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DARCY'S LAW AND EQUATION OF CONSOLIDATION

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The analysis of the mechanical behaviour of a deformable porous solid saturated and partly saturated with fluid is of importance in many engineering applications (soil mechanics, petroleum engineering, ...).

In this paper we consider a saturated and partly saturated porous medium as a two-phase and a three-phase mixture, respective. The equations of balance are formulated for each phase separately. Then we have derived, in accordance with the dissipation inequality for the mixture, nonlinear and linear constitutive relations. In the second part of the paper the Darcy's law and the equation of consolidation for this media are derived.

1. Introduction. The consolidation theory is also called the theory of flow through porous deformabile media. It's beginnings can be traced to 1923. when Terzaghi formulated the problem of settlement of soli columm under the action of a load applied to its surface [7].

In 1941. Biot generalized Terzaghi's problem, but in 1956. he formulated a new the theory of flow of a liquid through porous elastic deformable media. He stated the equations on the basic of general thermodynamical considerations.

We have extended the theory to viscoelastic media.

Further, much of the literature on porous media is concerned with Darcy's law which gives an empirical linear relation between the relative velocity V and the fluid pressure p.

Anyhow it is well-known that the Darcy's law is valid only when fluid is laminar [1].

The questions are:

(1) what is theoretical interpretation of this law?

(2) how to generalize this relation?

The answer to the first question can be found in monography by I. Müler [2]: ... "This law is a simplified version of the equation of balance of momenta" (for the fluid).

In our case we have: $\rho^f \acute{v}_i^f = \rho^f f_i^f + r_i^f + t_{ij,j}^f$.

This is also answer to the second question.

Anyhow, the above relation is general and therefore not suitable for the application.

Now we have to derive constitutive relations for constitutive functions r_i^f and t_{ij}^f.

2. Definitions and notations. We have introduced the following definitions

$$V^\alpha = \int_V n^\alpha \, dV, \qquad S^\alpha = \int_S \lambda^\alpha \, dS, \qquad m^\alpha = \int_V \rho_*^\alpha \, dV^\alpha$$

for the volume, the surface and the mass distributions, respectively, where: V is the volume of the body, S — the surface bounding V, m^α — the mass of V^α, and n^α, λ^α, ρ_*^α — denote the respective "densities".

The index α is referred to water (fluid), gas or solid, i.e. if the quantity is referred to water, gas or solid we will denote it by index "w", "g" and "s", respectively.

The quantity ρ_*^α represents the effective mass density of the constituent, with respect to its own volume V^α.

We also define partial densities: $\rho^\alpha = \rho_*^\alpha n^\alpha = dm^\alpha/dV$.

The volume porosity is defined by: $n = dV^p/dV$, where V^p is volume of void (pore).

In this paper we have assumed that the surface (dS^p/dS) and volume porosity are the same. This is, of course, not true in general, but reasonable for most application.

The ratio between the volume of voids filled with water and volume of total pore space is called the degree of saturation and is defined by $S_r = dV^w/dV$.

For any quantity $f^\alpha(x,t)$ we define

$$\frac{D^\alpha f^\alpha}{Dt} = \frac{\partial f^\alpha}{\partial t} + f_{,i}^\alpha v_i^\alpha \equiv \acute{f}^\alpha.$$

which are the material derivatives following the α-constituent.

3. Balance laws. At a regular point we postulate the balance laws, in local forms [3]:

— balance of mass for each constituent:

$$\acute{\rho}^\alpha + \rho^\alpha v_{i,i}^\alpha = 0, \tag{3.1}$$

— balance of momentum for each constituent:

$$\rho^\alpha \acute{v}_i^\alpha = \rho^\alpha f_i^\alpha + r_i^\alpha + t_{ij,j}^\alpha \tag{3.2}$$

— balance of moment of momentum:

$$\varepsilon_{ijk}(t_{kj}^\alpha + m_{jkl,l}^\alpha) = 0 \tag{3.3}$$

— balance of energy for each constituent:

$$\rho^\alpha \acute{\psi}^\alpha + \rho^\alpha T \acute{\eta}^\alpha + \rho^\alpha \eta^\alpha \acute{T} = t_{ij}^\alpha v_{j,i}^\alpha + q_{i,i}^\alpha + \rho^\alpha h^\alpha - \rho^\alpha M^\alpha \acute{n}^\alpha - r_i^\alpha v_i^\alpha \tag{3.4}$$

— entropy inequality for mixture as a whole:

$$\sum_{\alpha=1}^{3}\Big[-\rho^\alpha(\acute{\psi}^\alpha+\eta^\alpha\acute{T})+(\Phi_i^\alpha/T+\rho^\alpha\eta^\alpha u_i^\alpha)T_{,i}$$
$$+(q_i^\alpha-\Phi_i^\alpha-T\rho^\alpha\eta^\alpha u_i)_{,i}+M_{kj}^\alpha v_{j,k}^\alpha+m_{ijk}^\alpha v_{j,ki}^\alpha-\rho^\alpha M^\alpha\acute{n}-r_i^\alpha v_i^\alpha\Big]\geq 0,$$

where:
v_i^α denotes velocity of the constituent, f_i^α — body force density, r_i^α — interaction force density between the constituents, t_{ij}^α — partial stress tensor, q_i^α — heat vector, h^α — heat source density, η^α — entropy density, T — absolute temperature, equal for each constituent, $\underset{\sim}{M}^\alpha$ — generalized force, ψ^α — free energy, Φ_i^α — generalized flux, u_i^α — diffusion velocity, $V_i^{\alpha\beta}\equiv v_i^\alpha-v_i^\beta$, $\alpha,\beta=1,2,3$, $1\to s$, $2\to w$, $3\to g$.

4. Constitutive relations. We assume that the response of partly saturated porous media is essentially determined by following set of variables:

$$\underset{\sim}{A}=\{T;T_{,i};n;x_{k,K}^\alpha;x_{k,KL}^\alpha;\acute{x}_{k,K}^\alpha;\rho^\gamma;V_i^{\alpha\beta};S_r\},$$
$$\alpha,\beta=1,2,3;\quad \gamma=2,3.$$

In accordance with the principle of equipresence, the constitutive functions $\underset{\sim}{t}^\alpha$, $\underset{\sim}{q}^\alpha$, ψ^α, η^α, $\underset{\sim}{r}^\alpha$, $\underset{\sim}{\Phi}^\alpha$, are assumed to be dependent on the same list of variables.

After introducing the constitutive functions in the entropy inequality and carrying out indicated differentiations we obtain the following explicit forms of the entropy inequality:

$$\sum_{\alpha=1}^{3}\Big[-\rho^\alpha\left(\frac{\partial\psi^\alpha}{\partial T}+\eta^\alpha\right)\acute{T}-\rho^\alpha\left(\frac{\partial\psi^\alpha}{\partial T_{,i}}\acute{T}_{,i}+\frac{\partial\psi^\alpha}{\partial x_{k,K}^\alpha}\acute{x}_{k,K}^\alpha\right.$$
$$\left.+\frac{\partial\psi^\alpha}{\partial x_{k,KL}^\alpha}\acute{x}_{k,KL}^\alpha+\frac{\partial\psi^\alpha}{\partial\acute{x}_{k,K}^\alpha}\ddot{x}_{k,K}^\alpha+\frac{\partial\psi^\alpha}{\partial\rho^\gamma}\acute{\rho}^\gamma+\frac{\partial\psi^\alpha}{\partial V_i^{\alpha\beta}}\acute{V}_i^{\alpha\beta}+\frac{\partial\psi^\alpha}{\partial S_r}\acute{S}_r\right)\tag{4.1}$$
$$+\left(\frac{\Phi_i^\alpha}{T}+\rho^\alpha\eta^\alpha u_i^\alpha\right)T_{,i}+M_{kj}^\alpha V_{j,k}^\alpha+(q_i^\alpha-\Phi_i^\alpha-T\rho^\alpha\eta^\alpha u_i^\alpha)_{,i}$$
$$+m_{ijk}^\alpha V_{j,ki}^\alpha-\rho^\alpha\left(M^\alpha+\frac{\partial\psi^\alpha}{\partial n}\right)\acute{n}-r_i^\alpha V_i^\alpha\Big]\geq 0.$$

From thermodynamic restrictions and introducing dissipation function follow next relations:

$$\psi^\alpha=\psi^\alpha(T;x_{k,K}^\alpha;n);\qquad D=D\left(d_{ij}^\alpha;\frac{T_{,i}}{T};x_{k,KL}^\alpha;V_i^{\gamma s}\right)$$
$$t_{ij}^s=\rho^s\frac{\partial\psi^s}{\partial x_{j,k}^s}x_{i,k}^s+\lambda\frac{\partial D}{\partial d_{ij}^s}\tag{4.2}$$
$$t_{ij}^\gamma=\rho^\gamma\left(\frac{\partial\psi^\gamma}{\partial x_{j,K}^\gamma}x_{i,K}^\gamma+\delta_{ij}\rho^\gamma\frac{\partial\psi^\gamma}{\partial\rho^\gamma}\right)+\lambda\frac{\partial D}{\partial d_{ij}^\gamma};\quad r_i^\gamma=\lambda\frac{\partial D}{\partial V_i^{\gamma s}}\qquad\gamma=2,3.$$

From the objectivity condition we get:

$$\psi^\alpha = \psi^\alpha(E^\alpha_{KL}; T; n)$$

and

$$t^s_{ij} = \rho^s \frac{\partial \psi^s}{\partial E^s_{KL}} x^s_{i,K} x^s_{j,K} + \lambda \frac{\partial D}{\partial d^s_{ij}} \tag{4.3}$$

$$t^\gamma_{ij} = \rho^\gamma \left(\frac{\partial \psi^\gamma}{\partial E^\gamma_{KL}} x^\gamma_{i,K} x^\gamma_{j,L} + \delta_{ij} \rho^\gamma \frac{\partial \psi^\gamma}{\partial \rho^\gamma} \right) + \lambda \frac{\partial D}{\partial d^\gamma_{ij}}.$$

In the linear theory for t^γ_{ij} and r^γ_i we get [3]:

$$\begin{aligned} t^\gamma_{ij} = {} & \rho^\gamma_0 \left[(1 - e^\gamma_{kk}) \psi^\gamma_{ij} + \psi^\gamma_{ijkl} e^\gamma_{kl} + A^\gamma_{ij}(T - T_0) + B^\gamma_{ij}(n - n_0) \right] \\ & + S^\gamma_{ij}(S_r - S_0) + \lambda \Bigg[D^\gamma_{ij} + D^\gamma_{ijkl} d^\gamma_{kl} + \sum_{\alpha=1}^{3} (1 - \delta^{\alpha\gamma}) C^{\alpha\gamma}_{ijkl} d^\alpha_{kl} \\ & + \sum_{\beta=2}^{3} C^{\beta\gamma}_{ijk} V^{\beta 1}_k + \frac{T_{,i}}{T} N^\gamma_{ijk} \Bigg] - p^\gamma \delta_{ij}; \end{aligned} \tag{4.4}$$

$$r^\gamma_i = \lambda \left(C^\gamma_i + D^\gamma_{ij} V^{\gamma 1}_j + C^\gamma_{ij} \frac{T_{,i}}{T} + \sum_{\alpha=1}^{3} C^{\gamma 1}_{kji} d^\alpha_{kj} \right),$$

where:

D is dissipation function, d_{ij} — deformation rate tensor, ρ^γ_0 — reference density, e_{ij} — Eulerian strain tensor, and where $\underset{\sim}{\psi}^\gamma, \underset{\sim}{A}^\gamma, \underset{\sim}{B}^\gamma, \underset{\sim}{C}^\gamma, \underset{\sim}{D}^\gamma$, are constants, and

$$\delta_{ij} \rho^\gamma \frac{\partial \psi^\gamma}{\partial \rho^\gamma} \equiv -\frac{p^\gamma}{\rho^\gamma} \delta_{ij}.$$

5. Darcy's law and equation of consolidation for saturated media. Darcy's law is, like Fick's law of diffusion in homogenous mixtures and Ohm's law of diffusion of electrically charged constituents, simplified version of the equation of balance of momentum (for fluid) [2].

In special case, for saturated media, we have:

$$S_r = 0, \quad \alpha, \beta = 1, 2 \quad (s, w), \quad \gamma = w.$$

Then, when relations (4.4) are introduced in (3.2) it is obtained:

$$\rho^w \acute{v}^w_i = \rho^w f^w_i + \lambda D^w_{ij} V_j - p_{,i} + \bar{t}^*_{ij,j} + r^*_i, \tag{5.1}$$

where:

$$\begin{aligned} t^w_{ij} = {} & \rho^w_0 \left[(1 - e^w_{kk}) \psi^w_{ij} + \psi^w_{ijkl} e^w_{kl} + A^w_{ij}(T - T_0) + B^w_{ij}(n - n_0) \right] \\ & + \lambda \left[D^w_{ij} + D^w_{ijkl} d^w{}_{kl} + \sum_{\alpha=1}^{2} (1 - \delta^{\alpha w}) C^{\alpha w}_{ijkl} d^\alpha_{kl} + C^w_{ijk} V_k + \frac{T_{,i}}{T} N^w_{ijk} \right] - p^w \delta_{ij} \end{aligned}$$

$$\equiv \bar{t}^*_{ij} - p\delta_{ij},$$

$$r^w_i = \lambda\Big(C^w_i + D^w_{ij}V_j + C^w_{ij}\frac{T_{,i}}{T} + \sum_{\alpha=1}^{2} C^{ws}_{kji} d^\alpha_{kj}\Big) \equiv \lambda D_{ij}V_j + r^*_i,$$

$$V_j = v^w_i - v^s_i, \qquad p^w = p.$$

Here p is the water (fluid) pressure. Now we introduce, insted of p, pore pressure P (acting over the area fraction of a surface) defined by

$$P = \frac{p}{n}. \tag{5.2}$$

If ther hydraulic resistivity tensor (D_{ij}) is isotropic than [5]

$$\frac{D_{ij}}{n} = \frac{\mu}{k} n\delta_{ij}, \tag{5.3}$$

where

μ — is the dynamical viscosity of the pore fluid,

k — is the intrinisic permeability of the solid.

Now (5.1) becomes

$$\rho^w \acute{v}^w_i - \rho^w f^w_i - \bar{t}^*_{ij,j} - r^*_i + Pn_{,i} = n\big((\mu/k)nV_i - P_{,i}\big). \tag{5.4}$$

Equation (5.4) represents a generalization of Darcy's law.

Now, we shall assume that both constituents be incompressible in the sense that the densities are constants. Then we have, from (3.1):

$$\frac{\partial(1-n)}{\partial t} + \big((1-n)v^s_i\big)_{,i} = 0, \tag{5.5}$$

$$\frac{\partial n}{\partial t} + (nv^w_i)_{,i} = 0. \tag{5.6}$$

Substituting

$$\mathring{\hat{v}}_i = (1-n)v^s_i; \qquad \mathring{\bar{v}}_i = nv^w_i$$

and addition of equations (5.5) and (5.6) yields:

$$\mathring{\hat{v}}_{i,i} + \mathring{\bar{v}}_{i,i} = 0. \tag{5.7}$$

Now we divide (5.4) by $\mu n/k$, and differentiate with respect to x^α_i, to obtain:

$$\Big(\frac{k}{\mu}\rho^w \acute{v}^w_i\Big)_{,i} = \Big(\frac{k}{\mu}\rho^w f^w_i\Big)_{,i} + \Big(\frac{k}{\mu n}\bar{t}^*_{ij,j}\Big)_{,i} \tag{5.8}$$

$$- \Big(\frac{k}{\mu n}Pn_{,i}\Big)_{,i} - e_{,i}\mathring{\hat{v}}_i + (1+e)\mathring{\bar{v}}_{i,i} - \underline{\Big(\frac{k}{\mu}P_{,i}\Big)_{,i}}.$$

In the above relation we substitute n by e as

$$e = \frac{n}{1-n},$$

where e is the void ratio and

$$V_i = v_i^w - v_i^s = \frac{\bar{\mathring{v}}_i}{n} - \frac{\hat{\mathring{v}}_i}{1-n}.$$

In the above relation only the underlined part is well-known consolidation equation [6].

In this way derived relations (5.4) and (5.8) are generalizations of Darcy's law and consolidation equation derived in [6], for saturated media.

6. Darcy's law and equation of consolidation for partly saturated media. As we said before, the Darcy's law is given by the balance of momentum for fluid. In this case, for linear constitutive relations, when relations (4.4) is introduced in (3.2) it is obtained:

$$\rho^w \acute{v}_i^w = \rho^w f_i^w + \lambda D_{ij}^w V_j^{ws} - p_{,j}^w \delta_{ij} + R_i \tag{6.1}$$

where

$$\begin{aligned} R_i \equiv \lambda\Big(& C_i^w + C_{ij}^w \frac{T_{,j}}{T} + \sum_\alpha C_{kji}^{ws} d_{kj}^\alpha \Big) \\ & + \rho_0^w \left(\psi_{ijkl}^w e_{kl,j}^w + A_{ij}^w T_{,j} + B_{ij}^w n_{,j} + S_{ij}^w S_{r,j} \right) \\ & + \lambda \left[D_{ijkl}^w d_{kl,j}^w + \sum_\alpha (1-\delta^{\alpha w}) C_{ijkl}^{\alpha w} d_{kl,j}^\alpha + \sum_{\beta=2}^{3} C_{ijkl}^{\beta w} V_{k,j}^{\beta s} + \left(\frac{T_{,i}}{T}\right)_{,j} N_{ijk}^w \right] \end{aligned}$$

or

$$\lambda D_{ij}^w V_j^{ws} - p_{,j}^w \delta_{ij} = \bar{R}_i; \quad V_j^{ws} \equiv v_i^w - v_i^s; \quad \bar{R}_i \equiv \rho^w (\acute{v}_i^w - f_i^w) - R_i. \tag{6.2}$$

If we introduce relations [6], [5]:

$$p^w = n^w p; \quad \lambda D_{ij}^w = (\mu/k) n^w \delta_{ij}; \quad n^w v_i^w \equiv v_i; \quad n^s v_i^s \equiv \mathring{v}_i$$

where

μ — is the dynamical viscosity of the pore fluid,

k — is the intrinisic permeability of the solid,

we obtain:

$$\frac{\mu}{k} n^w \left(\frac{v_i}{n^w} - \frac{\mathring{v}_i}{1-n} \right) - (n^w p)_{,i} = F_i$$

or

$$\underline{v_i = \frac{n^w}{1-n} \mathring{v}_i = \frac{k}{\mu} p_{,i}} + \frac{k}{\mu} \frac{n_{,i}^w}{n^w} p + \frac{k}{\mu} \frac{F_i}{n^w}. \tag{6.3}$$

Equation (6.3) represents a generalization of the Darcy's law and only the underlined part is well-known Darcy's law in soil mechanics [6].

Now, we shall assume that both constituent (water, solid) are incompressible in the sense that densities ρ^α are constants. Then we have, from (3.1):

$$\frac{\partial(1-n)}{\partial t}+\mathring{v}_{i,i}=0 \tag{6.4}$$

$$\frac{\partial n^w}{\partial t}+v_{i,i}=0 \tag{6.5}$$

$$\frac{\partial(\rho^g n^g)}{\partial t}+(\rho^g n^g v_i^g)_{,i}=\rho\left[\frac{n^g}{\rho}\frac{\partial\rho}{\partial t}+\frac{\partial n^g}{\partial t}+w_{i,i}+\frac{w_i}{\rho}\rho_{,i}\right]=0, \tag{6.6}$$

where $w_i \equiv n^q v_i^g$.

If (6.6) and (6.5) are added to (6.4) we obtain:

$$\left(v_i+\mathring{v}_i+w_i\right)_{,i}+\frac{n^g}{\rho}\frac{\partial\rho}{\partial t}+\frac{w_i}{\rho}\rho_{,i}=0. \tag{6.7}$$

The relations (6.7) and Darcy's law (6.3) give:

$$\frac{1}{1-n}\mathring{v}_{i,i}+\frac{n^g}{\rho}\frac{\partial\rho}{\partial t}+\left(\frac{k}{\mu}p_{,i}\right)_{,i} \tag{6.8}$$

$$=-\left[\frac{k}{\mu}\frac{n^w_{,i}}{n^w}p+\frac{k}{\mu}\frac{F_i}{n^w}\right]_{,i}-\frac{w_i}{\rho}\rho_{,i}-\mathring{v}_i\left(\frac{n^g}{1-n}\right)_{,i}.$$

In the above relation only the underlined parttis well-known consolidation equation in [6].

The terms on the righthand side (6.8) represents the generalization of the equation of consolidation. These terms well be the subject of our further investigation.

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LE LOI DE DARCY ET L'ÉQUATION DE CONSOLIDATION

Le comportement mécanique d'un squelette déformé, poreux, saturé ou en partie saturé avec fluide est important en beaucoup des applications ingénieurique (le méchanique du sol, l'industrie du pétrole, ...).

Dans cet article on examine le millieu poreux, saturé et non saturé comme un mélange en deux-phase ou en trois-phase, respectivement.

On formule les équations de balance pour chaque phase séparément. Ensuite, en utilisant les inéquations dissipationnées, écritées pour un mélange comme une totalité, on deduit le loi de comportement non linéaire.

En second part de l'article on deduit le loi de Darcy et l'équation de consolidation par ce millieu.

DARSIJEV ZAKON I JEDNAČINA KONSOLIDACIJE

Mehaničko ponašanje deformabilnog poroznog tela, zasićenog ili nezasićenog fluidom je od važnosti u mnogim inženjerskim primenama (mehanika tla, naftna industrija, ...).

U ovom radu posmatra se zasićena i nezasićena porozna sredina kao dvofazna, odnosno trofazna mešavine, respektivno.

Formulišu se jednačine balansa za svaku fazu posebno. Dalje se, korišćenjem disipativne nejednačine napisane za mešavinu kao celinu, izvode nelinearne konstitutivne jednačine.

U drugom delu rada se izvode Darsijev zakon i jednačina konsolidacije za ovakvu sredinu.

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