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THE GENERALIZATION OF THE BASIC DYNAMIC THEOREMS IN THE CASE OF THE RELATIVE MOTION OF NON-HOLONOMIC SYSTEMS

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In relation to the one-parametric transformation of the coordinates the virtual displacements are defined and are derived the conditions under which they are in acccordance with the non-holonomic constraints which are put on the system.

If the D'Alembert-Lagrange's principle is followed as in [1], the relations are derived, for the case of the relative motion of the non-holonomic mechanical systems, out of which we can carry out the generalization of the basic dynamic theorems for the relative motion by analyzing the invariability characteristics in relation to the given transformation.

Let's consider the example of the rolling ball without slipping on the inside of the rotating sphere.

1. If the system of N material points is given with mases m_i $(i = 1, N)$, whose motion is defined in relation to the moving system of the reference $Oxyz$. Let's say the moving of the moving coordination system $Oxyz$ is defined by the velocity $\mathbf{v}$ the pole O and by the vector of its own angle velocity ω.

If the vector of the pole position O is marked with $\mathbf{r}_0$ in relation to the inmobile system of the coordinates $\tilde{O}\tilde{x}\tilde{y}\tilde{z}$ means that

$$\tilde{\mathbf{r}}_i = \mathbf{r}_{o(t)} + \mathbf{r}_{i(x_i,y_i,z_i)}.$$

Since $\mathbf{r}_{o(t)}$ is the given function of the time, the variations of the position vector in relation to the inmobile and mobile system of the references coincide, or:

$$\delta\tilde{\mathbf{r}}_i = \delta\mathbf{r}_i. \tag{1}$$

The non-holonomic constraints in the form:

$$\sum_{i=1}^{N} A_{i\nu(\mathbf{r}_i,t)} \frac{d_r\mathbf{r}_i}{dt} + A_{\nu(\mathbf{r}_i,t)} = 0. \qquad (\nu = 1, l), \tag{2}$$

are put above the system of the material points, where d_r/dt is used to mark the relative derivative of time.

The D'Alambert-Lagrange's principle for the case of the relative motion of the material system under the condition of the active forces $\mathbf{F}$ considering (1) can be written in the form:

$$\sum_{i=1}^{N}\left[m_i\mathbf{a}_{ri} - (\mathbf{F}_i - m_i\mathbf{a}_{pi} - m_i\mathbf{a}_{ci})\right]\delta\mathbf{r}_i = 0. \tag{3}$$

where the variations $\delta\mathbf{r}_i$ must be in agreement with the constraint (2), which means it must be:

$$\sum_{i=1}^{N} A_{i\nu}\delta\mathbf{r}_i = 0. \tag{4}$$

Let's consider the one-parametric transformation in the form:

$$\mathbf{r}_{i(x_i,y_i,z_i)} \longrightarrow \mathbf{r}_{i(x'_i,y'_i,z'_i,t,\alpha)}, \tag{5}$$

in relation to which the relative velocity would be transformed in the following way

$$\frac{d_r\mathbf{r}_i}{dt} = \overset{*}{\mathbf{r}}_i = \frac{\partial\mathbf{r}_i}{\partial t} + \sum_{j=1}^{N}\left(\frac{\partial\mathbf{r}_i}{\partial x'_j}\frac{dx'_j}{dt} + \frac{\partial\mathbf{r}_i}{\partial y'_j}\frac{dy'_j}{dt} + \frac{\partial\mathbf{r}_i}{\partial z'_j}\frac{dz'_j}{dt}\right). \tag{6}$$

Considering the transformation (5) variations can be defined in the form [1]

$$\delta\mathbf{r}_i = \frac{\partial\mathbf{r}_i}{\partial\alpha}\delta\alpha. \tag{7}$$

One-parametric transformation (5) would be in agreement to the non-holonomic constraints (2) if the following condition is satisfied considering (4) and (7):

$$\sum_{i=1}^{N} A_{i\nu}\frac{\partial\mathbf{r}_i}{\partial\alpha} = 0, \qquad (\nu = 1, l). \tag{8}$$

Considering D'Alambert-Lagrange's principle (3), as in [1] the following theorem can be proved, if we observe the relative motion of the mechanical non-holonomic system.

THEOREM 1. *If the one-parametric transformation is in agreement to the non-holonomic constraints, which means that the condition* (8) *is satisfied, the relation is in effect*

$$\frac{d_rS}{dt} - \frac{\partial T_r}{\partial\alpha} = \sum_{i=1}^{N}(\mathbf{F}_i + \mathbf{F}_{pi} + \mathbf{F}_{ci})\frac{\partial\mathbf{r}_i}{\partial\alpha}, \tag{9}$$

where

$$S = \sum_{i=1}^{N}\frac{\partial T_r}{\partial\mathbf{v}_{ri}}\frac{\partial\mathbf{r}_i}{\partial\alpha}, \qquad T_r = \frac{1}{2}\sum_{i=1}^{N} m_i\mathbf{v}_{ri}^2, \qquad \mathbf{v}_{ri} = \overset{*}{\mathbf{r}}_i,$$

$$\mathbf{F}_{pi} = -m_i\mathbf{a}_{pi}, \qquad \mathbf{F}_{ci} = -m_i\mathbf{a}_{ci}.$$

It should be considered that the relation is in effect [2]

$$\frac{d_r}{dt}(\delta \mathbf{r}_i) - \delta\left(\frac{d_r \mathbf{r}_i}{dt}\right) = 0. \tag{9'}$$

Considering the relation (9) where T_r is the relative kinetics energy of the material system, the following theorem can be proved:

THEOREM 2. *If T_r is the invariable value in the relation to one-parametric transformation* (5), *which is in agreement to constraints* (2), *means that*

$$\frac{d_r S}{dt} = \sum_{i=1}^{N} (\mathbf{F}_i + \mathbf{F}_{pi} + \mathbf{F}_{ci}) \frac{\partial \mathbf{r}_i}{\partial \alpha}. \tag{10}$$

The proof comes from the relation (9) and the condition of the invariability of the function T_r which can be shown in the form [1]:

$$\frac{\partial T_r}{\partial \alpha} = 0.$$

2. Considering the shape of the inertia forces $\mathbf{F}_{pi}$ and $\mathbf{F}_{ci}$ which are the result of the moving of the mobile system $Oxyz$, the relation (9) can be shown in a different shape which should be more convenient for analysis in some other cases.

Namely, considering that is [3]

$$\begin{aligned}
&\sum_{i=1}^{N} (\mathbf{F}_{pi} + \mathbf{F}_{ci}) \delta \mathbf{r}_i \\
&= \sum_{i=1}^{N} \big(-m_i \mathbf{a}_0 - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times m_i \mathbf{r}_i) - \boldsymbol{\varepsilon} \times m_i \mathbf{r}_i - \boldsymbol{\omega} \times m_i \mathbf{v}_{ri}\big) \delta \mathbf{r}_i \\
&= \left\{ -\frac{d_r}{dt}\left[(\boldsymbol{\omega} \times m_i \mathbf{r}_i) \frac{\partial \mathbf{r}_i}{\partial \alpha}\right] - \frac{\partial}{\partial \alpha}[\Pi^0 + \Pi^{\omega} - \boldsymbol{\omega}(\mathbf{r}_i \times m_i \mathbf{v}_{ri})] \right\} \delta\alpha,
\end{aligned}$$

where

$$\Pi^0 = \sum_{i=1}^{N} m_i \mathbf{a}_0 \mathbf{r}_i, \qquad \Pi^{\omega} = -\frac{1}{2} J \omega^2, \qquad \mathbf{a}_0 = \frac{d\mathbf{v}_0}{dt}, \qquad \boldsymbol{\varepsilon} = \frac{d\boldsymbol{\omega}}{dt},$$

in which J is the moment of the inertia of the material system in relation to the momentary axis of the rotation ω which goes through the mobile pole O, the relation (9) gets the form

$$\frac{d_r \bar{S}}{dt} - \frac{\partial \bar{T}}{\partial \alpha} = \sum_{i=1}^{N} \mathbf{F}_i \frac{\partial \mathbf{r}_i}{\partial \alpha}, \tag{11}$$

$$\bar{S} = S + \sum_{i=1}^{N} (\boldsymbol{\omega} \times m_i \mathbf{r}_i) \frac{\partial \mathbf{r}_i}{\partial \alpha}, \qquad \bar{T} = T_r - \Pi^0 - \Pi^\omega + \boldsymbol{\omega}(\mathbf{r}_i \times m_i \mathbf{v}_{ri}).$$

If the considered material system is in the field of the potential forces with the potential $\Pi(\mathbf{r}, t)$ the following theorem can be proved.

THEOREM 3. *If the function $\Lambda = \bar{T} - \Pi$ is invariable in the relation to the one-parametric transformation* (5), *which is in agreement with the non-holonomic constraints* (2), *then the equation of the motion have the first integral in the form*

$$\bar{S} = S + \sum_{i=1}^{N} (\boldsymbol{\omega} \times m_i \mathbf{r}_i) \frac{\partial r_i}{\partial \alpha} = \text{const}.$$

The proof comes from (11) and from the condition of the invariability of the function Λ

$$\frac{\partial \Lambda}{\partial \alpha} = 0.$$

3. In the case of the parallel motion alone the certain direction which is defined by the unit vector $\mathbf{l}$, whose cosines of direction in relation to the system $Oxyz$ can be the given functions of the time, the vector of the position $\mathbf{r}_i$ which is defined by

$$\mathbf{r}_i = \mathbf{r}_i' + \alpha \mathbf{l}, \qquad i = 1, N. \tag{12}$$

Considering that

$$\frac{\partial T_r}{\partial \alpha} = \sum_{i=1}^{N} \frac{\partial T_r}{\partial \mathbf{v}_{ri}} \frac{\partial \mathbf{v}_{ri}}{\partial \alpha} = \sum_{i=1}^{N} m_i \mathbf{v}_{ri} \frac{d_r \mathbf{l}}{dt} = \mathbf{Q}_r \cdot \overset{*}{\mathbf{l}},$$

where $\mathbf{Q}_r$ is the vector of the relative momentum of the system, the condition of the invariability of the function T_r is brought down to

$$\mathbf{Q}_r \cdot \overset{*}{\mathbf{l}} = 0. \tag{13}$$

Considering Theorem 2, if the condition (13) is satisfied and if the constraint (2) are agreement to the relation (12) which means thet they permit the parallel motion of the material system along the direction $\mathbf{l}$, from (11) follows:

$$\frac{d_r \mathbf{Q}_r}{dt} \cdot \mathbf{l} = \sum_{i=1}^{N} (\mathbf{F}_i + \mathbf{F}_{pi} + \mathbf{F}_{ci}) \cdot \mathbf{l}, \tag{14}$$

the law of the changing of the relative momentum of the material system along the direction $\mathbf{l}$.

Let's consider now the transformation which defines the rotation around the direction $\mathbf{l}$, which goes through a certain point A. Then is [1]

$$\frac{\partial \mathbf{r}_i}{\partial \alpha} = \mathbf{l} \times (\mathbf{r}_i - \mathbf{r}_A), \tag{15}$$

and then considering (9′), for $\partial T_r/\partial\alpha$ we get:

$$\frac{\partial T_r}{\partial\alpha}=\sum_{i=1}^{N} m_i\mathbf{v}_{ri}\frac{d}{dt}[\mathbf{l}\times(\mathbf{r}_i-\mathbf{r}_A)]=\mathbf{K}_r\cdot\overset{*}{\mathbf{l}}+\mathbf{Q}_r\cdot\frac{d_r}{dt}(\mathbf{r}_A\times\mathbf{l}),$$

in which $\mathbf{K}_r$ is the relative moment of momentum for the point O.

The condition of invariability of the function T_r now has the form:

$$\mathbf{K}_r\cdot\overset{*}{\mathbf{l}}+\mathbf{Q}_r\cdot\frac{d_r}{dt}(\mathbf{r}_A\times\mathbf{l})=0. \tag{16}$$

When the direction $\mathbf{l}$ goes through the center of the mass od the given system the condition (16) gets the form:

$$\mathbf{K}_{rC}\cdot\overset{*}{\mathbf{l}}=0, \tag{17}$$

in which $\mathbf{K}_{rC}$ is the relative moment of momentum for the center of the maas.

If the constraints (2) are in agreement with the transformation (15) and it the condition (16) is satisfied from the theorem 2 follows

$$\frac{d_r}{dt}(\mathbf{K}_{rA}\cdot\mathbf{l})=\mathbf{M}_A\cdot\mathbf{l}, \tag{18}$$

the low of the changing of the relative moment of momentum in relation to the direction $\mathbf{l}$.

4. Example. Let a ball of mass m and radius a, roll without sliding in the field of earth's gravitation inside of the sphere of radius R which rotates, with the angular speed ω around the vertical axis Oz. In relation to the coordinate system $Oxyz$ connected to the sphere the position of the centre of the ball is defined by the angles α and β:

$$x_C=r\sin\beta\cos\alpha,\quad y_C=r\sin\beta\sin\alpha,\quad z_C=r\cos\beta,\quad r=R-a.$$

The equation of the constraint is received form the conditions of rolling without sliding:

$$\mathbf{v}_{rC}+\boldsymbol{\omega}'\times\mathbf{a}=0, \tag{19}$$

in which $\boldsymbol{\omega}$ is the vector of its own angle speed of the ball with the coordinates p', q' and r' in relation to the system $Oxyz$.

If the one-parametric transformation is defined with (15) and we shall take for the direction $\mathbf{l}$ the axis Oz. It is not difficult to show that in this case function $\Lambda=T-\Pi$ is invariable in relation to the one-parametric transformation (15) which is in accordance with the constraints (19) and considering the theorem 3 follows the first integral of the equation of the motion in the form:

$$J_z r'+\omega(J_z+mr^2\sin^2\beta)+mr^2\dot{\alpha}\sin^2\beta=\text{const}.$$

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ОБОБЩЕНИЕ ОСНОВНЫХ ТЕОРЕМ ДИНАМИКИ ПРИ ОТНОСИТЕЛЬНОМ ДВИЖЕНИИ НЕГОЛОНОМНОЙ МЕХАНИЧЕСКОЙ СИСТЕМЫ

Из принципа Даламбера-Лагранжа получени утверждения, из которых, анализированием инвариантных особенностей относительно однопараметрических семейств преобразований, можно обобщить основные теоремы динамики при относительном движении неголономной механической системы. Рассматривается пример.

UOPŠTENJE OSNOVNIH TEOREMA DINAMIKE PRI RELATIVNOM KRETANJU NEHOLONOMNIH MEHANIČKIH SISTEMA

Polazeći od Dalamber-Lagranžovog principa, izvode se relacije iz kojih se analizom invarijantnih osobina u odnosu na datu jednoparametarsku transformaciju može izvršiti uopštenje osnovnih teorema dinamike za relativno kretanje neholonomnih mehaničkih sistema. Razmatra se primer kugle koja se kotrlja bez klizanja po unutrašnjosti rotirajuće sfere.

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