon}_P \cdot \mathbf{A} \cdot \mathbf{E}_e, \\
 \mathbf{H}_{31}^* &= \mathbf{E}_e \cdot \boldsymbol{\varepsilon}_P^2 \cdot \mathbf{A} - \mathbf{A} \cdot \boldsymbol{\varepsilon}_P^2 \cdot \mathbf{E}_e, & \mathbf{H}_{32}^* &= \mathbf{A} \cdot \mathbf{E}_e \cdot \boldsymbol{\varepsilon}_P^2 - \boldsymbol{\varepsilon}_P^2 \cdot \mathbf{E}_e \cdot \mathbf{A}, \\
 \mathbf{H}_{33}^* &= \mathbf{E}_e \cdot \mathbf{A} \cdot \boldsymbol{\varepsilon}_P^2 - \boldsymbol{\varepsilon}_P^2 \cdot \mathbf{A} \cdot \mathbf{E}_e, \\
 \mathbf{H}_{34}^* &= \boldsymbol{\varepsilon}_P^2 \cdot \mathbf{E}_e \cdot \boldsymbol{\varepsilon}_P \cdot \mathbf{A} - \mathbf{A} \cdot \boldsymbol{\varepsilon}_P \cdot \mathbf{E}_e \cdot \boldsymbol{\varepsilon}_P^2, \\
 \mathbf{H}_{35}^* &= \boldsymbol{\varepsilon}_P^2 \cdot \mathbf{A} \cdot \boldsymbol{\varepsilon}_P \cdot \mathbf{E}_e - \mathbf{E}_e \cdot \boldsymbol{\varepsilon}_P \cdot \mathbf{A} \cdot \boldsymbol{\varepsilon}_P^2, \\
 \mathbf{H}_{36}^* &= \boldsymbol{\varepsilon}_P^2 \cdot \mathbf{E}_e \cdot \mathbf{A} \cdot \boldsymbol{\varepsilon}_P - \boldsymbol{\varepsilon}_P \cdot \mathbf{A} \cdot \mathbf{E}_e \cdot \boldsymbol{\varepsilon}_P^2, \\
 \mathbf{H}_{37}^* &= \boldsymbol{\varepsilon}_P^2 \cdot \mathbf{A} \cdot \mathbf{E}_e \cdot \boldsymbol{\varepsilon}_P - \boldsymbol{\varepsilon}_P \cdot \mathbf{E}_e \cdot \mathbf{A} \cdot \boldsymbol{\varepsilon}_P^2, \\
 \mathbf{H}_{38}^* &= \mathbf{E}_e \cdot \boldsymbol{\varepsilon}_P^2 \cdot \mathbf{A} \cdot \boldsymbol{\varepsilon}_P - \boldsymbol{\varepsilon}_P \cdot \mathbf{A} \cdot \boldsymbol{\varepsilon}_P^2 \cdot \mathbf{E}_e, \\
 \mathbf{H}_{39}^* &= \mathbf{A} \cdot \boldsymbol{\varepsilon}_P^2 \cdot \mathbf{E}_e \cdot \boldsymbol{\varepsilon}_P - \boldsymbol{\varepsilon}_P \cdot \mathbf{E}_e \cdot \boldsymbol{\varepsilon}_P^2 \cdot \mathbf{A}.
 \end{aligned} \tag{36a}$$

The scalar coefficients in (33a) and (36a) are now the following functions of the invariants listed in (30)

$$\begin{aligned}
 b_k^* &= \sum_1^6 b_{k\alpha}^*(\gamma_\pi) i_\alpha, & k &\in \{1, 6\} \cup \{23, 24\} \quad \text{and} \\
 b_k^* &= b_k^*(\gamma_\pi), & k &\in \{7, 22\} \cup \{25, 39\}.
 \end{aligned} \tag{38}$$

By similar reasoning the evolution equation for the heat flux vector (7) might be represented by the following relationship

$$\begin{aligned}
 \tau_q Dq &= \sum_\alpha b_\alpha(\gamma_\pi) \mathbf{N}_\alpha \cdot \text{grad } \theta + \sum_\beta [-1 + b_\beta(\gamma_\pi) \mathbf{N}_\beta] \cdot q, \\
 \alpha &\in \{1, 6\}, \quad \beta \in \{7, 11\},
 \end{aligned} \tag{39}$$

where  $\tau_q$  is a heat relaxation time [6] while  $\mathbf{N}_\alpha$  and  $\mathbf{N}_\beta$  are the corresponding symmetric second rank tensors — generators given bellow

$$\begin{aligned}
 \mathbf{N}_1 &= \mathbf{1}, & \mathbf{N}_2 &= \mathbf{N}_7 = \boldsymbol{\varepsilon}_P, & \mathbf{N}_3 &= \mathbf{N}_8 = \boldsymbol{\varepsilon}_P^2, & \mathbf{N}_4 &= \mathbf{N}_9 = \mathbf{A}, \\
 \mathbf{N}_5 &= \mathbf{N}_{10} = \mathbf{A} \cdot \boldsymbol{\varepsilon}_P + \boldsymbol{\varepsilon}_P \cdot \mathbf{A}, & \mathbf{N}_6 &= \mathbf{N}_{11} = \mathbf{A} \cdot \boldsymbol{\varepsilon}_P^2 + \boldsymbol{\varepsilon}_P^2 \cdot \mathbf{A}.
 \end{aligned} \tag{40}$$

The equation (39) represents the Vernotte-Cattaneo relation (cf. [6, 33]) generalized in such a way to account for finite deformations of an thermoviscoplastic transversely isotropic body. Since the relaxation properties of a thermal field have been experimentally observed only at very low temperatures (such as liquid helium temperatures) putting  $\tau_q = 0$  this equation allows for a generalized Fourier law holding under assumptions (A1) and (A2) for all the other temperature ranges in the following way:

$$\bar{q} = - \sum_\alpha k_\alpha(\gamma_\pi) \mathbf{N}_\alpha \cdot \text{grad } \theta, \quad \alpha \in \{1, 6\}. \tag{41}$$

Consider, finally, the most special case given by the following assumption:

(A3) Suppose now that the plastic strain itself is small of the same order as thermoelastic strain.

In such a very restricted "small strains" case plastic stretching and plastic spin dramatically reduce their expressions to become

$$\begin{aligned} \mathbf{D}_P \approx \langle f \rangle [ & b_2^0 \boldsymbol{\varepsilon}_P + (b_3^0 + b_{31}^* i_1 + b_{34}^* i_4 + b_{31}^\# \pi_1 + b_{34}^\# \pi_4) \mathbf{A} \\ & + (b_5^0 + b_{51}^* i_1 + b_{54}^* i_4 + b_{51}^\# \pi_1 + b_{54}^\# \pi_4) (\mathbf{A} \cdot \boldsymbol{\varepsilon}_P + \boldsymbol{\varepsilon}_P \cdot \mathbf{A}) \\ & + b_7^0 \mathbf{E}_e + b_9^0 (\mathbf{E}_e \cdot \mathbf{A} + \mathbf{A} \cdot \mathbf{E}_e)], \end{aligned} \quad (42)$$

$$\mathbf{W}_P \approx \langle f \rangle [b_{23}^0 (\mathbf{A} \cdot \boldsymbol{\varepsilon}_P - \boldsymbol{\varepsilon}_P \cdot \mathbf{A}) + b_{26}^0 (\mathbf{E}_e \cdot \mathbf{A} - \mathbf{A} \cdot \mathbf{E}_e)], \quad (43)$$

where all the scalar coefficients are some functions of the temperature alone. The limitations  $\text{tr } \mathbf{D}_P \approx 0$  and  $\mathbf{D}_P \approx D\boldsymbol{\varepsilon}_P$  valid for the small strains case are taken into account in (42) whereas it is worthy of note that in the case of initially isotropic materials  $\mathbf{W}_P \approx \mathbf{O}$  (cf. also [29]).

Similarly the Fourier law for transversely isotropic materials takes its familiar form to read

$$q = -(k_1 \mathbf{1} + k_4 \mathbf{A}) \cdot \text{grad } \theta \quad (44)$$

and under the same assumption the response function for stress is reduced into

$$\mathbf{S} \approx (a_1^0 + a_{11}^* i_1 + a_{14}^* i_4) \mathbf{1} + (a_4^0 + a_{41}^* i_1 + a_{44}^* i_4) \mathbf{A} + a_5^0 \mathbf{E}_e + a_7^0 (\mathbf{E}_e \cdot \mathbf{A} + \mathbf{A} \cdot \mathbf{E}_e), \quad (45)$$

where the difference between the inherent (i.e. initial) and the induced anisotropy is also clear. The first is taken into account by the terms containing  $\mathbf{A}$  and the second by the terms with  $\boldsymbol{\varepsilon}_P$ . This note holds true for all the explicit invariants and the generators listed in this section (for the initial anisotropy representations see [30–31]).

**Concluding remarks.** The results accomplished by this paper are shortly stated in the following:

(a) the developed theory of anisotropic (e.g. composite) materials is of the non-associate type even in the case of small total strains as experimentally supported for cruciform specimens made of isotropic materials;

(b) the geometric picture of deformation and the extended thermodynamics allow for the theory which carefully recognizes between the inherent and the acquired — mechanical as well as thermal — anisotropy;

(c) tensorial representations for inelastic evolution equations and Hook's law are expressed by minimal sets of tensors generators and the corresponding principal and mixed invariants for small as well as for plastic strain cases.

(d) the special case of very stiff fibres idealized by their elastic as well as plastic inextensibility (as assumed in [3]) is easily derived from the general results presented herein.

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## ÜBER DIE VISKOPLASTIZITÄT DER TRANSVERSALISOTROPISCHEN WERKSTOFFE

In der Arbeit ist die Viskoplastizität von Composit mit einer Familie der Fasern, die einem inhomogenen und nichtstationären Temperaturfeld ausgelegt sind, besichtigt worden. Es sind die allgemeinen konstitutiven und Entwicklungsgleichungen postuliert worden und mit der Hilfe der Erweiterungsthermodynamik der irreversiblen Prozesses sind die konstitutiven Begrenzungen, vorgeschrieben durch das zweiten Gesetz der Thermodynamik, ausegeführt worden. Das Fließgesetz ist ausgeschlossen (d.h. die Geschwindigkeit der plastischen Verformung ist nicht senkrecht an die Fließoberfläche).

Besonders ist es den Fall von kleiner Verformungsgeschwindigkeit, Temperaturgradient und elastischen Verformungen, dann endlichen plastischen Verformungen analysiert worden. Neben der mechanischen Anisotropie erhielt man auch termische Anisotropie, verursacht durch plastische Verformungen.

## O VISOKOPLASTIČNOSTI TRANSVERZALNO IZOTROPNIH MATERIJALA

U radu se razmatra viskoplastičnost kompozita sa jednom porodicom vlakana izloženog nehomogenom nestacionarnom temperaturskom polju. Postuliraju se opšte konstitutivne i evolucione jednačine pa se pomoću proširene termodinamike nepovratnih procesa izvode konstitutivna ograničenja propisana drugim zakonom termodinamike. Zakon tečenja je nepridružen.

Posebno se analizira slučaj male brzine deformacije, temperaturskog gradijenta i elastične deformacije, a konačne plastične deformacije. Pored mehaničke anizotropije dobija se i termička anizotropija izazvana plastičnom deformacijom.

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