TEORIJSKA I PRIMENJENA MEHANIKA 16, pp 39–46, 1990, UDK 539.3

A TWO-DIMENSIONAL GENERALIZATION OF THE CONSIDÈRE CRITERION

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(Received 10.07.1990.)

Introduction. The Considère criterion for maximum load under uniaxial tension has been extended to biaxial deformation of a flat sheet. During the period shortly before maximum load is reached it is assumed that plane stress deformation occurs homogeneously throughout the sheet. The criterion for reaching the maximum load under biaxial deformation is derived for the major strain direction. It is shown that it is not possible for the major and minor direction loads to peak simultaneously except under equibiaxial stretching. The sheet material is assumed to be strain and strain rate dependent, and anisotropic. The results are summarized as follows. At the point of maximum load the major strain is accurately predicted in terms of the material strain hardening parameter, strain rate sensitivity parameter and deformation rates in the major strain direction. The major strain at maximum load is independent of the strain rate ratio.

A little more than a hundred years ago Considère (1885) published his classic analysis of the condition of maximum load in a tension test. The crucial importance of the maximum load point in such a test is that the onset of necking occurs close to this point, at least for ductile materials. Since necking, or strain localization, often leads fairly quickly to fracture, it is a highly significant feature of mechanical behavior. The general problem of strain localization is usually referred to as the analysis of plastic stability (or instability, depending upon one's point of view). In any case, the ideas presented in Considère's seminal paper have provided the fundamental basis for most subsequent stability analyses.

About twenty-five years ago this subject began to receive renewed emphasis when it was realized that material viscosity (strain rate sensitivity) had a substantial stabilizing influence upon plastic deformation. Backofen, Turner and Avery (1964) were among the first to attempt to quantify this effect. Subsequently, Hart (1967) developed a more general analysis of the tension test in which he studied the rate of growth of on initial inhomogeneity. Shortly afterwards, Campbell (1967) put forward a theory of plastic stability for rate-sensitive materials, in terms of axial strain gradients in the specimen. Most later analyses of the problem have used Hart's analysis, or Campbell's, or both as starting points. See, for example, Jonas and co-workers (1976, 1977), Argon (1973), Hutchinson and coworkers (1977a, 1977b) and Ghosh (1977).

About thirty-five years ago attention had focussed upon plastic stability in the process of forming sheet metals. This represents a problem of immense technological significance. Swift (1952) was among the first to suggest a Considère type relation in the two dimensional analysis of sheets.

Associated with this was the work of Hill (1950, 1952) in the theory of plasticity and in the area of stabillity. To a certain extent, however, sheet analyses have suffered in comparison with bar analyses because they lack a clearly developed criterion comparable to Considère's. It is the aim of this paper to rectify that situation by generalizing the Considère criterion to two dimensions.

Background. The Considère analysis (1885) of the prismatic bar under tension is so succinct that it can profitably be reviewed here. Let P denote the axial load on a bar, σ the true axial stress and A the current cross-sectional area. Then $P = \sigma A$ and $\dot{P} = \dot{\sigma}A + \sigma\dot{A}$ where the superposed dots denote derivatives with respect to time. Dividing through by σA gives

$$\dot{P}/P = \dot{\sigma}/\sigma + \dot{A}/A \tag{1}$$

and at maximum load $\dot{P}$ is zero.

In the vicinity of maximum load but prior to reaching it, the deformation of ductile metals is observed to be homogeneous and isochoric. As a volume element AL, where L denotes some axial length, remains constant $\dot{A}L + A\dot{L} = 0$. Consequently, $\dot{A}/A$ in (1) can be replaced by $-\dot{L}/L$ which is precisely $-\dot{\varepsilon}$, the axial (logarithmic) strain rate. Therefore, at maximum load (1) states that

$$\dot{\sigma}/\sigma = \dot{\varepsilon}. \tag{2}$$

This is the Considère criterion.

The only connection between (2) and the stability (or instability) of plastic flow is the aformentioned empirical observation that necking generally begins relatively soon after the maximum load is reached in a tension test.

Theory. Instead of a cylindrical bar being elongated, consider a flat sheet being deformed biaxially. Denote the direction of maximum stretching by the 1-axis and call it the major strain direction. Denote the perpendicular direction within the plane of the sheet by the 2-axis and call it the minor strain direction. Obviously, the 3-axis will be in the sheet thickness direction.

Assume that plane stress adequately describes conditions in the sheet. The sheet can be treated as anisotropic but it is assumed that the principal axes of material anisotropy and the principal axes of stress and strain coincide. Then, two stress components, σ_1 and σ_2, and the two corresponding strain rate components, $\dot{\varepsilon}_1$ and $\dot{\varepsilon}_2$, suffice to describe the sheet behavior. Because of the assumed alignment of principal directions, all shear components vanish, and from the previous assumption of isochoric deformation $\dot{\varepsilon}_3 = -(\dot{\varepsilon}_1 + \dot{\varepsilon}_2)$.

There are two loads applied to, or induced in, the sheet. The load in the major strain direction causes σ_1 whilst σ_2 results from the load in the minor strain direction. Throughout the remainder of this paper conditions will be discussed relating to the load in the major strain direction reaching a maximum. It will be shown that generally this will not coincide with the maximum load in the minor strain direction. Nevertheless, for brevity this condidtion will frequently be referred to simply as maximum load. During the period shortly before maximum load is reached it is assumed that deformation occurs homogeneously throughout the sheet. The criterion for reaching the maximum load in the major strain direction is simply

$$\dot{\sigma}_1/\sigma_1 = \dot{\varepsilon}_1. \tag{3}$$

The derivation of (3) is precisely equivalent to that of (2).

Although (3) then is not new, there are some new results that can be shown to follow from it. Suppose the sheet material is described by the Levy-Mises flow relations. (See, for example, Johnson and Mellor, 1973.) Then an effective stress and an effective strain rate can be expressed by

$$\sigma = \left\{\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2\right\}^{1/2} \tag{4}$$

$$\dot{\varepsilon} = \left\{(4/3)\left(\dot{\varepsilon}_1^2 + \dot{\varepsilon}_1\dot{\varepsilon}_2 + \dot{\varepsilon}_2^2\right)\right\}^{1/2}. \tag{5}$$

The corresponding material flow rules are

$$\dot{\varepsilon}_1 = (2\sigma_1 - \sigma_2)\dot{\varepsilon}/2\sigma, \tag{6}$$

$$\dot{\varepsilon}_2 = (2\sigma_2 - \sigma_1)\dot{\varepsilon}/2\sigma. \tag{7}$$

Inversion of (6) and (7) results in

$$\sigma_1 = (2\dot{\varepsilon}_1 + \dot{\varepsilon}_2)(2\sigma/3\dot{\varepsilon}), \tag{8}$$

$$\sigma_2 = (2\dot{\varepsilon}_2 + \dot{\varepsilon}_1)(2\sigma/3\dot{\varepsilon}). \tag{9}$$

Equations (4)–(9) describe the general behavior of the sheet material. Denote the strain rate ratio $\dot{\varepsilon}_2/\dot{\varepsilon}_1$ by ρ. In the vicinity of maximum load assume ρ to be a constant. Then from (5)

$$\dot{\varepsilon} = C_0\dot{\varepsilon}_1 \tag{10}$$

where $C_0 = \left\{(4/3)(1+\rho+\rho^2)\right\}^{1/2}$ is a constant. Using this result in (8) and (9) shows that

$$\sigma_1 = C_1\sigma, \tag{11}$$

$$\sigma_2 = C_2\sigma, \tag{12}$$

where $C_1 = (2+\rho)\{3(1+\rho+\rho^2)\}^{-1/2}$ and $C_2 = (1+2\rho)\{3(1+\rho+\rho^2)\}^{-1/2}$ are also constants.

Now (11) can be used in the left hand side of (3). Because $\dot{\sigma}_1 = C_1\dot{\sigma}$

$$\dot{\sigma}_1/\sigma_1 = \dot{\sigma}/\sigma. \tag{13}$$

Substituting for the right hand side of (3) from (10) gives

$$\dot{\sigma}/\sigma = \dot{\varepsilon}/C_0. \tag{14}$$

This is the criterion for maximum load in the major strain direction expressed in terms of effective stress, effective stress rate, effective strain rate, the set of isotropic material constants and the strain rate ratio.

Because most sheet materials are anisotropic it is worthwile to note the effect of such behavior on the foregoing analysis. Jones and Gillis (1984) proposed a plastic description for a generally anisotropic sheet material having a quadratic flow function. In fact, this description does not even require the strain increment vector to be normal to the flow surface. However, the analysis follows through in the same way as for the Levy-Mises material.

The effective stress and effective strain rate become

$$\sigma = \{b_{11}\sigma_1^2 - (b_{12}+b_{21})\sigma_1\sigma_2 + b_{22}\sigma_2^2\}^{1/2}, \tag{15}$$

$$\dot{\varepsilon} = \{[b_{22}\dot{\varepsilon}_1^2 + (b_{12}+b_{21})\dot{\varepsilon}_1\dot{\varepsilon}_2 + b_{11}\dot{\varepsilon}_2^2]/\Delta\}^{1/2}. \tag{16}$$

Here the b_{ij} are anisotropic material constants and Δ denotes $b_{11}b_{22} - b_{12}b_{21}$. The corresponding flow rules are

$$\dot{\varepsilon}_1 = (b_{11}\sigma_1 - b_{12}\sigma_2)\dot{\varepsilon}/\sigma, \tag{17}$$

$$\dot{\varepsilon}_2 = (b_{22}\sigma_2 - b_{21}\sigma_1)\dot{\varepsilon}/\sigma. \tag{18}$$

Those can be inverted to obtain

$$\sigma_1 = (b_{22}\dot{\varepsilon}_1 + b_{12}\dot{\varepsilon}_2)(\sigma/\Delta\dot{\varepsilon}), \tag{19}$$

$$\sigma_2 = (b_{11}\dot{\varepsilon}_2 + b_{21}\dot{\varepsilon}_1)(\sigma/\Delta\dot{\varepsilon}). \tag{20}$$

Under the previous assumption of a constant strain rate ratio near maximum load, (16) gives the result expressed by (10) but with $C_0 = \{[b_{22} + (b_{12}+b_{21})\rho + b_{11}\rho^2]/\Delta\}^{1/2}$ as the operative constant. Using that result in (19) and (20) gives the results expressed by (11) and (12) with new values for C_1 and C_2. In this case $C_1 = (b_{22} + b_{12}\rho)/C_0\Delta$ and $C_2 = (b_{11}\rho + b_{21})/C_0\Delta$ are the new constants. Of course, in cases where the material has isotropy in the plane of the sheet the values of the b_{ij} are such as to reduce this anisotropic case back to the Levy-Mises results.

In any case, with (10) and (10) reestabilished for this generally anisotropic material having a quadratic yield function the analysis leading to (14) follows directly. Therfore, (14) is the criterion for maximum load in the major strain direction for a wide range of sheet materials.

Meanwhile, one might ask what is going on in the minor strain direction. Manipulating (12) in the same way as (11) leads to

$$\dot{\sigma}_2/\sigma_2 = \dot{\sigma}/\sigma. \tag{21}$$

However, at maximum load $\dot{\sigma}/\sigma = \dot{\varepsilon}_1 = \dot{\varepsilon}_2/\rho$. Hence when the load maximum is reached in the major strain direction

$$\dot{\sigma}_2/\sigma_2 = \dot{\varepsilon}_2/\rho. \tag{22}$$

Therefore, it is not possible for the major and minor direction loads to peak simultaneously except for equibiaxial stretching ($\rho = 1$).

Some further interesting representations come from this analysis. Suppose that the effective stress is some function of effective strain, ε and effective strain rate, $\dot{\varepsilon}$.

$$\sigma = \sigma(\varepsilon, \dot{\varepsilon}). \tag{23}$$

Let the material strain hardening and strain rate sensitivity parameters, n and m respectively, be defined in the usual manner.

$$n = \partial \ln \sigma / \partial \ln \varepsilon, \tag{24}$$

$$m = \partial \ln \sigma / \partial \ln \dot{\varepsilon}. \tag{25}$$

Form the quotient $\dot{\sigma}/\sigma$ from the constitutive law (23). Using the chain rule of differentiation

$$\dot{\sigma} = (\partial \sigma / \partial \varepsilon)\dot{\varepsilon} + (\partial \sigma / \partial \dot{\varepsilon})\ddot{\varepsilon}. \tag{26}$$

Dividing both sides of (26) by σ and taking account of (24) and (25) leads to

$$\dot{\sigma}/\sigma = n\dot{\varepsilon}/\varepsilon + m\ddot{\varepsilon}/\dot{\varepsilon}. \tag{27}$$

From equation (10) $\ddot{\varepsilon}/\dot{\varepsilon} = \ddot{\varepsilon}_1/\dot{\varepsilon}_1$ and, if it is assumed that ρ remains constant from the beginning of loading, $\dot{\varepsilon}/\varepsilon = \dot{\varepsilon}_1/\varepsilon_1$. Furthermore, at maximum load $\dot{\sigma}/\sigma = \dot{\varepsilon}_1$, as shown by (3) and (13). Substituting all of these relations into (27) gives

$$\dot{\varepsilon}_1 = n\dot{\varepsilon}_1/\varepsilon_1 + m\ddot{\varepsilon}_1/\dot{\varepsilon}_1 \tag{28}$$

and this can be solved for the strain at maximum load

$$\varepsilon_1 = n\left[1 - m\ddot{\varepsilon}_1/\dot{\varepsilon}_1^2\right]^{-1}. \tag{29}$$

Equation (29) is an extremely interesting result. It indicates that the strain at maximum load depends upon the strain rate and strain rate sensitivity parameters but not upon the strain rate ratio, providing this last quantity has been always constant. For a rate insensitive material, the well-known relation, $\varepsilon_1 = n$, is recovered.

Now (29) can be evaluated for the cases of two simple types of deformation.

(1) *Constant logarithmic (true) strain rate test*. In this test $\dot{\varepsilon}_1$ is a constant. Then, $\ddot{\varepsilon}_1 = 0$ and (29) becomes

$$\varepsilon_1 = n. \tag{30}$$

This result shows that the longitudinal strain at maximum load is the strain hardening parameter, n, over the entire range of the strain ratio, ρ.

(2) *Constant crosshead speed (constant engineering strain rate) test*. In this test $\dot{L}$ is a constant, thus $\ddot{L} = 0$. Then $\ddot{\varepsilon}_1 = \ddot{L}/L - \dot{L}^2/L^2 = 0 - (\dot{L}/L)^2 = -\dot{\varepsilon}_1^2$ and (29) becomes

$$\varepsilon_1 = n/(1 + m). \tag{31}$$

This result shows that ε_1 is somewhat less than in the previous case; the reduction depends on the strain rate sensitivity parameter, m.

Comparison with one-dimensional case. It is straightforward to compare these biaxial sheet results with uniaxial results from the analysis of Hart (1967). He derived as the basic relationship for uniform deformation, the equation

$$\dot{P}/P = -(\dot{L}/L)[1 - \gamma + m] + (\ddot{L}/\dot{L})m. \tag{32}$$

Here γ is defined as

$$\gamma = \partial \ln \sigma / \partial \varepsilon \tag{33}$$

which is an alternative strain hardening parameter related to n by

$$\gamma = n/\varepsilon. \tag{34}$$

As in the previous section, (32) can be applied to describe the two simplest types of tension test.

For the case of a constant logarithmic strain rate test, $\ddot{\varepsilon} = 0$ and $\ddot{L}/L = (L/\dot{L})^2$ so that $\ddot{L}/\dot{L} = \dot{L}/L$. Then

$$\dot{P}/P = -(\dot{L}/L)[1 - \gamma + m - m] = -(\dot{L}/L)[1 - \gamma]. \tag{35}$$

Therefore, maximum load, characterized by $\dot{P}/P = 0$, occurs when $\gamma = 1$. Then from (34) the effective strain at maximum load is $\varepsilon = n$. Keeping in mind that for the tension test the axial strain is identical to the effective strain, this is exactly the same result as (30).

For the case of a constant crosshead speed test $\ddot{L} = 0$ and then

$$\dot{P}/P = -(\dot{L}/L)[1 - \gamma + m]. \tag{36}$$

Therefore, maximum load occurs when $\gamma = 1 + m$. Then again from (34)

$$\varepsilon = n/(1 + m). \tag{37}$$

This corresponds exactly to the biaxial result (31).

Discussion. The main result of the present analysis is given in (3). Equation (3) is simply a kinetic condition that must be satisfied when the load in the major strain direction reaches a maximum. It applies to the stress rate, stress and strain rate in the major strain direction.

Subsidiary results are given in (14), (22) and (29). Equation (14) is an analogous condition that must be simultaneously satisfied by the effective stress rate, effective stress and effective strain rate. This condition involves a constant that is determined in part by the strain rate ratio. Equation (22) is also analogous to (3) and must be simultaneously satisfied by the stress rate, stress and strain rate in the minor strain direction. It contains the strain rate ratio; and this generally prevents the load in the minor strain direction from reaching its maximum value simultaneously with the major load.

Equations (3), (14) and (22) are equivalent expressions for the condition of reaching maximum load in the major strain direction. They merely express this condition in terms of different sets of variables. In two of these the strain rate ratio appears and is stated to be a constant. However, the derivation makes it clear that this ratio need not be held constant along the entire loading path. It only has to be constant in some neighborhood of maximum load.

The condition expressed by these three equations is that the load in the 1-direction reaches a maximum value. This condition per se does not relate to the stability of plastic deformation.

Equation (29) expresses the major strain component at maximum load in terms of material constitutive properties and rate parameters of the axial deformation. Under constant strain rate ratio $\varepsilon_2 = \rho\varepsilon_1$ and under monotonic loading $\varepsilon = C_0\varepsilon_1$ so that equivalent expressions could be constructed for these strain values as well. For any constant strain rate ratio, the value of major strain is independent of the strain rate ratio. This result was confirmed recently by Choi (1987) in the numerical calculation of forming limit diagrams. For a given material the deformation program for the 1-direction was held constant; the 2-direction deformation was varied over a set of discrete cases from uniaxial tension through plane strain to equibiaxial stretching. At maximum load the major strain had the same value in every case and that value was precisely as predicted by (29). When different material parameters were used and different deformation programs simulated the results were always in accord with (29).

Conclusions. The Considère criterion for the occurrence of maximum load in a tension test has been extended to biaxial tension in a sheet.

The biaxial criterion is identical in form to Considère's uniaxial criterion but with the addition of appropriate subscripts. Equation (3).

For a broad class of anisotropic material behavior it has been shown that maximum load in the minor strain direction cannot concide with maximum load in the major strain direction except under equibiaxial stretching. Equation (22).

At the point of maximum load the major strain is accurately predicted in terms of the material strain hardening parameter, strain rate sensitivity parameter and deformation rates in the major strain direction. Equation (29).

The major strain at maximum load is independent of the (constant) strain rate ratio. Equation (29).

REFERENCES

[1] Argon, A. S., 1973, *Stability of Plastic Deformation*, Chapter 7 of *The Inhomogeneity of Plastic Deformation*, American Society for Metals, 161–189.

[2] Backofen, W. A., Turner, I. R., and Avery, D. H., 1964, *Superplasticity in an Al-Zn Alloy*, Transactions of the ASM, Vol. **57**, 980–990.

[3] Campbell, J. D., 1967, *Plastic Instability in Rate-dependent Materials*, Journal of the Mechanics and Physics of Solids, Vol. **15**, 359–370.

[4] Choi, W., 1987, *Mathematical Modeling of Sheet Metal Deformation*, Ph.D. Dissertation, University of Kentucky.

[5] Considère, A., 1885, *Memoire sur l'emploi du fer et de l'acier dans les constructions*, Annales des Ponts Chaussees, Vol. **9**, 574–775.

[6] Ghosh, A. K., 1977, *Tensile Instability and Necking in Materials with Strain Hardening and Strain-rate Hardening*, Acta Metallurgica, Vol. **25**, 1413–1424.

[7] Hart, E. W., 1967, *Theory of the Tensile Test*, Acta Metallurgica, Vol. **15**, 351–355.

[8] Hill, R., 1950, *The Mathematical Theory of Plasticity*, Oxford University Press.

[9] Hill, R., 1952, *On Discontinuous Plastic States, with Special Reference to Localized Necking in Thin Sheets*, Journal of the Mechanics and Physics of Solids, Vol. **1**, 19–30.

[10] Hutchinson, J. W., and Obrecht, H., 1977a, *Tensile Instabilities in Strain-rate Dependent Materials*, Fracture 1977, (edited by D. M. R. Taplin), University of Waterloo Press, Vol. **1**, 101–116.

[11] Hutchinson, J. W., and Neale, K. W., 1977b, *Influence of Strain-rate Sensitivity on Necking under Uniaxial Tension*, Acta Metallurgica, Vol. **25**, 839–846.

[12] Johnson, W., and Mellor, P. B., 1973, *Engineering Plasticity*, Van Nostrand Reinhold Company, 83–85.

[13] Jonas, J. J., Holt, R. A., and Coleman, C. E., 1976, *Plastic Stability in Tension and Compresion*, Acta Metallurgica, Vol. **24**, 911–918.

[14] Jonas, J. J., and Baudelet, B., 1977, *Effect of Crack and Cavity Generation on Tensile Stability*, Acta Metallurgica, Vol. **25**, 43–50.

[15] Jones, S. E., and Gillis, P. P., 1984, *A Generalized Quadratic Flow Law for Sheet Metals*, Metallurgical Transactions A, Vol. **15A**, 129–132.

[16] Swift, H. W., 1952, *Plastic Instability under Plane Stress*, Journal of the Mechanics and Physics of Solids, Vol. **1**, 1–18.

UNE GÉNÉRALISATION EN DEUX DIMENSIONS DU CRITÈRE DE CONSIDÈRE

La critère de Considère pour atteindre la charge maximale sous tension uniaxiale a été prolongé au cas de la déformation biaxiale d'une surface plate. Peu avante que la charge maximale soit atteinte, on suppose que la constrainte de déformation de la surface plane est répartie de manière homogène sur toute la surface. La critère pour atteindre la charge maximale sous déformation biaxiale est obtenu pour la direction principale de la tension. On démontre qu'à moins d'opérer avec une force *equi-biaxiale*, les maxima des charges dans les directions principale et secondaire ne peuvent apparaître simultanément. La matière de la surface est supposée tendue. On suppose de plus que c'est une fonction du taux de tension et que'elle est anisotrope. Le résultats peuvent être résumés de la manière principale peut être prédite de manière précise en fonction du paramètre de durcissement de la matière, du taux de tension et des vitesses de déformation dans la direction principale de la tension.

DVODIMENZIJSKO UOPŠTENJE USLOVA CONSIDÈRE-A

Considère-ov kriterijum koji određuje najveće opterećenje pri jednoosnom zatezanju proširen je na slučaj ravnog stanja napona. Pretpostavljeno je da, neposredno pre nego što se dostigne najveće opterećenje, u celoj ploči imamo homogeno ravno stanje napona. Kriterijum za određivanje maksimalnog opterećenja povezan je sa glavnim pravcima tenzora deformacija. Pokazano je da nije moguće da opterećenja u glavnim pravcima tenzora deformacija imaju istovremeno ekstremne vrednosti, izuzev u slučaju jednakih dilatacija u oba pravca. Usvojeno je da napon u materijalu od koga je načinjena ploča zavisi od deformacije i brzine deformacije. Centralni zaključak analize je da za najveće opterećenje, glavna maksimalna dilatacija se može precizno predvideti preko karakteristika materijala i to parametra ojačanja i parametra osetljivosti na brzinu deformacije u glavnim pravcima tenzora deformacije. Glavna maksimalna dilatacija za maksimalno opterećenje ne zavisi od odnosa brzina deformacije u glavnim pravcima.

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