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CALCULATION OF 2-D TRANSONIC FLOWS ABOUT AXI-SYMMETRIC BODIES

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Introduction. We shall show in this work general procedure for the solutions of transonic two dimensional flows, which can be sufficiently good described with isentropic flow equation. It is accepted that isentropic flow assumption is valid when Mach number does not exceed 1.3, ($M_\ell \leq 1.3$). With this limitation it is ensured that shocks are of the weak intensity.

When the flow equations are in the conservative form, it is necessary to add mass in the supersonic part of flow domain, in order to obtain convergent and stable solution. Here we have applied, so called, artificial compressibility method. With this method density should be modified in the supersonic flow domain.

Calculation of transonic flows around biconvex airfoil, and around axi-symmetric body of revolution, obtained from biconvex airfoil, are presented.

Governing equations and boundary conditions. Equations which describes two dimensional, inviscid, isentropic flow are given in [4] or in [1]:

$$\frac{\partial(y^m \varrho u)}{\partial x} + \frac{\partial(y^m \varrho v)}{\partial y} = 0,$$

$$\varrho = \left[1 - \frac{\kappa - 1}{2} M_\infty^2 (u^2 + v^2 - 1)\right]^{1/(\kappa-1)}, \tag{1}$$

where all velocities are nondimensionalized with unperturbed flow velocity, at infinity. Lengths are divided with some chosen length, while densities are divided with unperturbed density. Axi-symmetric flow equations are obtained for $m = 1$, and two dimensional when parameter is zero, $m = 0$.

The existence of the potential requires that the flow be irrotational:

$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0. \tag{2}$$

Free stream boundary conditions can be specified:

$$x, y \mapsto \pm\infty, \qquad u \mapsto 1, \qquad v \mapsto 0, \tag{3}$$

while the impermeability of the body is defined with:

$$\vec{V} \cdot \vec{n} = 0 \,. \tag{4}$$

Equations (1) and (2) are conservative form of flow equations. If, by definition, scalar quantity Φ is introduced:

$$u = \Phi_x = \frac{\partial \Phi}{\partial x}, \qquad v = \Phi_y = \frac{\partial \Phi}{\partial y},$$

it is possible to reduce problem from two partial differential equations of the first order to one of the second order (1) and (2):

$$(y^m \varrho \Phi_x)_x + (y^m \varrho \Phi_y)_y = 0$$

$$\varrho = \left[1 - \frac{\kappa - 1}{2} M_\infty^2 (\Phi_x^2 + \Phi_y^2 - 1)\right]^{1/(\kappa-1)} \,. \tag{5}$$

This partial differential equation is reduced to Laplace equation when the free stream Mach number is $M_\infty = 0$. Pressure coefficient, C_p, around airfoil can be calculated when the densities, ϱ are known:

$$C_p = \frac{2}{\kappa M_\infty^2}(\varrho^\kappa - 1)\,,$$

local Mach number can be determined from the known densities on the computational mesh nodes, using formulae:

$$M^2 = \frac{2}{\kappa - 1}\left(\frac{1 + \frac{\kappa-1}{2} M_\infty^2}{\varrho^{\kappa-1}} - 1\right) .$$

Substituting 1 instead of M, ϱ^* is obtained:

$$\varrho^* = \left(\frac{1 + \frac{\kappa-1}{2} M_\infty^2}{1 + \frac{\kappa-1}{2}}\right)^{1/(\kappa-1)}$$

by direct substitution in the equation for C_p, C_p^* is obtained. All values of C_p which are less than C_p^* correspond to supersonic part of the flow domain.

Grid generation. Computational mesh around biconvex symmetric, as well as around corresponding body of revolution can be easily simplified. Also the way of boundary conditions specification is simplified. We have adopted, for this purpose, complex mapping of bump above x axis to ξ axis without bump, [2], figure (1), which is defined:

$$\zeta = w(z) = \frac{a\pi}{\pi - \alpha}\left\{1 - \left(1 - \frac{a}{z}\right)^{\pi/(\pi-\alpha)}\right\}^{-1} . \tag{6}$$

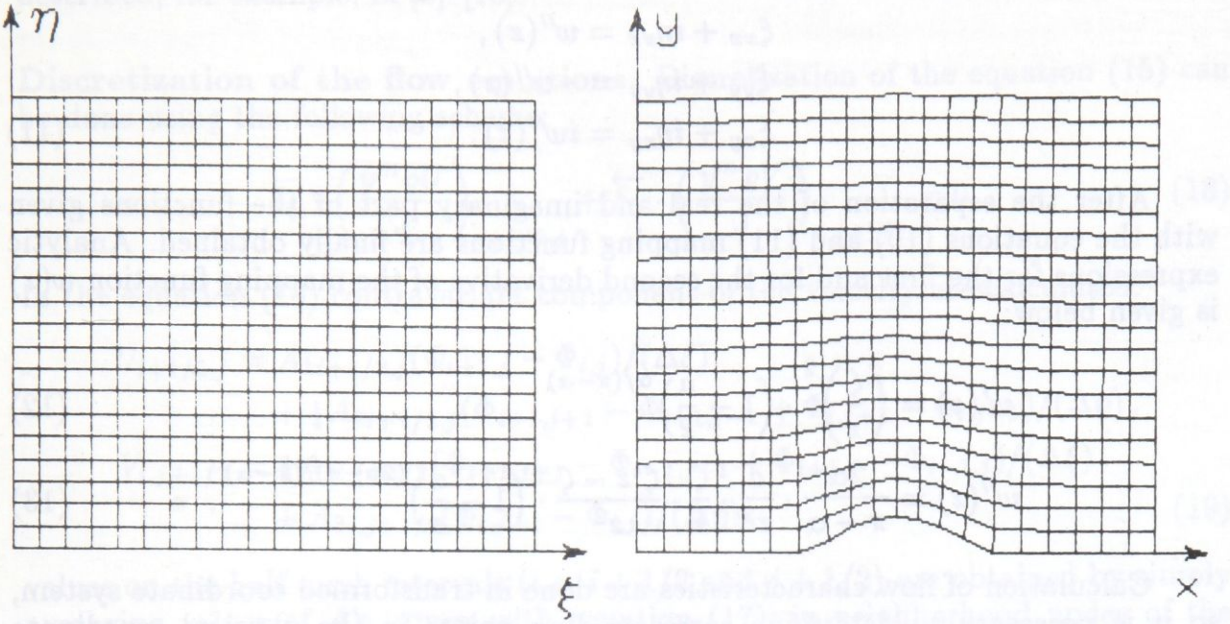

Figure 1: Calculation field in computational and physical plane

Real and imaginary part of the $\zeta = \xi(x,y) + i\eta(x,y)$ are functions of the coordinate of the point in the z-plane. Transformation of partial derivatives of the equation (5) is specified in the book [3]:

$$\begin{aligned}
\Phi_x &= \Phi_\xi \xi_x + \Phi_\eta \eta_x \\
\Phi_y &= \Phi_\xi \xi_y + \Phi_\eta \eta_y \\
\Phi_{xx} &= \Phi_{\xi\xi}\xi_x^2 + 2\Phi_{\xi\eta}\xi_x\eta_x + \Phi_{\eta\eta}\eta_x^2 + \Phi_\xi \xi_{xx} + \Phi_\eta \eta_{xx} \\
\Phi_{yy} &= \Phi_{\xi\xi}\xi_y^2 + 2\Phi_{\xi\eta}\xi_y\eta_y + \Phi_{\eta\eta}\eta_y^2 + \Phi_\xi \xi_{yy} + \Phi_\eta \eta_{yy} \\
\Phi_{xy} &= \Phi_{\xi\xi}\xi_x\xi_y + \Phi_{\xi\eta}(\xi_x\eta_y + \xi_y\eta_x) + \Phi_{\eta\eta}\eta_x\eta_y + \Phi_\xi \xi_{xy} + \Phi_\eta \eta_{xy} \\
J &= \xi_x\eta_y - \xi_y\eta_x
\end{aligned} \tag{7}$$

Mapping function is known function given with equation (6),

$$\zeta = \xi + i\eta = w(z),$$

using the rules for derivative calculation, it is easily possible to determine derivatives of mapping function given with equation (7):

$$d\zeta = d\xi + id\eta = w'(z)dz \tag{8}$$

$$d^2\zeta = d^2\xi + id^2\eta = w''(z)dz^2 . \tag{9}$$

Dividing equation (8) with ∂x, and with ∂y it is obtained:

$$\begin{aligned}
\xi_x + i\eta_x &= w'(z) \\
\xi_y + i\eta_y &= iw'(z) .
\end{aligned} \tag{10}$$

Dividing equation (9) with ∂x^2, ∂y^2 and with $\partial x \partial y$ second derivatives of the mapping function are obtained:

$$\begin{aligned} \xi_{xx} + i\eta_{xx} &= w''(z)\,, \\ \xi_{yy} + i\eta_{yy} &= -w''(z)\,, \\ \xi_{xy} + i\eta_{xy} &= iw''(z)\,. \end{aligned} \tag{11}$$

After the separation of the real and imaginary part of the functions given with the equations (10) and (11) mapping functions are finally obtained. Analytic expressions for the first and for the second derivative of the mapping function $w(z)$ is given below:

$$w'(z) = \left(\frac{\zeta}{z}\right)^2 \cdot \left(1 - \frac{a}{z}\right)^{\alpha/(\pi-\alpha)} \tag{12}$$

$$w''(z) = \frac{2\alpha}{\pi - \alpha} \cdot \frac{a}{z^2} \cdot \frac{\zeta}{z} \cdot \frac{\zeta' z - \zeta}{z^2} \cdot \left(1 - \frac{a}{z}\right)^{(2\alpha-\pi)/(\pi-\alpha)} . \tag{13}$$

Calculation of flow characteristics are done in transformed coordinate system, so it is necessary to determine corresponding points in the physical coordinate system which correspond to nodes of the computational mesh. That is equivalent to find inverse transformation of the transformation (6). For the flows around biconvex airfoil it is possible to find analytic expression for the inverse mapping function, by which can be coordinates in the z-plane obtained which corresponds to points in the ζ-plane:

$$z = a\left\{1 - \left(1 - \frac{a\pi}{(\pi-\alpha)\zeta}\right)^{(\pi-\alpha)/\pi}\right\}^{-1} . \tag{14}$$

Transformed form of equations. Equation (5) after general coordinate transformation, with which physical domain (x, y) is transformed in the computational domain (ξ, η), and after substitution of the $\Phi_x = \Phi_\xi \xi_x + \Phi_\eta \eta_x$ and the $\Phi_y = \Phi_\xi \xi_y + \Phi_\eta \eta_y$ and after grouping of members derived with ξ and η is transformed to:

$$\begin{aligned} \left(y^m \varrho \frac{U}{J}\right)_\xi + \left(y^m \varrho \frac{V}{J}\right)_\eta &= 0\,, \\ \varrho = \left[1 - \frac{\kappa - 1}{2} M_\infty^2 (U\Phi_\xi + V\Phi_\eta - 1)\right]^{1/(\kappa-1)} &, \end{aligned} \tag{15}$$

where U and V are the contravariant velocity components given by:

$$U = A_1\Phi_\xi + A_2\Phi_\eta\,, \qquad V = A_2\Phi_\xi + A_3\Phi_\eta\,, \tag{16}$$

while coefficients are:

$$\begin{aligned} A_1 = \xi_x^2 + \xi_y^2\,, \quad A_2 = \xi_x\eta_x + \xi_y\eta_y\,, \quad A_3 = \eta_x^2 + \eta_y^2\,, \\ J = \frac{\partial(\xi, \eta)}{\partial(x, y)} = \xi_x\eta_y - \xi_y\eta_x\,. \end{aligned} \tag{17}$$

We have solved formerly formulated problem with finite difference method, as it is described, for example, in [5]–[10].

Discretization of the flow equations. Discretization of the equation (15) can be done using the following scheme:

$$\overleftarrow{\delta_\xi} \cdot \left(\frac{y^m \varrho U}{J}\right)_{i+1/2,j} + \overleftarrow{\delta_\eta} \cdot \left(\frac{y^m \varrho V}{J}\right)_{i,j+1/2} = 0\,. \tag{18}$$

In the equation (18) contravariant component of the velocity are calculated:

$$\begin{aligned} U_{i+1/2,j} &= A_{1i+1/2,j}(\Phi_{i+1,j} - \Phi_{i,j})/(\Delta\xi) \\ &\quad + \tfrac{1}{4} A_{2i+1/2,j}(\Phi_{i+1,j+1} - \Phi_{i+1,j-1} + \Phi_{i,j+1} - \Phi_{i,j-1})/(\Delta\eta)\,, \\ V_{i,j+1/2} &= \tfrac{1}{4} A_{2i,j+1/2}(\Phi_{i+1,j+1} - \Phi_{i-1,j+1} + \Phi_{i+1,j} - \Phi_{i-1,j})/(\Delta\xi) \\ &\quad + A_{3i,j+1/2}(\Phi_{i,j+1} - \Phi_{i,j})/(\Delta\eta)\,, \end{aligned} \tag{19}$$

values on the half mesh intervals (t.e. $i + 1/2$ and $j + 1/2$) are obtained by simply averaging values of A's, given with equation (17), in neighborhood nodes of the computational grid. Such discretization is suitable for completely subsonic flow. But it is necessary to apply concept of artificial compressibility [9] for the supersonic flow domain:

$$\overleftarrow{\delta_\xi} \left[y^m \tilde{\varrho}_i \left(\frac{U}{J}\right)_{i+1/2,j} \right] + \overleftarrow{\delta_\eta} \left(\frac{y^m \varrho V}{J}\right)_{i,j+1/2} = 0\,, \tag{20}$$

where $\tilde{\varrho}$ is defined by:

$$\tilde{\varrho}_i = (1 - \nu_{i+k,j})\varrho_{i+1/2,j} + \nu_{i+k,j}\,\varrho_{i+2k-1/2,j}\,, \tag{21}$$

while:

$$k = \begin{cases} 0, & U_{i+1/2,j} > 0 \\ 1, & U_{i+1/2,j} < 0 \end{cases} \tag{22}$$

Artificial compressibility coefficient is obtained by expression:

$$\nu = \max\left[0, C_1\left(1 - \frac{1}{M^2}\right)\right], \tag{23}$$

where C_1 is constant close to unity for small supersonic speed, and is increased with rise of the supersonic speeds. With such defined scheme is possible to calculate completely subsonic flow problems, as well as flows with supersonic pockets.

Calculation procedure. There are various procedure for the solution of the obtained equations. We have used, so called AF2 scheme developed by Holst and Ballhaus, [8], [10]. An AF scheme, as most others, for the full potential equation my be written:

$$NC^n + \omega L\Phi^n = 0\,, \tag{24}$$

for relaxation problem governed by partial differential equation of the form $L\Phi = 0$, where L is a differential operator. The ω is a relaxation parameter, C^n is the correction term $(\Phi^{n+1} - \Phi^n)$, while $L\Phi^n$ represents the residual because partial differential equation is not satisfied with approximate solution. With N is assigned operator which determines the iteration method. In AF schemes, N represents the product of two or more operators:

$$N = N_1 \cdot N_2 \, .$$

The operators N_1 and N_2 must be selected so that their product approximates L, only simple matrix operations are allowed, and overall scheme should be stable. AF2 scheme developed in [8], [10] has the form:

$$\alpha N C^n_{i,j} = -\left[\alpha - \overrightarrow{\delta_\eta}\left(\frac{y^m \varrho A_3}{J}\right)_{i,j-1/2}\right]\left[\alpha\overleftarrow{\delta_\eta} - \overleftarrow{\delta_\xi}\,\tilde{\varrho}_i\left(\frac{y^m A_1}{J}\right)_{i+1/2,j}\overrightarrow{\delta_\xi}\right] C^n_{i,j} \, . \tag{25}$$

The α is free parameter which may be interpreted as $(\Delta t)^{-1}$. A sequence of alphas is used during calculations in order to reduce both high and low frequency errors in the solution.

AF2 scheme given with equation (25) is implemented in two steps:

$$\left[\alpha - \overrightarrow{\delta_\eta}\left(\frac{y^m \varrho A_3}{J}\right)_{i,j-1/2}\right] f^n_{i,j} = \alpha\omega L\Phi^n_{i,j} \, ,$$
$$\left[\alpha\overleftarrow{\delta_\eta} - y^m\overleftarrow{\delta_\xi}\,\tilde{\varrho}_i\left(\frac{A_1}{J}\right)_{i+1/2,j}\overrightarrow{\delta_\xi}\right] C^n_{i,j} = f^n_{i,j} \, , \tag{26}$$

where $f^n_{i,j}$ is an intermediate result, required by numerical scheme, and is obtained in the first step by solution of two diagonal system of equations. In the second step correction term is calculated by the solution of tridiagonal system of equations. For this numerical routine, the sweep direction required is outward away from the body in step 1 and inward for the step 2. There is no limit for this scheme for sweep direction due to the flow direction.

After application of the operator N_1 on the first of the equations (26) the following system of two diagonal equations are obtained:

$$\left[\alpha + \frac{y^m}{\Delta\eta}\left(\frac{\varrho A_3}{J}\right)_{i,j-1/2}\right] f^n_{i,j} - \frac{y^m}{\Delta\eta}\left(\frac{\varrho A_3}{J}\right)_{i,j+1/2} f^n_{i,j+1} = \alpha\omega L\Phi^n_{i,j} \, , \tag{27}$$

where $\Delta\eta = 0.5 \times (\eta_{j+1} - \eta_{j-1})$.

Application of the N_2 operator on the second of equation (26) the following tridiagonal system of equations are obtained:

$$a_i C^n_{i-1} + b_i C^n_i + c_i C^n_{i+1} = d_i \, , \tag{28}$$

where the coefficients in above equation are:

$$a_i = -\frac{\tilde{\varrho}_{i-1,j}}{\overleftarrow{\Delta\xi}^2_i}\left(\frac{y^m A_1}{J}\right)_{i-1/2,j}$$

$$b_i = \left[\frac{\alpha}{\Delta \overleftarrow{\eta}_j} + \frac{\tilde{\varrho}_{i-1,j}}{\Delta \overrightarrow{\xi}_i^2}\left(\frac{y^m A_1}{J}\right)_{i-1/2,j} + \frac{\tilde{\varrho}_{i,j}}{\Delta \overrightarrow{\xi}_i \Delta \overleftarrow{\xi}_i}\left(\frac{y^m A_1}{J}\right)_{i+1/2,j}\right]$$

$$c_i = -\frac{\tilde{\varrho}_{i,j}}{\Delta \overrightarrow{\xi}_i \Delta \overleftarrow{\xi}_i}\left(\frac{y^m A_1}{J}\right)_{i+1/2,j}$$

$$d_i = f^n_{i,j} + \frac{\alpha}{\Delta \overleftarrow{\eta}_j} C^n_{i,j-1} .$$

Problem which arises in calculation of coefficients for the line $\eta = 0$, which corresponds to streamlined body, can be solved by reflection. So the needed values below the lowest computational grid line can be taken from the line just above these line ($j = 2$). Fully developed operator $L\Phi^n_{i,j}$, equation (20), is given with:

$$\Delta\xi = \xi_{+1/2} - \xi_{-1/2} \qquad \Delta\eta = \eta_{+1/2} - \eta_{-1/2} ,$$

$$\frac{1}{\Delta\xi}\left[\tilde{\varrho}\left(\frac{y^m U}{J}\right)_{+1/2,} - \tilde{\varrho}_{-1}\left(\frac{y^m U}{J}\right)_{-1/2,}\right]$$

$$+\frac{1}{\Delta\eta}\left[\left(\frac{y^m \varrho V}{J}\right)_{,+1/2} - \left(\frac{y^m \varrho V}{J}\right)_{,-1/2}\right] = 0 , \tag{29}$$

in the last equation we have avoided subscripts (i, j) in all terms.

There are three parameters with which is possible to control convergence of computational procedure: ω, α i C_1, their careful specification, as well as the way of variation of parameter α can significantly reduce the needed computational time.

Computed Results. Two solutions are here presented. First is the solution of transonic flow around biconvex 20% thick airfoil, at $M = 0.8$, and at the incidence angle $\alpha = 0°$. Second solution is for axi-symmetrical problem, for which body is obtained by rotation of biconvex airfoil. The flow parameters are the same as for the airfoil.

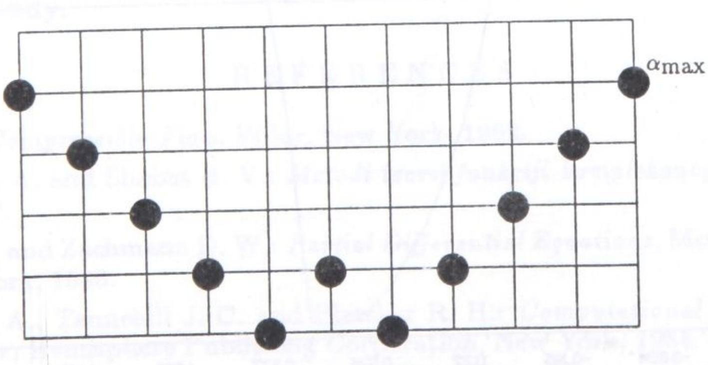

Figure 2: Convergence parameter cycle

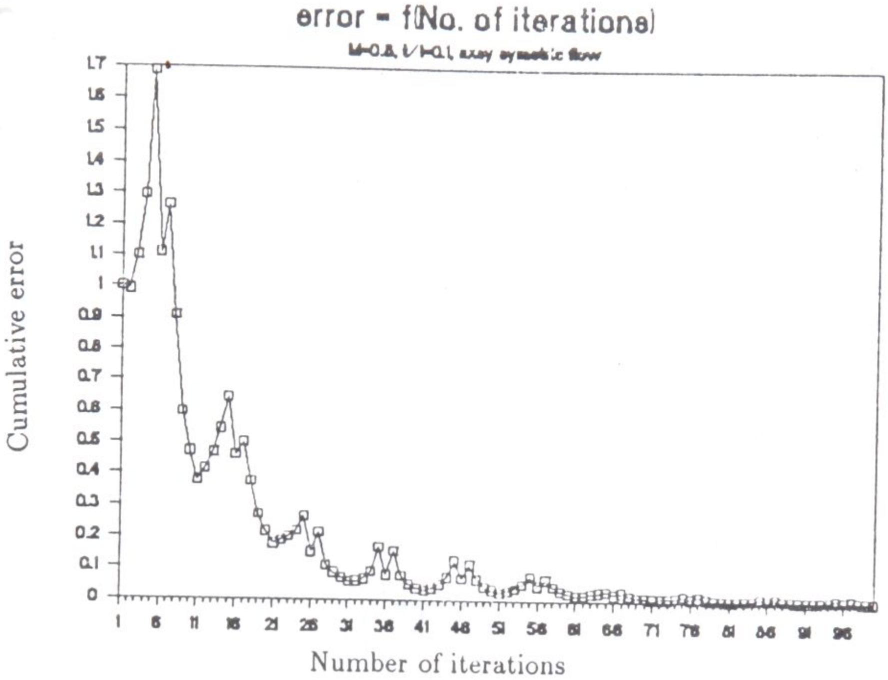

Figure 3: Convergence history for axi-symmetric body

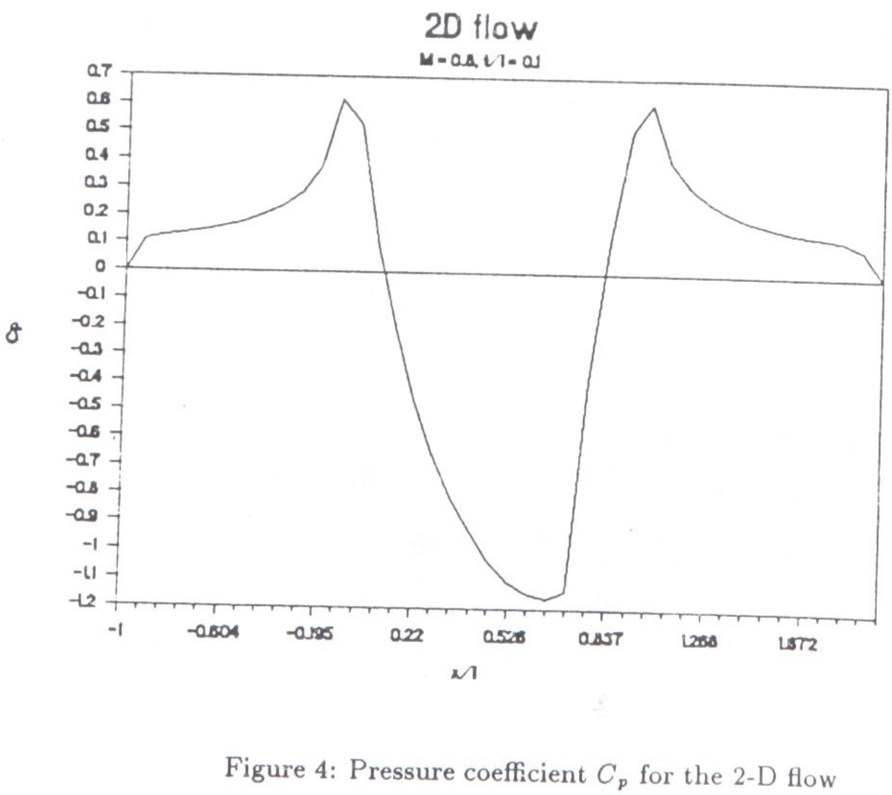

Figure 4: Pressure coefficient C_p for the 2-D flow

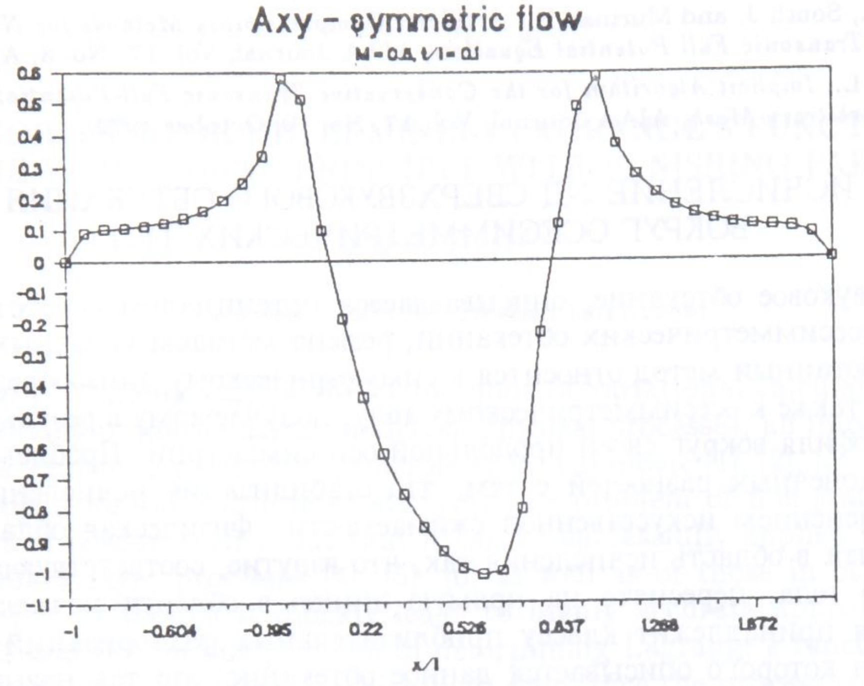

Figure 5: Pressure coefficient C_p for axi-symmetric flow

Calculations are done on the (40 × 20) computational grid where only 20 calculation points fall on the body. Convergence parameter α is cyclically varied as it is shown on the figure 2. Convergence history is displayed on the next figure 3. Computational error is defined as the sum of absolute values of the correction terms:

$$\epsilon^n = \sum_{i,j} |C^n_{i,j}| \,.$$

Pressure coefficient for twodimensional flow for the line $\eta = 0$ is given on the figure 4, while with figure 5 is represented pressure coefficient distribution for the axi-symmetric body.

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ИСЧИСЛЕНИЕ 2-Д СВЕРХЗВУКОВОГО ОБТЕКАНИЯ ВОКРУГ ОСЕСИММЕТРИЧЕСКИХ ТЕЛ

Сверхзвуковое обтекание, описывающееся потенциалом скорости для двумерных и осесимметрических обтеканий, решено методом конечных разностей. Этот разработанный метод относится к симметрическому линзообразному аэропрофилю, а также к осесимметрическому телу, получаемому в результате вращения аэропрофиля вокруг своей продольной оси симметрии. Проблема решалась на основе конечных разностей с тем, что стабилизация исчисления осуществлена применением искусственной сжимаемости. Физическая область обтекания переснята в область исчисления так, что вздутие, соответствующее контуру обтекаемого тела, переснято на прямую линию в области исчисления. Метод решения принадлежит классу приблизительных факторизаций оператора, при помощи которого описывается данное обтекание; это так называемый АФ 2 способ. Предоставляются в распоряжение диаграммы распределения коэффициента давления по контуру тела, а также диаграмма сходимости для данного примера.

PRORAČUN 2-D TRANSONIČNOG STRUJANJA OKO OSNOSIMETRIČNIH TELA

U radu je rešeno transonično strujanje opisano preko potencijala brzine za dvodimenziona i osno simetrična strujanja metodom konačnih razlika. Razrađeni postupak je primenjen na simetrični sočivasti aeroprofil i na osno simetrično telo koje je dobijeno rotacijom aeroprofila oko svoje uzdužne ose simetrije. Problem je rešavan konačnim razlikama, dok je stabilizacija proračun izvršen primenom veštačke kompresibilnosti. Fizička oblast strujanja preslikana je u proračunsku oblast tako da se ispupčenje koje odgovara konturi obstrujavanog tela preslikalo na pravu liniju u proračunskoj oblasti. Postupak rešavanja pripada klasi približnih faktorizacija operatora kojim se opisuje ovakvo strujanje, poznatim pod imenom AF2 postupak. Dati su dijagrami raspodele koeficijenta pritiska po konturi tela, kao i dijagram konvergencije za dati primer.

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