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PARTICULAR SOLUTION FOR THERMAL STRESSES IN A DISK OF HYPERBOLIC SHAPE

A. Alujevič, Dj. Žebeljan

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1. Introduction. As the governing equations to be used, there are the following sets of

$$\begin{aligned} &\text{equilibrium} && \vec{\nabla}\cdot\underline{\sigma}+\vec{f}=0 \\ &\text{kinematics} && \underline{\varepsilon}=(\vec{\nabla}\vec{u}+\vec{\nabla}^{*}\vec{u})/2 \qquad (1) \\ &\text{stress-strain} && \underline{\sigma}=2\mu\underline{\varepsilon}+[\lambda\varepsilon_v-(3\lambda+2\mu)\alpha\Theta]\underline{\delta}, \end{aligned}$$

where Lamé constants are $\mu=E/[2(1+\nu)]$ and $\lambda=\nu E/[(1+\nu)(1-2\nu)]$.

The above system of equations may be transcribed into Navier-Lamé equation between displacements and loading forces

$$\mu\nabla^2\vec{u}+(\mu\varepsilon_v'-\alpha E\Theta')/(1-2\nu)+\vec{f}=0. \qquad (2)$$

For a relatively thin disk, these relationships simplify considerably. Assuming non-isotropic disk of constant thickness, the corresponding second order equation is

$$u_R''+\frac{1}{r}u_R'-\frac{1}{r^2}u_R+l\omega^2r=\kappa\Theta'+\frac{1}{r}\chi\Theta \qquad (3)$$

where l, κ and χ are functions of elastic properties and thermal expansion. For the isotropic disk case $\kappa=(1+n)/(1-\nu)$, $\chi=0$. An alternative formulation is known in terms of radial and circumferential stresses (for a disk with rotation and without thermal loading)

$$\sigma_R'+\frac{1}{r}(\sigma_R-\sigma_\phi)+\rho\omega^2r=0, \qquad (4)$$

or by eliminating the hoop component, the radial stress obeys

$$\sigma_R''+\frac{3}{r}\sigma_R'+(3+\nu)\rho\omega^2=0. \qquad (5)$$

2. Evaluation. Extending the above differential equation for disk of variable thickness (neglecting the centrifugal forces), the corresponding expression is

$$\sigma_R''+\left(\frac{3}{r}+\frac{t'}{t}\right)\sigma_R'+\left[\frac{t''}{t}+\frac{(2+\nu)}{r}\frac{t'}{t}-\left(\frac{t'}{t}\right)^2\right]\sigma_R+\frac{\alpha E}{r}\Theta'=0 \qquad (6)$$

and due to $\sigma_\phi = r\sigma_R' + (1 + rt'/t)\sigma_R$ for the hyperbolic shape, defined by

$$t = t_0\left(\frac{a}{r}\right)^n \qquad t' = -nt_0\frac{a^n}{r^{n+1}} \qquad t^n = n(n+1)t_0\frac{a^n}{r^{n+2}} \tag{7}$$

a nonhomogeneous equation is obtained

$$\sigma_R'' + \frac{3-n}{r}\sigma_R' - \frac{n(1+\nu)}{r^2}\sigma_R + \frac{\alpha E}{r}\Theta' = 0 \tag{8}$$

where its homogeneous part is Euler's equation, which may be solved using r^w, yielding the characteristic polynomial

$$w^2 + (2-n)w - n(1+\nu) = 0 \tag{9}$$

with roots

$$_1w_2 = \frac{n-2 \pm \sqrt{n^2+4\nu n+4}}{2} = (A, B) \tag{10}$$

and the general solution is

$$\sigma_R = C_1 r^A + C_2 r^B. \tag{11}$$

Functions C_1 and C_2 have to be determined by the variation of constants, giving the radial stress formula

$$\sigma_R = r^A\left(\int_a^r \frac{\alpha E\Theta'}{B-A}r^{-A}\,dr + D_1\right) - r^B\left(\int_a^r \frac{\alpha E\Theta'}{B-A}r^{-B}\,dr + D_2\right) \tag{12}$$

where D_1 and D_2 are integration constants to be determined from the boundary conditions, and assuming no additional load ($p_a = p_b = 0$) on the inner and outer perimeters

$$D_1 = \frac{a^B}{b^Aa^B - b^Ba^A}\frac{\alpha E}{B-A}\left(b^B\int_a^b \Theta' r^{-B}\,dr - b^A\int_a^b \Theta' r^{-A}\,dr\right) \tag{13}$$

$$D_2 = \frac{a^A}{b^Aa^B - b^Ba^A}\frac{\alpha E}{B-A}\left(b^A\int_a^b \Theta' r^{-A}\,dr - b^B\int_a^b \Theta' r^{-B}\,dr\right). \tag{14}$$

Performing the integration by parts, e.g.

$$\int_a^b \Theta' r^{-A}\,dr = \Theta(b)b^{-A} - \Theta(a)a^{-A} + A\int_a^b \Theta(r)r^{-A-1}\,dr \tag{15}$$

the radial and circumferential stress components are finally given as

$$\begin{aligned}\sigma_R = \frac{\alpha E}{B-A}\Bigg\{ & r^A\left[\Theta(r)r^{-A} - \Theta(a)a^{-A} + A\int_a^r \Theta(r)r^{-A-1}\,dr\right] \\ & - r^B\left[\Theta(r)r^{-B} - \Theta(a)a^{-B} + B\int_a^r \Theta(r)r^{-B-1}\,dr\right] \\ & + \frac{a^Br^A - a^Ar^B}{b^Aa^B - b^Ba^A}\left[b^B\left(\Theta(b)b^{-B} - \Theta(a)a^{-B} + B\int_a^b \Theta(r)r^{-B-1}\,dr\right)\right.\end{aligned}$$

$$-b^A\left(\Theta(b)b^{-A}-\Theta(a)a^{-A}+A\int_a^b\Theta(r)r^{-A-1}\,dr\right)\Bigg]\Bigg\} \tag{16}$$

and due to

$$\sigma_\phi = r\sigma_R' + (1-n)\sigma_R \tag{17}$$

$$\begin{aligned}\sigma_\phi = \frac{\alpha E}{B-A}\Bigg\{&(A+1-n)r^A\left[\Theta(r)r^{-A}-\Theta(a)a^{-A}+A\int_a^r\Theta(r)r^{-A-1}\,dr\right]\\ &-(B+1-n)r^B\left[\Theta(r)r^{-B}-\Theta(a)a^{-B}+B\int_a^r\Theta(r)r^{-B-1}\,dr\right]\\ &+\frac{(A+1-n)a^Br^A-(B+1-n)a^Ar^B}{b^Aa^B-b^Ba^A}\times\\ &\times\Bigg[b^B\left(\Theta(b)b^{-B}-\Theta(a)a^{-B}+B\int_a^b\Theta(r)r^{-B-1}\,dr\right)\\ &-b^A\left(\Theta(b)b^{-A}-\Theta(a)a^{-A}+A\int_a^b\Theta(r)r^{-A-1}\,dr\right)\Bigg]\Bigg\}.\end{aligned} \tag{18}$$

In the case of constant thickness ($n = 0$) known form of stres distribution can be deduced (Timoshenko [5]).

Temperature distribution is given by the differential equation

$$t\Theta'' + \left(t' + \frac{t}{r}\right)\Theta' - \frac{h}{k}\Theta = 0 \tag{19}$$

(h = transmissivity, k = conductivity), which for the hyperbolic disk reads

$$\Theta'' + \frac{1-n}{r}\Theta' - \frac{h}{kt}\Theta = 0, \qquad t = t_0\left(\frac{a}{r}\right)^n \tag{20}$$

and generally cannot be solved analytically ($k/h \neq$ const), but it must be evaluated numerically. Determining thermal efficiency

$$\eta = \frac{2}{\Theta_0(b^2-a^2)}\int_a^b \Theta(r)r\,dr \tag{21}$$

thermal stresses may also be obtained subject to a corresponding temperature dependent efficiency. For the hyperbolic disk if k/h = const. with the temperature distribution (where $M = m/\sqrt{a^n}$, $m = \sqrt{h/kt_0}$, $N = 2/(n+2)$)

$$\begin{aligned}\Theta(r) = \Theta_0\sqrt{\left(\frac{r}{a}\right)^n}\;\times&\\ \times\;\frac{I_N(NMb^{1/N})I_{1-N}(NMr^{1/N})-I_{-N}(NMb^{1/N})I_{N-1}(NMr^{1/N})}{I_N(NMb^{1/N})I_{1-N}(NMa^{1/N})-I_{-N}(NMb^{1/N})I_{N-1}(NMa^{1/N})}&\end{aligned} \tag{22}$$

thermal efficiency is

$$\begin{aligned}\eta = \frac{2a}{m(b^2-a^2)}\times&\\ \times\;\frac{I_{-N}(NMb^{1/N})I_N(NMa^{1/N})-I_N(NMb^{1/N})I_{-N}(NMa^{1/N})}{I_N(NMb^{1/N})I_{1-N}(NMa^{1/N})-I_{-N}(NMb^{1/N})I_{N-1}(NMa^{1/N})}&\end{aligned} \tag{23}$$

while within the disk of constant thickness ($n = 0$) its temperature

$$\Theta(r) = \Theta_0 \frac{K_1(mb)I_0(mr) + I_1(mb)K_1(mr)}{K_1(mb)I_0(ma) + I_1(mb)K_0(ma)} \tag{24}$$

and efficiency

$$\eta = \frac{2a}{m(b^2 - a^2)} \frac{I_1(mb)K_1(ma) - K_1(mb)I_1(ma)}{K_1(mb)I_0(ma) + I_1(mb)K_0(ma)}. \tag{25}$$

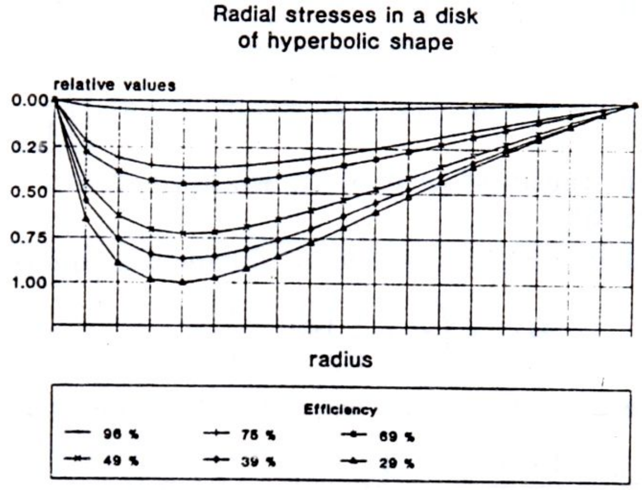

Fig. 1

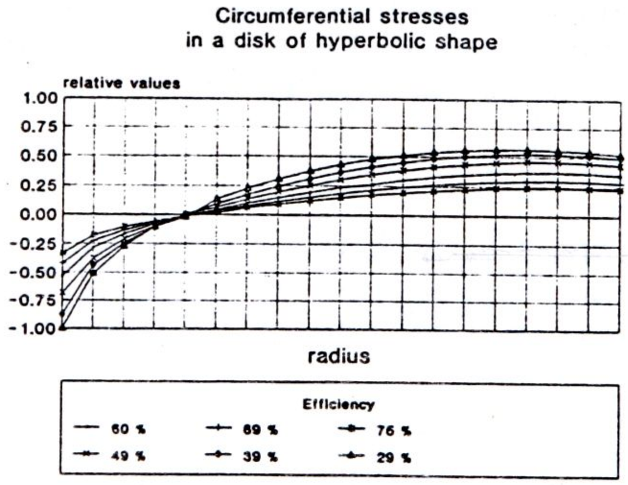

Fig. 2

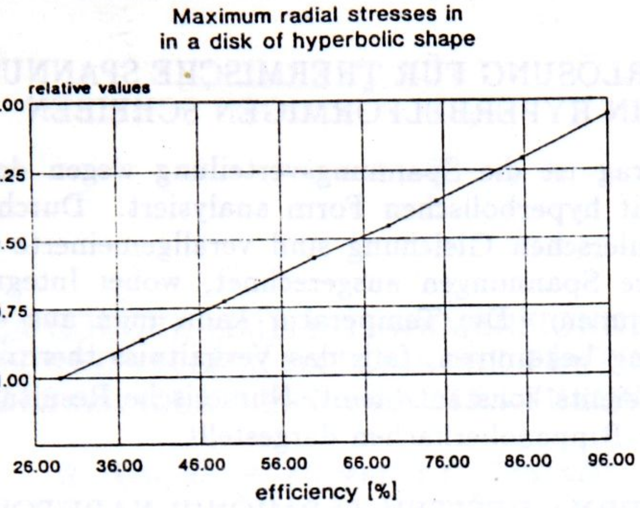

Fig. 3

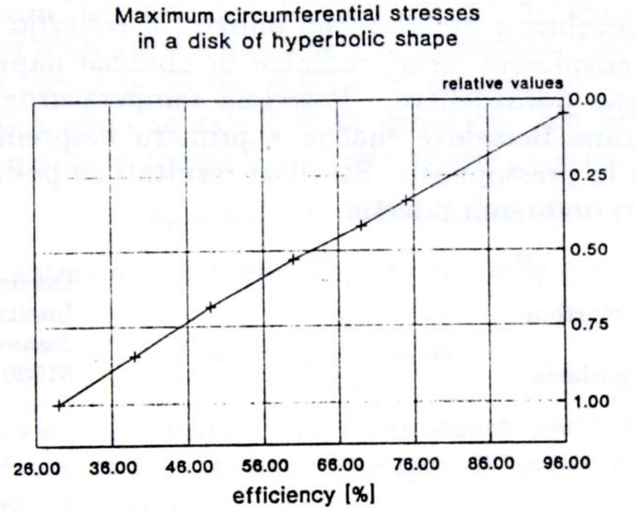

Fig. 4

3. Case study. A numerical case has been run for thermal stresses in the hyperbolic disk with $n = 1$. Fig's 1 and 2 give normalised relative radial and circumferential stresses with respect to radii, while thermal efficiency is used parametrically. Fig's 3 and 4 show the maximum values of radial and circumferential stress respectively vs. efficiency for a selected radius, where maximum values do appear. The correlations turn to be almost linear.

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SONDERLÖSUNG FÜR THERMISCHE SPANNUNGEN IN HYPERBELFÖRMIGEN SCHEIBEN

In diesem Beitrag ist die Spannungsverteilung wegen des Wärmefeldes in dünnen Scheiben mit hyperbolischen Form analysiert. Durch die Lösung einer nicht homogenen Eulerschen Gleichung sind verallgemeinerte Ausdrücke für die radiale und zirkuläre Spannungen ausgerechnet, wobei Integrale der Temperaturverteilung vorkommen. Die Temperatur kann man aus einer modifizierten Besselschen Gleichung bekommen, falls das Verhältniss thermischer Leitfähigkeit und Übergangskoeffizients konstant bleibt. Numerische Resultate sind in Relation zur Wirksamkeit der Rippenoberflächen dargestellt.

POSEBNA REŠITEV TERMIČNIH NAPETOSTI V DISKU HIPERBOLIČNE OBLIKE

V prispevku je podana porazdelitev napetosti zaradi toplotnega polja v tankem disku spremenljive debeline s hiperbolično obliko. Z rešitvijo ustrezne Eulerjeve enačbe so izpeljani posplošeni izrazi radialne in obročne napetosti, ki vsebujejo integrale temperaturne porazdelitve. Potrebna temperaturna distribucija pa je dobljena iz modificirane Besselove enačbe v primeru nespremenljivega razmerja toplotne prevodnosti in prestopnosti. Števílski rezultati so podani v odvisnosti od ustrezne učinkovitosti orebrenih površin.

Andro Alujevič
Tehniška fakulteta Maribor
Smetanova 17
62000 Maribor, Yugoslavia

Đorđe Žebeljan
Institut Jožef Stefan
Jamova 39
61000 Ljubljana, Yugoslavia