TEORIJSKA I PRIMENJENA MEHANIKA 15, pp 83–92, 1989, UDK 533.951

ON GENERALIZED EQUATIONS OF THE IONISED GAS BOUNDARY LAYER

Branko R. Obrović

(Received 15.03.1989)

1. Introduction, equations of the ionised gas boundary layer. Investigated in the paper is the flow of ionised gas, the corresponding equations of the boundary layer of this flow being transformed into the so-called universal, i.e. generalized form.

At high temperatures that are characteristical for the boundary layer in e.g. supersonic flow around a body (missile, plane), it is well known that the gas (air) first dissociates, which is followed by ionisation. The "homogeneous" gas becomes a mixture containing positive particles — ions, negative particles — electrons, and neutral particles — atoms, the resulting gas being referred to as the plasma. The degree, or the ratio of ionisation (α_j) of the plasma is determined by the ratio of the electron concentration n_e to the sum of the respective concentrations of ions n_j and atoms n_a. This coefficient represents one of the principal characteristics of the ionised gas.

At sufficiently high ionisation and recombination (three-particle, radiation, etc.) rates, thermodynamic equilibrium of the concentrations of individual components is estabilished in the plasma flow. In the conditions of full thermodynamic (thermochemical) equilibrium, concentrations of individual components are related by the well known [5] equation of Sah, i.e. the ionisation coefficient is a function of temperature T.

Due to the ionisation the gas becomes electro-conductive. If the ionised gas flows through magnetic field, the flow of electric current is set up, which interacti gwith the external magnetic field of intensity $\vec{B}_m$ (analogously to the electro-conductive liquid) creates the so-called Lorentz force. In addition, due to the electric current flowing through the gas, Joule heat is generated. These two effects cause new terms to appear in the boundary layer equations of a electro-conductive gas, i.e. plasma. This is why the equations of a laminary, steady, plane boundary layer have the following form for the flow of ionised gas in the magnetic field in the conditions of ideal ionisation [1]:

$$\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)=0;\quad \rho u\frac{\partial u}{\partial x}+\rho v\frac{\partial u}{\partial y}=-\frac{dp}{dx}+\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right)-\sigma B_m^2 u;$$

$$\rho u\frac{\partial h}{\partial x}+\rho v\frac{\partial h}{\partial y}=u\frac{dp}{dx}+\mu\left(\frac{\partial u}{\partial y}\right)^2+\frac{\partial}{\partial y}\left(\frac{\mu}{\Pr}\frac{\partial h}{\partial y}\right)+\sigma B_m^2u^2; \tag{1}$$

with the corresponding boundary conditions:

$$u=v=0,\quad h=h_w\quad \text{at } y=0;\qquad u\to u_e(x),\quad h\to h_e(x)\quad \text{at } y\to\infty;$$
$$u=u_0(y),\quad h=h_0(v)\quad \text{at } x=x_0; \tag{2}$$

assuming that there is no external electric field, that the external magnetic field with induction $\vec{B}_m$ is perpendicular to the submerged body contour, such that $B_{mx}=0$, and $B_{my}=B_m$, that because of the relatively small thickness of the boundary layer one may consider that $B_m=B_m(x)$, and that there is no internal magnetic field.

The equation system (1) differs from, e.g. the equation system [4] for the ideally-dissociated gas, because it contains the terms $\sigma B_m^2 u$ and $\sigma B_m^2 u^2$ that characterize the effects of magnetic field, i.e. the Lorentz force in the dynamic equation, and the Joule heat in the energy equation.

The notation customary in the boundary layer theory has been used in the system of equations (1) and (2), x and y representing the longitudinal and lateral coordinates, respectively, $u(x,y)$ — the longitudinal projection of velocity in the boundary layer, $v(x,y)$ — the lateral projection, h — enthalpy, μ — dynamic viscosity, ρ — density, Pr — Prandtl number, p — pressure, and σ — electro-conductivity of the plasma. The indices refer to as follows: w — submerged body wall, 0 — distribution of the physical quantities over a boundary layer cross-section defined by $x=x_0$.

Electro-conductivity of the plasma σ is in general a variable quantity depending upon temperature, i.e. upon enthalpy. However, in this paper it is assumed that analogously to the magnetic field intensity B_m, $\sigma=\sigma(x)$ (or $\sigma=$ const, the solutions being valid for small temperature shanges in the ionised gas boundary layer).

Since at the external boundary of the boundary layer $u(x,y)\to u_e(x)$ and $(\partial u/\partial y)_e\to 0$, the dynamic equation at this boundary reduces to:

$$\rho_e u_e\frac{du_e}{dx}=-\frac{dp}{dx}-\sigma B_m^2 u_e.$$

In on the basis of this equation pressure is excluded from the system of equations (1), then the system of equations for the ionised gas boundary layer may be written in the form:

$$\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)=0;\quad \rho u\frac{\partial u}{\partial x}+\rho v\frac{\partial u}{\partial y}=\rho_e u_e\frac{du_e}{dx}+\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right)+\sigma B_m^2(u_e-u);$$
$$\rho u\frac{\partial h}{\partial x}+\rho v\frac{\partial h}{\partial y}=-u\rho_e u_e\frac{du_e}{dx}+\mu\left(\frac{\partial u}{\partial y}\right)^2+\frac{\partial}{\partial y}\left(\frac{\mu}{\Pr}\frac{\partial h}{\partial y}\right)+\sigma B_m^2(u_e^2-uu_e); \tag{1'}$$

with unchanged boundary conditions (2).

2. Variable Transformation, the Impulse Equation. When considering the problem of ionised gas flow, in analogy with the dissociated gas flow, instead of physical coordinates x, y one introduces new transformations in the form of variables:

$$s(x) = \frac{1}{\rho_n \mu_n} \int_0^x \rho_w \mu_w \, dx; \qquad z(x,y) = \frac{1}{\rho_n} \int_0^y \rho \, dy, \tag{3}$$

where ρ_n and μ_n represent known values of density and the coefficient of dynamic viscosity. Clearly, the values ρ_0 and μ_0 may be taken as ρ_n and μ_n.

Introducing the stream function $\psi(x,y)$ by way of relations:

$$u = \frac{\partial \psi}{\partial z}; \qquad \widetilde{V} = \frac{\rho_n \mu_n}{\rho_w \mu_w}\left(u \frac{\partial z}{\partial x} + v \frac{\rho}{\rho_n}\right) = -\frac{\partial \psi}{\partial s}, \tag{4}$$

whereby the continuity equation is identically satisfied, the initial system of equations (1') is transformed into the form:

$$\frac{\partial \psi}{\partial z}\frac{\partial^2 \psi}{\partial s \partial z} - \frac{\partial \psi}{\partial s}\frac{\partial^2 \psi}{\partial z^2} = \frac{\rho_e}{\rho} u_e \frac{du_e}{ds} + \nu_n \frac{\partial}{\partial z}\left(\widetilde{N}\frac{\partial^2 \psi}{\partial z^2}\right) + \frac{\sigma B_m^2}{\rho}\frac{\rho_n \mu_n}{\rho_w \mu_w}\left(u_e - \frac{\partial \psi}{\partial z}\right);$$

$$\frac{\partial \psi}{\partial z}\frac{\partial h}{\partial s} - \frac{\partial \psi}{\partial s}\frac{\partial h}{\partial z} = -\frac{\rho_e}{\rho} u_e \frac{du_e}{ds}\frac{\partial \psi}{\partial z} + \nu_n \widetilde{N}\left(\frac{\partial^2 \psi}{\partial z^2}\right)^2 + \nu_n \frac{\partial}{\partial z}\left(\frac{\widetilde{N}}{\Pr}\frac{\partial h}{\partial z}\right) \tag{5}$$

$$+ \frac{\rho_n \mu_n}{\rho_w \mu_w}\frac{\sigma B_m^2}{\rho}(u^2 - u u_e);$$

with the corresponding boundary conditions:

$$\psi = \frac{\partial \psi}{\partial z} = 0, \quad h = h_w \quad \text{at } z = 0; \qquad \frac{\partial \psi}{\partial z} \to u_e(x), \quad h \to h_e(x) \quad \text{at } z \to \infty;$$

$$\frac{\partial \psi}{\partial z} = u_0(z), \quad h = h_0(z) \quad \text{at } s = s_0. \tag{6}$$

The nondimensional function $\widetilde{N}$ and Prandtl number Pr are determined by the equations:

$$\widetilde{N} = \frac{\rho \mu}{\rho_w \mu_w}; \quad \widetilde{N} = 1 \quad \text{at } z = 0;$$

$$\widetilde{N} \to \frac{\rho_e \mu_e}{\rho_w \mu_w} = \widetilde{N}(s) \quad \text{at } z \to \infty; \qquad \Pr = \frac{\mu c_p}{\lambda}. \tag{7}$$

By means of the variables (3), and assuming that $\sigma = \sigma(x)$, applying the standard procedure from the incompressible gas flow, one can derive the corresponding impulse equation, writing it down in one of the following three forms:

$$\frac{dZ^{**}}{ds} = \frac{F_m}{u_e}; \quad \frac{df}{ds} = \frac{u_e'}{u_e}F_m + \frac{u_e''}{u_e'}f; \quad \frac{\Delta^{**\prime}}{\Delta^{**}} = \frac{F_m}{2f}\frac{u_e'}{u_e}. \tag{8}$$

In the above equations, and in the remaining text, the prime (') symbol denotes derivation with respect to the variable s.

In deriving the impulse equation, we have introduced the parameter of the form f, and the following customary characteristic quantities and functions of the boundary layer:

$$Z^{**}(s)=\frac{\Delta^{**2}}{\nu_n};\quad f(s)=u_e'Z^{**};\quad F_m=F-2H_1g;\quad F=2[\zeta-(2+H)f]$$

$$\zeta=\left[\frac{\partial(u/u_e)}{\partial(z/\Delta^{**})}\right]_{z=0};\quad H=\frac{\Delta^*}{\Delta^{**}};\quad H_1=\frac{\Delta_1^*}{\Delta^{**}}. \tag{8'}$$

In the above relations, the conditional impulse displacement thickness $\Delta^*(s)$, the impulse loss thickness $\Delta^{**}(s)$, and the conditional thickness $\Delta_1^*(s)$ are defined by the expressions:

$$\Delta^*(s)=\int_0^\infty(\rho_e/\rho-u/u_e)\,dz;\quad \Delta^{**}(s)=\int_0^\infty(u/u_e)(1-u/u_e)\,dz;$$

$$\Delta_1^*(s)=\int_0^\infty(\rho_e/\rho)(1-u/u_e)\,dz,$$

whereas the magnetic parameter $g(s)$ is determined by the expression:

$$g(s)=NZ^{**};\qquad N=\frac{\rho_n\mu_n}{\rho_w\mu_w}\overline{N},\qquad \overline{N}=\frac{\sigma B_m^2}{\rho_e}. \tag{10}$$

It should be stressed that the first equation of the system (5), in the case of constant temperature (i.e. density and viscosity), and constant electro-conductivity σ, agrees in form with the boundary layer equation for an electro-conductive liquid [2]. Since, however, the variables (3), introduced under these conditions, reduce to $s(x)=x$ and $z(x,y)=y$, the first equation of the system fully reduces to the boundary layer equation for an electro-conductive liquid. Therefore, generally speaking, the transformations (3) and (4) convert the boundary layer equations for compressible fluids into the form indentical with the form of the equation for the corresponding boundary layer of incompressible fluid.

Pursuing further the idea developed in [3], one more variable transformation is applied to the consideration of the problem of plasma flow in the boundary layer:

$$s=s;\quad \eta(s,z)=u_e^{b/2}\left(a\nu_n\int_0^s u_e^{b-1}\,ds\right)^{-1/2}z; \tag{11}$$

$$\psi(s,z)=u_e^{1-b/2}\left(a\nu_n\int_0^s u_e^{b-1}\,ds\right)^{1/2}\varphi(s,\eta);\quad h(s,z)=h_1\cdot\bar{h}(s,\eta);\quad h_1=\text{const.},$$

where $\eta(s,z)$ is the new non-dimensional lateral coordinate, $\varphi(s,\eta)$ is non-dimensional stream function, $\bar{h}$ is non-dimensional enthalpy, h_1 is total enthalpy (stagnation enthalpy) in the outer flow around the submerged body, and a and b are arbitrary constants.

The above introduced boundary layer thicknesses (9) make it possible to write the new transformation in a more convenient form:

$$\eta(s,z)=\frac{B(s)}{\Delta^{**}(s)}z;\quad \psi(s,z)=\frac{u_e(s)\Delta^{**}(s)}{B(s)}\varphi(s,\eta), \tag{11'}$$

while the individual characteristic quantities of the boundary layer may be written as:

$$H=\frac{\Delta^*}{\Delta^{**}}=\frac{A}{B};\quad H_1=\frac{\Delta_1^*}{\Delta^{**}}=\frac{A_1}{B};\quad \zeta=\left[\frac{\partial(\partial\varphi/\partial\eta)}{\partial(\eta/B)}\right]_{\eta=0}=B\left(\frac{\partial^2\varphi}{\partial\eta^2}\right)_{\eta=0}. \tag{12}$$

The quantities A, A_1 and B in the expressions (12) replace the integrals:

$$A=\int_0^\infty\left(\frac{\rho_e}{\rho}-\frac{\partial\varphi}{\partial\eta}\right)d\eta;\quad A_1=\int_0^\infty\frac{\rho_e}{\rho}\left(1-\frac{\partial\varphi}{\partial\eta}\right)d\eta;$$
$$B=\int_0^\infty\frac{\partial\varphi}{\partial\eta}\left(1-\frac{\partial\varphi}{\partial\eta}\right)d\eta \tag{13}$$

and are assumed to be continuous functions of the coordinate s.

In addition, by means of the transformations (11) one easily arrives at the relation:

$$\frac{f}{B^2}=\frac{au_e'}{u_e^b}\int_0^s u_e^{b-1}\,ds, \tag{14}$$

which plays the decisive role in the application of the so-called universal solutions to the solution of each concrete problem of the plasma boundary layer. The characteristic function F_m of the boundary layer reads now:

$$F_m=aB^2-bf+\frac{2u_eB'f}{u_e'B}. \tag{15}$$

Using the transformations (11), the system of equations (5) is, after a somewhat involved algebra, reduced to the form:

$$\frac{\partial}{\partial\eta}\left(\tilde{N}\frac{\partial^2\varphi}{\partial\eta^2}\right)+\frac{aB^2+(2-b)f}{2B^2}\varphi\frac{\partial^2\varphi}{\partial\eta^2}+\frac{f}{B^2}\left[\frac{\rho_e}{\rho}-\left(\frac{\partial\varphi}{\partial\eta}\right)^2\right]+\frac{g}{B^2}\frac{\rho_e}{\rho}\left(1-\frac{\partial\varphi}{\partial\eta}\right)$$
$$=\frac{u_eZ^{**}}{B^2}\left(\frac{\partial\varphi}{\partial\eta}\frac{\partial^2\varphi}{\partial s\partial\eta}-\frac{\partial\varphi}{\partial s}\frac{\partial^2\varphi}{\partial\eta^2}\right); \tag{16}$$
$$\frac{\partial}{\partial\eta}\left(\frac{\tilde{N}}{\mathrm{Pr}}\frac{\partial\bar{h}}{\partial\eta}\right)+\frac{aB^2+(2-b)f}{2B^2}\varphi\frac{\partial\bar{h}}{\partial\eta}-\frac{2\kappa f}{B^2}\frac{\rho_e}{\rho}\frac{\partial\varphi}{\partial\eta}+2\kappa\tilde{N}\left(\frac{\partial^2\varphi}{\partial\eta^2}\right)^2$$
$$+\frac{2\kappa g}{B^2}\frac{\rho_e}{\rho}\frac{\partial\varphi}{\partial\eta}\left(\frac{\partial\varphi}{\partial\eta}-1\right)=\frac{u_eZ^{**}}{B^2}\left(\frac{\partial\varphi}{\partial\eta}\frac{\partial\bar{h}}{\partial s}-\frac{\partial\varphi}{\partial s}\frac{\partial\bar{h}}{\partial\eta}\right);$$

with boundary conditions:

$$\varphi=\frac{\partial\varphi}{\partial\eta}=0,\quad \bar{h}=\bar{h}_w\quad \text{at } \eta=0;\quad \frac{\partial\varphi}{\partial\eta}\to 1,\quad \bar{h}\to\bar{h}_e=1-\kappa\quad \text{at } \eta\to\infty;$$
$$\varphi=\varphi_0(\eta),\quad \bar{h}=\bar{h}_0(\eta)\quad \text{at } s=s_0. \tag{17}$$

In the energy equation of the system (16), and in the boundary conditions (17), one can notice the quantity:

$$\kappa=f_0=u_e^2/2h_1, \tag{18}$$

which, indentically to the case of dissociated gas, may be termed local parameter of gas compressibility. This parameter is a function of the coordinate s, specified in advance.

In addition, the system of equations (16) contains the external velocity $u_e(s)$, and the solution shall therefore depend upon the concrete distribution of this velocity, and, certainly, also upon a number of parameters and distributions, such as Pr, $\widetilde{N}$, ρ_e/ρ, that should be additionally determined. The system of equations may however be made independent of the external velocity $u_e(s)$, that is it may be brought into a universal form, which is the aim of this study.

3. Universalisation of the Equations of the Problem Under Consideration. Further analysis of the system of equations (16) reveals that the variable s enters explicitly only the right-hand side of these equations, and the last boundary condition. The quantities κ, f, and g in these equations, determined by the equalities (18), (8), and (10), may in analogy with incompressible fluid [2] be termed "similarity parameters", and included into the "similarity variables". Namely, if the parameters κ, f, and g are considered as independent variables, then in order to "universalise" the equations (5) of the plasma boundary layer, one should apply the similarity transformations in the form:

$$\eta(s,z) = \frac{B}{\Delta^{**}}z; \quad \psi(s,z) = \frac{u_e\Delta^{**}}{B}\varphi(\eta,\kappa,f,g); \quad \bar{h}(s,z) = h_1\bar{h}(\eta,\kappa,f,g), \tag{19}$$

whereby the variable s does not enter the functions φ and $\bar{h}$ explicitly, but by way of parameters κ, f, and g, which can be seen by comparing with the equation (11).

Having calculated (by means of transformation (19)) the expressions for individual derivatives that enter the left and right hand sides of the basic system of equations (5), we obtain:

$$\begin{aligned}
&\frac{\partial}{\partial\eta}\left(\widetilde{N}\frac{\partial^2\varphi}{\partial\eta^2}\right) + \frac{aB^2+(2-b)f}{2B^2}\varphi\frac{\partial^2\varphi}{\partial\eta^2}\\
&+\frac{f}{B^2}\left[\frac{\rho_e}{\rho} - \left(\frac{\partial\varphi}{\partial\eta}\right)^2\right] + \frac{g}{B^2}\frac{\rho_e}{\rho}\left(1-\frac{\partial\varphi}{\partial\eta}\right)\\
&= \frac{u_eZ^{**}}{B^2}\left[\left(\frac{\partial\varphi}{\partial\eta}\frac{\partial^2\varphi}{\partial\eta\partial\kappa} - \frac{\partial\varphi}{\partial\kappa}\frac{\partial^2\varphi}{\partial\eta^2}\right)\kappa' + \left(\frac{\partial\varphi}{\partial\eta}\frac{\partial^2\varphi}{\partial\eta\partial f} - \frac{\partial\varphi}{\partial f}\frac{\partial^2\varphi}{\partial\eta^2}\right)f'\right.\\
&\left.+\left(\frac{\partial\varphi}{\partial\eta}\frac{\partial^2\varphi}{\partial\eta\partial g} - \frac{\partial\varphi}{\partial g}\frac{\partial^2\varphi}{\partial\eta^2}\right)g'\right];\\
&\frac{\partial}{\partial\eta}\left(\frac{\widetilde{N}}{\Pr}\frac{\partial\bar{h}}{\partial\eta}\right) + \frac{aB^2+(2-b)f}{2B^2}\varphi\frac{\partial\bar{h}}{\partial\eta} - \frac{2\kappa f}{B^2}\frac{\rho_e}{\rho}\frac{\partial\varphi}{\partial\eta}\\
&+2\kappa\widetilde{N}\left(\frac{\partial^2\varphi}{\delta\eta^2}\right)^2 + \frac{2\kappa g}{B^2}\frac{\rho_e}{\rho}\frac{\partial\varphi}{\partial\eta}\left(\frac{\partial\varphi}{\partial\eta}-1\right)\\
&= \frac{u_eZ^{**}}{B^2}\left[\left(\frac{\partial\varphi}{\partial\eta}\frac{\partial\bar{h}}{\partial\kappa} - \frac{\partial\varphi}{\partial\kappa}\frac{\partial\bar{h}}{\partial\eta}\right)\kappa' + \left(\frac{\partial\varphi}{\partial\eta}\frac{\partial\bar{h}}{\partial f} - \frac{\partial\varphi}{\partial f}\frac{\partial\bar{h}}{\partial\eta}\right)f'\right.\\
&\left.+\left(\frac{\partial\varphi}{\partial\eta}\frac{\partial\bar{h}}{\partial g} - \frac{\partial\varphi}{\partial g}\frac{\partial\bar{h}}{\partial\eta}\right)g'\right].
\end{aligned} \tag{20}$$

The equations (20) do not satisfy the conditions of "generalized similarity", since the terms $u_eZ^{**}f'$ and $u_eZ^{**}g'$ on the respective right hand sides depend upon s, and can not be expressed in terms of the parameters f and g. While, namely, the term $u_e Z^{**}\kappa'$ may be written as:

$$u_e Z^{**}\kappa' = 2\kappa f = \theta_0, \tag{21}$$

the other two terms $u_e Z^{**} f'$ and $u_e Z^{**} g'$, due to the relations $f = u_e' Z^{**}$; $u_e Z^{**\prime} = F_m$; $g = NZ^{**}$, reduce to:

$$u_e Z^{**} f' = fF_m + u_e u_e'' Z^{**2}; \qquad u_e Z^{**} g' = gF_m + u_e N' Z^{**2}, \tag{22}$$

and can not be expressed only in terms of f, g, and F_m, as they contain in other terms (22) the quantity $u_e(s)$. If index "1" is added to the parameters f and g, the other terms in (22) may be interpreted as the following parameters:

$$f_2 = u_e u_e'' Z^{**2}; \qquad g_2 = u_e N' Z^{**2} \tag{23}$$

the equations (22) being then:

$$u_e Z^{**} f_1' = f_1 F_m + f_2; \qquad u_e Z^{**} g_1' = g_1 F_m + g_2. \tag{24}$$

The introduction of new variables f_2, and g_2 into the function φ and the nondimensional enthalpy $\bar{h}$ would give rise to the appearance of new products (terms) in the equations (20), the later containing in accordance with the equations analogous to (22) the following new variables f_3 and g_3:

$$f_3 = u_e^2 u_e''' Z^{**3}; \qquad g_3 = u_e^2 N'' Z^{**3}, \tag{25}$$

which are related by equations:

$$u_e Z^{**} f_2' = (f_1 + 2F_m) f_2 + f_3; \qquad u_e Z^{**} g_2' = (f_1 + 2F_m) g_2 + g_3. \tag{26}$$

Further repetition of the above procedure leads to the conclusion that the variables introduced (the similarity parameters) satisfy general laws in accordance with which they were created:

$$f_k = u_e^{k-1} \frac{d^k u_e}{ds^k} Z^{**k}; \qquad g_k = u_e^{k-1} \frac{d^{k-1} N}{ds^{k-1}} Z^{**k}; \quad (k = 1, 2, \ldots), \tag{27}$$

whereby on the basis of (24) and (26) one may conclude that these parameters also satisfy the following recurrent ordinary differential equations:

$$u_e Z^{**} f_k' = [(k-1) f_1 + kF_m] f_k + f_{k+1} \equiv \theta_k;$$
$$u_e Z^{**} g_k' = [(k-1) f_1 + kF_m] g_k + g_{k+1} \equiv G_k; \quad (k = 1, 2, \ldots). \tag{29}$$

The introduction of two infinite sets of parameters (27) and the compressibility parameter (18) right from the beginning into the transformations (19), makes it possible to obtain the equations of the plasma boundary layer in terms of the so-called general similarity variables. If the following transformations are applied to the system of equations (5):

$$\eta = \frac{B}{\Delta^{**}} z; \quad \psi = \frac{u_e \Delta^{**}}{B} \varphi[\eta; \kappa; (f_k), (g_k)];$$

$$\bar{h} = h_1 \cdot \bar{h}[\eta; \kappa; (f_k), (g_k)]; \qquad (k = 1, 2, \ldots), \tag{29}$$

the later is transformed into the form:

$$\begin{aligned}
&\frac{\partial}{\partial \eta}\left(\widetilde{N}\frac{\partial^2 \varphi}{\partial \eta^2}\right) + \frac{aB^2 + (2-b)f_1}{2B^2}\varphi\frac{\partial^2 \varphi}{\partial \eta^2} \\
&+ \frac{f_1}{B^2}\left[\frac{\rho_e}{\rho} - \left(\frac{\partial \varphi}{\partial \eta}\right)^2\right] + \frac{g_1}{B^2}\frac{\rho_e}{\rho}\left(1 - \frac{\partial \varphi}{\partial \eta}\right) \\
&= \frac{1}{B^2}\left[\sum_{k=0}^{\infty}\theta_k\left(\frac{\partial \varphi}{\partial \eta}\frac{\partial^2 \varphi}{\partial \eta \partial f_k} - \frac{\partial \varphi}{\partial f_k}\frac{\partial^2 \varphi}{\partial \eta^2}\right) + \sum_{k=1}^{\infty}G_k\left(\frac{\partial \varphi}{\partial \eta}\frac{\partial^2 \varphi}{\partial \eta \partial g_k} - \frac{\partial \varphi}{\partial g_k}\frac{\partial^2 \varphi}{\delta \eta^2}\right)\right]; \\
&\frac{\partial}{\partial \eta}\left(\frac{\widetilde{N}}{\Pr}\frac{\partial \bar{h}}{\partial \eta}\right) + \frac{aB^2 + (2-b)f_1}{2B^2}\varphi\frac{\partial \bar{h}}{\partial \eta} - \frac{2\kappa f_1}{B^2}\frac{\rho_e}{\rho}\frac{\partial \varphi}{\partial \eta} \\
&+ 2\kappa\widetilde{N}\left(\frac{\partial^2 \varphi}{\partial \eta^2}\right)^2 + \frac{2\kappa g_1}{B^2}\frac{\rho_e}{\rho}\frac{\partial \varphi}{\partial \eta}\left(\frac{\partial \varphi}{\partial \eta} - 1\right) \\
&= \frac{1}{B^2}\left[\sum_{k=0}^{\infty}\theta_k\left(\frac{\partial \varphi}{\partial \eta}\frac{\partial \bar{h}}{\partial f_k} - \frac{\partial \varphi}{\partial f_k}\frac{\partial \bar{h}}{\partial \eta}\right) + \sum_{k=1}^{\infty}G_k\left(\frac{\partial \varphi}{\partial \eta}\frac{\partial \bar{h}}{\partial g_k} - \frac{\partial \varphi}{\partial g_k}\frac{\partial \bar{h}}{\partial \eta}\right)\right];
\end{aligned} \tag{30}$$

with corresponding boundary conditions:

$$\begin{gathered}
\varphi = \frac{\partial \varphi}{\partial \eta} = 0; \quad \bar{h} = \bar{h}_w = \text{const.} \quad \text{at } \eta = 0; \\
\frac{\partial \varphi}{\partial \eta} \to 1; \quad \bar{h} \to \bar{h}_e = 1 - \kappa \quad \text{at } \eta \to \infty; \\
\varphi = \varphi_0(\eta); \quad \bar{h} = \bar{h}_0(\eta) \quad \text{at } f_0 = \kappa_0 = \text{const}, \ f_1 = f_2 = \ldots = g_1 = g_2 = \ldots = 0.
\end{gathered} \tag{31}$$

Obviously, by introducing the appropriate parameter sets, one obtains the equations (30), which may be understood as a *universal mathematical model of the problem of ionised gas flow in the boundary layer*. Namely, the dynamic system (30) has the same form for every special case of the flow, since neither in the system of equations, nor in the corresponding boundary conditions one finds the velocity distribution u_e at the external border of the bondary layer. System of equations (30) is in this sense universal. Of course, it is expected that the quantitiy $\widetilde{N}$ and the density ratio ρ_e/ρ are expressed in terms of the non-dimensional enthalpy, the way it is done in similar flow problems. The characteristic boundary layer function F_m is then expressed by:

$$F_m = aB^2 - bf_1 + \frac{2}{B}\left(\sum_{k=0}^{\infty}\theta_k\frac{\partial B}{\partial f_k} + \sum_{k=1}^{\infty}G_k\frac{\partial B}{\partial g_k}\right). \tag{32}$$

Concluding this considerations, it should be pointed out that in the case of constant temperature, i.e. constant density and viscosity, and when B = const., the universal dynamic equation of the system (30) reduces to the boundary layer equation for an electro-conductive liquid [6], with non-porous submerged body contour.

Since two sets of parameters appear on the right-hand sides of the system (30), numerical solution of this system of universal equations is possible only when the number of parameters is relatively low. Assuming that all the parameters, starting with the second one, are zero, and neglecting the derivatives with respect to the compressibility parameter $\kappa = f_0$ and the magnetic parameter g_1, that is if:

$$\begin{gathered}\kappa \neq 0; \quad f_1 \neq 0; \quad f_2 = f_3 = \ldots = 0; \quad \partial/\partial\kappa = 0, \\ g_1 \neq 0; \quad g_2 = g_3 = \ldots = 0; \quad \partial/\partial g_1 = 0,\end{gathered} \tag{33}$$

then under these conditions the system (30) is simplified, to read in the three-parameter, twice-localised approximation, as follows:

$$\begin{gathered}\frac{\partial}{\partial\eta}\left(\widetilde{N}\frac{\partial^2\varphi}{\partial\eta^2}\right) + \frac{aB^2+(2-b)f_1}{2B^2}\varphi\frac{\partial^2\varphi}{\partial\eta^2} + \frac{f_1}{B^2}\left[\frac{\rho_e}{\rho} - \left(\frac{\partial\varphi}{\partial\eta}\right)^2\right] + \frac{g_1}{B^2}\frac{\rho_e}{\rho}\left(1-\frac{\partial\varphi}{\partial\eta}\right) \\ = \frac{F_m f_1}{B^2}\left(\frac{\partial\varphi}{\partial\eta}\frac{\partial^2\varphi}{\partial\eta\partial f_1} - \frac{\partial\varphi}{\partial f_1}\frac{\partial^2\varphi}{\partial\eta^2}\right); \\ \frac{\partial}{\partial\eta}\left(\frac{\widetilde{N}}{\Pr}\frac{\partial\bar{h}}{\partial\eta}\right) + \frac{aB^2+(2-b)f_1}{2B^2}\varphi\frac{\partial\bar{h}}{\partial\eta} - \frac{2\kappa f_1}{B^2}\frac{\rho_e}{\rho}\frac{\partial\varphi}{\partial\eta} + 2\kappa\widetilde{N}\left(\frac{\partial^2\varphi}{\partial\eta^2}\right)^2 \\ + \frac{2\kappa g_1}{B^2}\frac{\rho_e}{\rho}\frac{\partial\varphi}{\partial\eta}\left(\frac{\partial\varphi}{\partial\eta}-1\right) = \frac{F_m f_1}{B^2}\left(\frac{\partial\varphi}{\partial\eta}\frac{\partial\bar{h}}{\partial f_1} - \frac{\partial\varphi}{\partial f_1}\frac{\partial\bar{h}}{\partial\eta}\right);\end{gathered} \tag{34}$$

$$\begin{gathered}\varphi = \frac{\partial\varphi}{\partial\eta} = 0; \quad \bar{h} = \bar{h}_w = \text{const.} \quad \text{at } \eta = 0, \quad \frac{\partial\varphi}{\partial\eta} \to 1; \quad \bar{h} \to 1-\kappa \quad \text{at } \eta \to \infty, \\ \varphi = \varphi_0(\eta); \quad \bar{h} = \bar{h}_0(\eta) \quad \text{at } f_0 = \kappa_0 = \text{const.} \quad f_1 = g_1 = 0.\end{gathered}$$

The expression for the characteristic function F_m is also simplified:

$$F_m = \frac{aB^2 - bf_1}{1 - \dfrac{2}{B}f_1\dfrac{dB}{df_1}}. \tag{35}$$

Determination of analytical relations for particular physical distributions $(\widetilde{N}, \rho_e/\rho, \Pr)$, and numerical colution of the three-parameter, twice-localised system of boundary layer equations for ionised gas will be the subject of our further investigations.

REFERENCES

[1] Лойцянский Л. Г.: *Ламинарный пограничный слой*, Физматгиз, Москва 1962.

[2] Лойцянский Л. Г.: *Механика жидкости и газа*, Наука, Москва 1973 и 1978.

[3] Saljnikov, V. N.: *A contribution to universal solutions of the boundary layer theory*, Teorijska i primenjena mehanika, br. 4, Beograd, 1978.

[4] Obrović, B.: *Strujanje disociranog gasa — problem tzv. zamrznutog graničnog sloja*, Beograd, 1979.

[5] Лукьянов Г. А.: *Сверхзвуковые струи плазмы*, Машиностроение, Ленинград 1985.

[6] Saljnikov, V. N., Ivanović, D.: *MHD granični sloj na aeroprofilima sa poroznom konturom*, Zbornik radova, Mašinski fakultet, Titograd, 1988.

ÜBER DIE VERALLGEMEINERTEN GRENZSCHICHTGLEICHUNGEN EINES IONISATIONSGASES

In diesem Beitrag wird die Strömung eines elektrischen Leitfähigkeitsgases untersucht. Die entsprechenden Grenzschichtgleichungen führt man an die sogenannte universelle Form zu. Dabei wird die neuen Veränderlichen, die Impulsgleichung und zwei Ähnlichkeitsparametermengen eingeführt.

O UOPŠTENIM JEDNAČINAMA GRANIČNOG SLOJA JONIZOVANOG GASA

U ovom radu se istražuje strujanje elektroprovodnog gasa. Odgovarajuće jednačine graničnog sloja dovode se na tzv. univerzalni oblik. Pri tome se uvode nove promenljive, jednačina impulsa i dva skupa parametara sličnosti.

Branko Obrović
Mašinski fakultet
Sestre Janjić 6, 34000 Kragujevac