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ON THE PRINCIPLE OF VIRTUAL WORK IN THE THEORY OF IMMISCIBLE MIXTURES CONTAINING INTERFACE

Z. Golubović, P. Cvetković

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1. Introduction. The theory of immiscible mixtures has recently been presented in the papers of Bedford and Drumheller [1], [2], [3] and other autors. Bedford and Drumheller's theory is based upon Hamilton's extended variational principle and includes a number of effects such as: the sedmintation of rigid particles in an incompressible liquid, the bubble liquid, the dusty gas, the fluid-saturated porous solid, a mixture of ideal gases, etc.

The problem of the interface, on the other hand, has been the subject of extensive research of several authors [4], [5], [6], [7]. In those papers different thoretical approaches are given in the study of the interface problems.

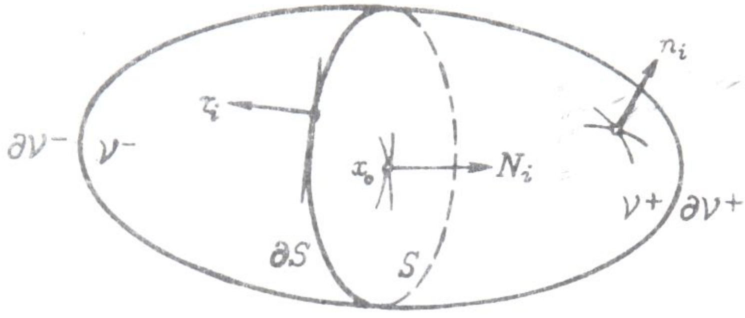
Of a particular interest, form the point of view of the real membrane processes are the problems of mixtures which contain the interface.

In this paper the material body in which the material interface of an arbitrary shape is embedded is investigated.

The body is an immiscible mixture with the property that the individual constituents of the mixture remain physically separated all the time. The fact that the constituents remain separated, in the local sense, from the other constituents, has several implications for the development of a continuum theory. The motion of a given constituent will be kinematically constrained by the presence of the other constituents. The change in the local volume of the constituent is measured in the theory by the volume fraction.

In the first section of the paper preliminary considerations have been given. The balance of mass has been presented in section 2. Following the principle of virtual work we investigate the balance of momentum in section 3. In section 4, we have considered the first principle of thermodynamics, obtaining the balance of energy. The second principle of thermodynamics has been investigated and entropy production deduced in section 6. In the section 7 will be demonstrated how, from the results derived in the paper, as special cases, the results alredy known in literature can be obtained.

2. Preliminaries. a) We consider the three-dimensional body $\mathcal{B}(t)$ in which a material interface is embedded. The interface $S(t)$ divides the bulk material into two parts $\mathcal{V}^+(t)$ and $\mathcal{V}^-(t)$ for which $S(t)$ is common boundary so that $\mathcal{V}(t) = \mathcal{V}^+(t) \cup S(t) \cup \mathcal{V}^-(t)$. The boundary of the body $\mathcal{B}(t)$ is $\partial\mathcal{V}(t) = \partial\mathcal{V}^+(t) \cup \partial S(t) \cup \partial\mathcal{V}^-(t)$.



We shall suppose that both bulk materials are immiscible mixtures. The M constituents of mixtures are treated as superimposed continua.

Using Cartesian tensor notation a motion of each constituent in a spatial coordinate system is written

$$x_k = \chi_{(\alpha)k}(X_{(\alpha)K};t), \tag{1}$$

where $X_{(\alpha)K}$ is a material particle of the α-th constituent. The Jacobian of the motion (1) is

$$J_{(\alpha)} = \det\left(\frac{\partial\chi_{(\alpha)k}}{\partial X_{(\alpha)K}}\right). \tag{2}$$

The velocity $v_{(\alpha)k}$ and acceleration $a_{(\alpha)k}$ of the α-th constituent are

$$v_{(\alpha)k} = \frac{\partial}{\partial t}\chi_{(\alpha)k}(X_{(\alpha)K};t), \tag{3}$$

$$a_{(\alpha)k} = \frac{\partial^2}{\partial t^2}\chi_{(\alpha)k}(X_{(\alpha)K};t). \tag{4}$$

The mass of the α-th constituent per unit volume of the mixture is partial density of the α-th constituent, i.e.

$$\rho_{(\alpha)} = \rho_{(\alpha)}(x_k;t). \tag{5}$$

In an immiscible mixture each constituent has an actual or local density $\overline{\rho}_{(\alpha)}$ (the mass of the α-th constituent per unit volume of the α-th constituent), i.e.

$$\overline{\rho}_{(\alpha)} = \overline{\rho}_{(\alpha)}(x_k;t). \tag{6}$$

Partial densities, local densities and volume fractions are related by

$$\rho_{(\alpha)} = \varphi_{(\alpha)}\overline{\rho}_{(\alpha)}, \tag{7}$$

where the volume fraction is assumed to satisfy

$$\sum_{\alpha}\varphi_{(\alpha)} = 1. \tag{8}$$

b) We assume that the material interface of an arbitrary shape is embedded in the three-dimensional immiscible mixture. We consider the interface without any singular line. Thermomechanical properties of the interface are completely different form the property of the constituents of the bulk material.

The velocity $\hat{v}_i$ of material particle that belongs to interface $S(t)$ and the absolute velocity ν_i of $S(t)$ are related by

$$\hat{v}_i - \nu_i = {}^S\hat{v}_i. \tag{9}$$

On the other hand it can be written

$$(\hat{v}_i - \nu_i)N_i = 0, \tag{10}$$

where N_i is the unit normal vector on S (Fig. 1), and ${}^S\hat{v}_i$ is tangential to S.

c) A variation is added to the motion (1) by writing

$$x_k = \chi_{(\alpha)k}(X_{(\alpha)K};t) + \delta x_{(\alpha)k}(X_{(\alpha)K};t). \tag{11}$$

The corresponding variations of the velocity of the α-th constituent is

$$\delta v_{(\alpha)k} = \delta \dot{x}_{(\alpha)k}. \tag{12}$$

The variation of a function $\Phi_{(\alpha)}$ holding the spatial point x_i fixed is given by

$$\delta\Phi_{(\alpha)}|_{x_k} = \delta\Phi_{(\alpha)} - \Phi_{(\alpha),i}\delta x_{(\alpha)i}. \tag{13}$$

The material derivatives for volumes and surfaces are

$$\frac{d\Phi_{(\alpha)}}{dt} = \frac{\partial\Phi_{(\alpha)}}{\partial t} + \Phi_{(\alpha),i}v_{(\alpha)i}, \tag{14}$$

$$\frac{\hat{d}\hat{\Phi}}{dt} = \frac{\partial\hat{\Phi}}{\partial t} + \hat{\Phi}_{,i}\hat{v}_i. \tag{15}$$

Transport theorems for volumes and surfaces are

$$\frac{d}{dt}\int_{\mathcal{V}^+\cup\mathcal{V}^-}\Phi_{(\alpha)}\,d\mathcal{V} = \int_{\mathcal{V}^+\cup\mathcal{V}^-}\left(\frac{d\Phi_{(\alpha)}}{dt} + \Phi_{(\alpha)}v_{(\alpha)i,i}\right)d\mathcal{V} + \int_S [\![\,\Phi_{(\alpha)}(v_{(\alpha)i} - \nu_i)\,]\!]\,N_i\,da, \tag{16}$$

$$\frac{\hat{d}}{dt}\int_S \hat{\Phi}\,da = \int_S\left(\frac{\hat{d}\hat{\Phi}}{dt} + \hat{\Phi}\hat{v}_{i,i}\right)da, \tag{17}$$

where

$$[\![\,\psi\,]\!] \equiv \psi^+ - \psi^-, \tag{18}$$

indicated the jump of ψ across S at x_0. Divergence theorems in volume and on a surface are

$$\int_{\partial\mathcal{V}^+\cup\partial\mathcal{V}^-} A_{(\alpha)i}n_i\,da = \int_{\mathcal{V}^+\cup\mathcal{V}^-} A_{(\alpha)i,i}\,d\mathcal{V} + \int_s [\![\,A_{(\alpha)i}\,]\!]\,N_i\,da. \tag{19}$$

$$\int_{\partial S} \hat{A}_i \tau_i \, dl = \int_S (\hat{A}_{i,i} + 2\Omega \hat{A}_i N_i) \, da, \tag{20}$$

where n_i is the unit outward normal to $\partial\mathcal{V}^+ \cup \partial\mathcal{V}^-$, τ_i is the unit binormal to ∂S (Fig. 1), and Ω means curvature of S.

3. The Balance of Mass. The mass containted in a material volume $\mathcal{V}$ is constant with time. This is expressed in a global form as

$$\frac{d}{dt}\left(\sum_\alpha \int_{\mathcal{V}^+ \cup \mathcal{V}^-} \rho_{(\alpha)} \, d\mathcal{V}\right) + \frac{\hat{d}}{dt} \int_S \hat{\rho} \, d\mathcal{V} = 0. \tag{21}$$

Using the transport theorems (16–17), after the localization yields the equations

$$\frac{d\rho_{(\alpha)}}{dt} + \rho_{(\alpha)} v_{(\alpha)i,i} = 0, \qquad \text{in } \mathcal{V}^+ \cup \mathcal{V}^- \tag{22}$$

$$\frac{\hat{d}\hat{\rho}}{dt} + \hat{\rho}\hat{v}_{i,i} + \left[\!\left[\sum_\alpha m_{(\alpha)} \right]\!\right] = 0, \qquad \text{on } S \tag{23}$$

with the definition

$$m_{(\alpha)} \equiv \rho_{(\alpha)}(v_{(\alpha)i} - \nu_i) N_i. \tag{24}$$

We can also write the equation (22) in the alternative form

$$J_{(\alpha)} = \frac{\rho_{(\alpha)R}}{\rho_{(\alpha)}} = \frac{\varphi_{(\alpha)R} \bar{\rho}_{(\alpha)R}}{\varphi_{(\alpha)} \bar{\rho}_{(\alpha)}}, \tag{25}$$

where the subscript R denotes that the variable is evaluated in the referent configuration.

In the case of incompressible constituents this yields

$$J_{(\alpha)} = \frac{\varphi_{(\alpha)R}}{\varphi_{(\alpha)}}. \tag{26}$$

4. The Principle of Virtual Work and Balance of Momentum. We derive the balance of momentum from the principle of virtual work. To this end we must modify this principle for the application to the case of immiscible mixture which contains the interface.

First of all we define the total inertial quantity as follows

$$I_k = \frac{d}{dt}\left(\sum_\alpha \int_{\mathcal{V}^+ \cup \mathcal{V}^-} \rho_{(\alpha)} v_{(\alpha)k} \, d\mathcal{V}\right) + \frac{\hat{d}}{dt} \int_S \hat{\rho}\hat{v}_k \, d\mathcal{V}. \tag{27}$$

After using the transport theorems (16–17) and balances of mass (22–23) we obtain from (27) the following equation

$$I_k = \sum_\alpha \int_{\mathcal{V}^+ \cup \mathcal{V}^-} \rho_{(\alpha)} \frac{dv_{(\alpha)k}}{dt} \, d\mathcal{V} + \int_S \left[\!\left[\sum_\alpha m_{(\alpha)} (v_{(\alpha)k} - \hat{v}_k) \right]\!\right] da + \int_S \hat{\rho} \frac{\hat{d}\hat{v}_k}{dt} \, da. \tag{28}$$

The virtual work of these forces is

$$\delta A^I = \sum_\alpha \int_{\mathcal{V}^+ \cup \mathcal{V}^-} \rho_{(\alpha)} \frac{dv_{(\alpha)i}}{dt} \delta x_{(\alpha)i} \, d\mathcal{V} + \int_S \left[\!\left[\sum_\alpha m_{(\alpha)} (v_{(\alpha)i} - \hat{v}_i) \right]\!\right] \delta \hat{x}_i \, da + \int_S \hat{\rho} \frac{d\hat{v}_i}{dt} \delta \hat{x}_i \, da. \tag{29}$$

The principle of virtual work can be written in a compact form

$$\delta A^I = \delta A^*, \tag{30}$$

where

$$\delta A^* = \delta A_1(\mathcal{V}^+ \cup S \cup \mathcal{V}^-) + \delta A_2(\partial\mathcal{V}^+ \cup \partial S \cup \partial\mathcal{V}^-) + \delta A_3(\partial_1 S). \tag{31}$$

The expressions of the virtual works are

$$\delta A_1 = -\sum_\alpha \int_{\mathcal{V}^+ \cup \mathcal{V}^-} \sigma_{(\alpha)ij} \delta x_{(\alpha)i,j} \, d\mathcal{V} + \sum_\alpha \int_{\mathcal{V}^+ \cup \mathcal{V}^-} [\Phi_{(\alpha)ij} \delta x_{(\alpha)i,j} + (\rho_{(\alpha)} f_{(\alpha)i} + b_{(\alpha)i}) \delta x_{(\alpha)i}] \, d\mathcal{V} - \int_S \hat{\sigma}_{ij} \delta \hat{x}_{i,j} \, da + \int_S (\hat{\Phi}_{ij} \delta \hat{x}_{i,j} + \hat{\rho} \hat{f}_i \delta \hat{x}_i) \, da. \tag{32}$$

$$\delta A_2 = \sum_\alpha \int_{\partial\mathcal{V}^+ \cup \partial\mathcal{V}^-} T_{(\alpha)i} \delta x_{(\alpha)i} \, da - \int_S \left[\!\left[\sum_\alpha T_{(\alpha)i} (\delta x_{(\alpha)i} - \delta \hat{x}_i) \, da \right]\!\right], \tag{33}$$

$$\delta A_3 = \int_{\partial S} \hat{T}_i \delta x_i \, dl, \tag{34}$$

where $\sigma_{(\alpha)ij}$, $\Phi_{(\alpha)ij}$, $f_{(\alpha)i}$, $T_{(\alpha)i}$ and $T_{(\alpha)i}$ are, respectively, the symmetric stress tensor, "double" forces, specific external body forces, the momentum supply, the "internal" surface traction and the surface traction of the α-th constituent. Quantities $\hat{\sigma}_{ij}$, $\hat{\Phi}_{ij}$ and $\hat{f}_i$, are corresponding fields defined on S, and $\hat{T}_i$ is the line traction.

Note that nonsymmetric stress tensors are defined by

$$t_{(\alpha)ij} = \sigma_{(\alpha)ij} - \Phi_{(\alpha)ij}, \qquad \hat{t}_{ij} = \hat{\sigma}_{ij} - \hat{\Phi}_{ij} \tag{35}$$

and by taking the skewsymmetric part of (35), it follows

$$t_{(\alpha)[ij]} = \Phi_{(\alpha)[ji]}, \qquad \hat{t}_{[ij]} = \hat{\Phi}_{[ji]}. \tag{36}$$

Taking into account the equations (29) and (31–34) equation (30) can be written in the form

$$\sum_\alpha \int_{\mathcal{V}^+ \cup \mathcal{V}^-} \rho_{(\alpha)} \frac{dv_{(\alpha)i}}{dt} \delta x_{(\alpha)i} \, d\mathcal{V} + \int_S \left[\!\left[\sum_\alpha m_{(\alpha)} (v_{(\alpha)i} - \hat{v}_i) \right]\!\right] \delta \hat{x}_i \, da + \int_S \hat{\rho} \frac{d\hat{v}_i}{dt} \delta \hat{x}_i \, da = -\sum_\alpha \int_{\mathcal{V}^+ \cup \mathcal{V}^-} \sigma_{(\alpha)ij} \delta x_{(\alpha)i,j} \, d\mathcal{V}$$

$$
\begin{aligned}
&+\sum_\alpha \int_{\mathcal{V}^+\cup\mathcal{V}^-} \left[\Phi_{(\alpha)ij}\delta x_{(\alpha)i,j} + (\rho_{(\alpha)}f_{(\alpha)i} + b_{(\alpha)i})\delta x_{(\alpha)i}\right] d\mathcal{V} \\
&= \int_S \hat{\sigma}_{ij}\delta\hat{x}_{ij}\, da + \int_S (\hat{\Phi}_{ij}\delta\hat{x}_{i,j} + \hat{\rho}\hat{f}_i\delta\hat{x}_i)\, da \\
&+\sum_\alpha \int_{\partial\mathcal{V}^+\cup\mathcal{V}^-} T_{(\alpha)i}\delta x_{(\alpha)i}\, da - \int_S \left[\!\left[\sum_\alpha T_{(\alpha)i}(\delta x_{(\alpha)i} - \delta\hat{x}_i)\right]\!\right] da + \int_{\partial S}\hat{T}_i\delta x_i\, dt \\
&- \int_{\mathcal{V}^+\cup\mathcal{V}^-} \lambda(\delta\varphi_{(\alpha)} - \varphi_{(\alpha),i}\delta\hat{x}_{(\alpha)i})\, d\mathcal{V} \\
&+ \int_{\mathcal{V}^+\cup\mathcal{V}^-} \left[-(\mu_{(\alpha)}J_{(\alpha)})_{,i}\delta x_{(\alpha)i} + \frac{\mu_{(\alpha)}J_{(\alpha)}}{\varphi_{(\alpha)}}\delta\varphi_{(\alpha)}\right] d\mathcal{V}. \qquad (37)
\end{aligned}
$$

The terms λ and $\mu_{(\alpha)}$ are the Lagrange multipliers associated with the volume fraction and the conservation of mass of the constituents. Using the transport theorems (16–17), and the condition

$$\delta J_{(\alpha)} = J_{(\alpha)}\delta x_{(\alpha)k,k}\,, \qquad (38)$$

after localization of (37) the following equations are obtained

$$
\begin{aligned}
&\rho_{(\alpha)}\frac{dv_{(\alpha)i}}{dt} = t_{(\alpha)ij,j} + \rho_{(\alpha)}f_{(\alpha)i} \\
&\qquad + b_{(\alpha)i} + \lambda\varphi_{(\alpha),i} - \left(\mu_{(\alpha)}J_{(\alpha)}\right)_{,i} && \text{in } \mathcal{V}^+\cup\mathcal{V}^- && (39)\\
&\lambda = \frac{\mu_{(\alpha)}J_{(\alpha)}}{\varphi_{(\alpha)}} && \text{in } \mathcal{V}^+\cup\mathcal{V}^- && (40)\\
&T_{(\alpha)i} = t_{(\alpha)ij}n_j && \text{on } \partial\mathcal{V}^+\cup\partial\mathcal{V}^- && (41)\\
&T^\pm_{(\alpha)i} - t^\pm_{(\alpha)ij}N_j = 0 && \text{on } S^\pm && (42)\\
&\hat{T}_i = \hat{t}_{ij}\tau_j && \text{on } \partial S && (43)\\
&\hat{\rho}\frac{\hat{d}\hat{v}_i}{dt} + \left[\!\left[\sum_\alpha m_{(\alpha)}(v_{(\alpha)i} - \hat{v}_i) - \sum_\alpha T_{(\alpha)i}\right]\!\right] \\
&\qquad = \hat{t}_{ij,j} + 2\Omega\hat{t}_{ij}N_j + \hat{\rho}\hat{f}_i && \text{on } S. && (44)
\end{aligned}
$$

Using (42) and the transversality condition

$$\hat{t}_{ij}N_j = 0 \qquad \text{on } S$$

the equation (44) can be written in the form

$$\hat{\rho}\frac{\hat{d}\hat{v}_i}{dt} + \left[\!\left[\sum_\alpha \rho_\alpha(v_{(\alpha)i} - \hat{v}_i)(v_{(\alpha)j} - \nu_j) - \sum_\alpha t_{(\alpha)ij}\right]\!\right] N_j = \hat{T}_{ij,j} + \hat{\rho}\hat{f}_i \quad \text{on } S. \quad (46)$$

The expressions (39) and (46) represent the balance laws of momentum of the α-th constituent of the surrounding materials and the interface.

5. The First Principle of Thermodynamics and the Balance of Energy. The balance of the total energy in the global form is

$$\frac{d}{dt}(E+K)+\frac{\hat{d}}{dt}(\hat{E}+\hat{K})=\frac{dA^0}{dt}+Q, \tag{47}$$

with the definitions

$$E=\sum_\alpha\int_{\mathcal{V}^+\cup\mathcal{V}^-}\rho_{(\alpha)}e_{(\alpha)}\,d\mathcal{V}, \tag{48}$$

$$\hat{E}=\int_S\hat{\rho}\hat{e}\,da, \tag{49}$$

$$K=\sum_\alpha\int_{\mathcal{V}^+\cup\mathcal{V}^-}\frac{1}{2}\rho_{(\alpha)}v_{(\alpha)}^2\,d\mathcal{V}, \tag{50}$$

$$\hat{K}=\int_S\frac{1}{2}\hat{\rho}\hat{v}^2\,da, \tag{51}$$

$$\begin{aligned}\frac{dA^0}{dt}&=\sum_\alpha\int_{\mathcal{V}^+\cup\mathcal{V}^-}\left(\rho_{(\alpha)}f_{(\alpha)i}v_{(\alpha)i}+b_{(\alpha)i}v_{(\alpha)i}+\Phi_{(\alpha)ij}v_{(\alpha)i,j}\right)d\mathcal{V}\\&+\sum_\alpha\int_{\partial\mathcal{V}^+\cup\partial\mathcal{V}^-}T_{(\alpha)i}v_{(\alpha)i}\,da+\int_S\left(\hat{\rho}\hat{f}_i\hat{v}_i+\hat{\Phi}_{ij}\hat{v}_{ij}\right)da+\int_{\partial S}\hat{T}_i\hat{v}_i\,dl.\end{aligned} \tag{52}$$

$$\begin{aligned}Q&=\sum_\alpha\int_{\mathcal{V}^+\cup\mathcal{V}^-}\left(\rho_{(\alpha)}h_{(\alpha)}+u_{(\alpha)}\right)d\mathcal{V}-\sum_\alpha\int_{\partial\mathcal{V}^+\cup\partial\mathcal{V}^-}q_{(\alpha)i}n_i\,da\\&+\int_S\hat{\rho}\hat{h}\,da-\int_{\partial S}\hat{q}_i\tau_i\,dl.\end{aligned} \tag{53}$$

where $e_{(\alpha)}$, $h_{(\alpha)}$, $q_{(\alpha)}$ and $u_{(\alpha)}$, are, the specific internal energy, the specific external heat supply and the heat flux vector of the α-th constituent, respectively, and the energy supply to the α-th constituent due to the interaction with the other constituents. Quantities $\hat{e}$, $\hat{h}$ and $\hat{q}_i$ are corresponding fields defined on S.

Making use of the transport theorems (16–17) and balances of mass (22–23) we have

$$\begin{aligned}\frac{dE}{dt}+\frac{\hat{d}\hat{E}}{dt}&=\sum_\alpha\int_{\mathcal{V}^+\cup\mathcal{V}^-}\rho_{(\alpha)}\frac{de_{(\alpha)}}{dt}\,d\mathcal{V}+\int_s\left[\!\left[\sum_\alpha e_{(\alpha)}m_{(\alpha)}\right]\!\right]da\\&+\int_S\hat{\rho}\frac{\hat{d}\hat{e}}{dt}\,da-\int_S\left[\!\left[\hat{e}\sum_\alpha m_{(\alpha)}\right]\!\right]da,\end{aligned} \tag{54}$$

$$\begin{aligned}\frac{dK}{dt}+\frac{\hat{d}\hat{K}}{dt}&=\sum_\alpha\int_{\mathcal{V}^+\cup\mathcal{V}^-}\rho_{(\alpha)}v_{(\alpha)i}\frac{dv_{(\alpha)i}}{dt}\,d\mathcal{V}\\&-\int_S\left[\!\left[\sum_\alpha\frac{1}{2}m_{(\alpha)}v_{(\alpha)}^2\right]\!\right]da+\int_S\hat{\rho}\hat{v}_i\frac{\hat{d}\hat{v}_i}{dt}\,da+\int_S\left[\!\left[\sum_\alpha\frac{1}{2}\hat{v}^2m_{(\alpha)}\right]\!\right]da.\end{aligned} \tag{55}$$

The expression (29), written for real velocity fields, has the form

$$\frac{dA^I}{dt}=\sum_\alpha\int_{\mathcal{V}^+\cup\mathcal{V}^-}\rho_{(\alpha)}\frac{v_{(\alpha)i}}{dt}v_{(\alpha)i}\,d\mathcal{V}+\int_S\hat{\rho}\frac{\hat{d}\hat{v}_i}{dt}\hat{v}_i\,da \tag{56}$$

$$+\int_S \left[\!\left[\sum_\alpha m_{(\alpha)} v_{(\alpha)i}\hat{v}_i \right]\!\right] da - \int_S \left[\!\left[\sum_\alpha m_{(\alpha)} \hat{v}^2 \right]\!\right] da.$$

With regard to (55) and (56) we obtain

$$\frac{dK}{dt} + \frac{\hat{d}\hat{K}}{dt} = \frac{dA^I}{dt} + \int_S \left[\!\left[\frac{1}{2} \sum_\alpha m_{(\alpha)} (v_{(\alpha)i} - \hat{v}_i)^2 \right]\!\right] da. \qquad (57)$$

Upon substituting this into (47), we get

$$\frac{dE}{dt} + \frac{\hat{d}\hat{E}}{dt} + \frac{dA^I}{dt} - \frac{dA^0}{dt} = -\int_S \left[\!\left[\frac{1}{2} \sum_\alpha m_{(\alpha)} (v_{(\alpha)i} - \hat{v}_i)^2 \right]\!\right] da + Q. \qquad (58)$$

Regarding the principle of virtual work (3) for the real velocity fields, the following can be obtained

$$\begin{aligned}
\frac{dA^I}{dt} = &-\sum_\alpha \int_{\mathcal{V}^+ \cup \mathcal{V}^-} \sigma_{(\alpha)ij} v_{(\alpha)i,j}\, d\mathcal{V} \\
&+ \sum_\alpha \int_{\mathcal{V}^+ \cup \mathcal{V}^-} \left[\Phi_{(\alpha)ij} v_{(\alpha)i,j} + \left(\rho_{(\alpha)} f_{(\alpha)i} + b_{(\alpha)i} \right) v_{(\alpha)i} \right] d\mathcal{V} \qquad (59) \\
&- \int_S \hat{\sigma}_{ij} \hat{v}_{i,j}\, da + \int_S \left(\hat{\Phi}_{ij} \hat{v}_{i,j} + \hat{\rho} \hat{f}_i \hat{v}_i \right) da - \sum_\alpha \int_{\partial\mathcal{V}^+ \cup \partial\mathcal{V}^-} T_{(\alpha)i} v_{(\alpha)i}\, da \\
&- \int_S \left[\!\left[\sum_\alpha \mathcal{T}_{(\alpha)i} \left(v_{(\alpha)i} - \hat{v}_i \right) \right]\!\right] da + \int_{\partial S} \hat{T}_i v_i\, dl.
\end{aligned}$$

Using (52–54), (56), (58–59), and transport theorems (16–17), (47) can be written in the form

$$\begin{aligned}
&\sum_\alpha \int_{\mathcal{V}^+ \cup \mathcal{V}^-} \rho_{(\alpha)} \frac{de_{(\varepsilon)}}{dt}\, d\mathcal{V} + \int_S \left[\!\left[\sum_\alpha m_{(\alpha)} \left(e_{(\alpha)} - \hat{e} \right) \right]\!\right] da + \int_S \hat{\rho} \frac{\hat{d}\hat{e}}{dt}\, da \qquad (60) \\
&\sum_\alpha \int_S \left[\!\left[\frac{1}{2} m_{(\alpha)} \left(v_{(\alpha)i} - \hat{v}_i \right)^2 \right]\!\right] da = \sum_\alpha \int_{\mathcal{V}^+ \cup \mathcal{V}^-} \sigma_{(\alpha)ij} v_{(\alpha)i,j}\, d\mathcal{V} \\
&+ \int_S \left[\!\left[\sum_\alpha \mathcal{T}_{(\alpha)i} \left(v_{(\alpha)i} - \hat{v}_i \right) \right]\!\right] da + \int_S \hat{\sigma}_{ij} \hat{v}_{i,j}\, da \\
&+ \sum_\alpha \int_{\mathcal{V}^+ \cup \mathcal{V}^-} \left(\rho_{(\alpha)} h_{(\alpha)} + u_{(\alpha)} \right) d\mathcal{V} + \int_S \hat{\rho} \hat{h}\, da \\
&- \sum_\alpha \int_{\mathcal{V}^+ \cup \mathcal{V}^-} q_{(\alpha)i,i}\, d\mathcal{V} - \sum_\alpha \int_S \left[\!\left[q_{(\alpha)i} \right]\!\right] N_i\, da - \int_S \hat{q}_{i,i}\, da + \int_S 2\Omega \hat{q}_i N_i\, da.
\end{aligned}$$

Localization of the equation (6) yields equations

$$\rho_{(\alpha)} \frac{de_{(\alpha)}}{dt} = \sigma_{(\alpha)ij} v_{(\alpha)i,j} + q_{(\alpha)k,k} + \rho_{(\alpha)} h_{(\alpha)} + u_{(\alpha)}, \qquad \text{in } \mathcal{V}^+ \cup \mathcal{V}^-, \qquad (61)$$

$$
\hat{\rho}\frac{\hat{d}\hat{e}}{dt} + \left[\!\left[\sum_\alpha m_{(\alpha)} \left[(e_{(\alpha)} - \hat{e}) + \frac{1}{2}(v_{(\alpha)i} - \hat{v}_i)^2 \right] \right.\right.
$$

$$
\left.\left. - \sum_\alpha \mathcal{T}_{(\alpha)i}(v_{(\alpha)i} - \hat{v}_i) + \sum_\alpha q_{(\alpha)i} N_i \right]\!\right]
$$

$$
= \hat{\sigma}_{ij}\hat{V}_{i,j} + \hat{\rho}\hat{h} - \hat{q}_{i,i} - 2\Omega\hat{q}_i N_i \qquad \text{on } S. \tag{62}
$$

With regard to the transfersality condition

$$
\hat{q}_i N_i = 0, \tag{63}
$$

and equation (42), the expression (62) has the form

$$
\hat{\rho}\frac{\hat{d}\hat{e}}{dt} + \left[\!\left[\sum_\alpha \rho_{(\alpha)} \left[(e_{(\alpha)} - \hat{e}) + \frac{1}{2}(v_{(\alpha)i} - \hat{v}_i)^2 \right] (v_{(\alpha)i} - \nu_j) \right.\right.
$$

$$
\left.\left. - \sum_\alpha t_{(\alpha)ij}(v_{(\alpha)i} - \hat{v}_i) + \sum_\alpha q_{(\alpha)j} \right]\!\right] N_j = \hat{\sigma}_{ij}\hat{v}_{i,j} + \hat{\rho}\hat{h} - \hat{q}_{i,i} \qquad \text{on } S. \tag{64}
$$

The relations (61) and (64) represent the balance laws of the internal energy of the α-th constituent of the surrounding materials and the interface.

6. The Second Principle of Thermodynamics and Clausius-Duhem Inequality. The second principle of thermodynamics in the global form is

$$
\frac{dN}{dt} + \frac{\hat{d}\hat{N}}{dt} \geq \mathcal{N}, \tag{65}
$$

with

$$
N = \sum_\alpha \int_{\mathcal{V}^+ \cup \mathcal{V}^-} \rho_{(\alpha)}\eta_{(\alpha)}\, d\mathcal{V}, \tag{66}
$$

$$
\hat{N} = \int_S \hat{\rho}\hat{\eta}\, da, \tag{67}
$$

and

$$
\mathcal{N} = \sum_\alpha \int_{\mathcal{V}^+ \cup \mathcal{V}^-} \frac{\rho_{(\alpha)} h_{(\alpha)}}{\theta_{(\alpha)}}\, d\mathcal{V} - \sum_\alpha \int_{\partial\mathcal{V}^+ \cup \partial\mathcal{V}^-} \frac{1}{\theta_{(\alpha)}} q_{(\alpha)i} n_i\, da
$$

$$
+ \int_S \frac{\hat{\rho}\hat{h}}{\hat{\theta}}\, da - \int_{\partial S} \frac{1}{\hat{\theta}} \hat{q}_i \tau_i\, dl. \tag{68}
$$

where $\eta_{(\alpha)}$ and $\theta_{(\alpha)}$ are specific entropy and thermodynamical temperature of α-th constituents, while $\hat{\eta}$ and $\hat{\theta}$ are the corresponding quantities defined on S.

Using the transport and divergence thorems (16–17) and (19–20), and balances of mass (22–23), form (65) the following local inequalities can be deduced

$$\sum_\alpha \rho_{(\alpha)}\frac{d\eta_{(\alpha)}}{dt} - \sum_\alpha \frac{1}{\theta_{(\alpha)}}\left(\rho_{(\alpha)}h_{(\alpha)} - q_{(\alpha)i,i}\right) - \sum_\alpha \frac{q_{(\alpha)i}\theta_{(\alpha),i}}{\theta^2_{(\alpha)}} \geq 0 \qquad \text{in } \mathcal{V}^+ \cup \mathcal{V}^- \quad (69)$$

$$\hat{\rho}\frac{d\hat{\eta}}{dt} - \frac{1}{\hat{\theta}}\left(\hat{\rho}\hat{h} - \hat{q}_{i,i}\right) - \frac{\hat{q}_i\hat{\theta}_{,i}}{\hat{\theta}^2} + \left[\!\left[\sum_\alpha \rho_{(\alpha)}\left(\eta_{(\alpha)} - \hat{\eta}\right)\left(v_{(\alpha)i} - \nu_i\right) + \sum_\alpha \frac{q_{(\alpha)i}}{\theta_{(\alpha)}}\right]\!\right] N_i \geq 0 \qquad \text{on } S. \quad (70)$$

Introducing $\psi_{(\alpha)}$ and $\hat{\psi}$ i.e. Helmholtz free energies of the constitutent and a corresponding quantity defined on S, as

$$\psi_{(\alpha)} = e_{(\alpha)} - \eta_{(\alpha)}\theta_{(\alpha)} \qquad \text{in } \mathcal{V}^+ \cup \mathcal{V}^- \quad (71)$$

$$\hat{\psi} = \hat{e} - \hat{\eta}\hat{\theta} \qquad \text{on } S \quad (72)$$

By solving (61) and (64) for $\rho_{(\alpha)}h_{(\alpha)}$ and $\hat{\rho}\hat{h}$ and substituting it together with (71–72) into (69) and (70) we obtain Clausius-Duhem inequalities

$$\sum_\alpha \left(-\frac{\rho_{(\alpha)}}{\theta_{(\alpha)}}\frac{d\psi_{(\alpha)}}{dt} - \frac{\rho_{(\alpha)}\eta_{(\alpha)}}{\theta_{(\alpha)}}\frac{d\theta_{(\alpha)}}{dt} + \frac{1}{\theta_{(\alpha)}}\sigma_{(\alpha)ij}v_{(\alpha)i,j} + \frac{u_{(\alpha)}}{\theta_{(\alpha)}} - \frac{q_{(\alpha)}\theta_{(\alpha),i}}{\theta^2_{(\alpha)}}\right) \geq 0, \qquad \text{in } \mathcal{V}^+ \cup \mathcal{V}^- \quad (73)$$

$$-\hat{\rho}\left(\frac{d\hat{\psi}}{dt} + \hat{\eta}\frac{d\hat{\theta}}{dt}\right) + \hat{\sigma}_{ij}\hat{v}_{i,j} - \frac{1}{\hat{\theta}}\hat{q}_i\hat{\theta}_{,i} - \left[\!\left[\sum_\alpha m_{(\alpha)}\left\{\eta_{(\alpha)}\left(\theta_{(\alpha)} - \hat{\theta}\right) + \left(\psi_{(\alpha)} - \hat{\psi}\right) + \frac{1}{2}\left(v_{(\alpha)i} - \hat{v}_i\right)^2\right\}\right]\!\right] + \left[\!\left[\sum_\alpha t_{(\alpha)ij}\left(v_{(\alpha)i} - \hat{v}_i\right) - \sum_\alpha q_{(\alpha)j}\left(1 - \frac{\hat{\theta}}{\theta_{(\alpha)}}\right)\right]\!\right] N_j \geq 0 \qquad \text{on } S. \quad (74)$$

7. Discussion. In this paper it has been shown how the validity of the principal of virtual work can be extended on the cases of immiscible mixtures containing the interface. In this section, it will be demonstrated how the results already known in literature can be obtained from those derived in the present paper as special cases.

1) Bulk material balance laws (22), (39) and (61) are consistent with Bedford and Drumheller's, [1], [2], corresponding balance laws, if the existence of the interface in immiscible mixture is neglected in our considerations.

2) Taking that

$$\rho_{(\alpha)} = \overline{\rho}_{(\alpha)}, \quad (75)$$

or that there is no difference between partial and loacl density of constituents, it follows from (7) that

$$\varphi_{(\alpha)} = 1, \quad (76)$$

and from (40)

$$\mu_{(\alpha)}J_{(\alpha)} = \lambda. \tag{77}$$

Then the balance laws of the bulk immiscible material (22), (39) and (61) are reduced to the known balance laws of classical mixtures

$$\frac{d\rho_{(\alpha)}}{dt} + \rho_{(\alpha)}v_{(\alpha)i,i} = 0, \qquad \rho\frac{dv_{(\alpha)i}}{dt} = t_{(\alpha)ij,j} + \rho_{(\alpha)}f_{(\alpha)i} + b_{(\alpha)i}, \tag{78}$$

$$\rho_{(\alpha)}\frac{de_{(\alpha)}}{dt} = t_{(\alpha)ij}v_{(\alpha)i,j} + q_{(\alpha)i,i} + \rho_{(\alpha)}h_{(\alpha)} + u_{(\alpha)}.$$

Using the definitions [8]

$$\sum_\alpha \rho_{(\alpha)} = \rho, \quad \sum_\alpha \rho_{(\alpha)}v_{(\alpha)i} = \rho v_i, \quad \sum_\alpha \big(t_{(\alpha)ij} - \rho_{(\alpha)}u_{(\alpha)i}u_{(\alpha)j}\big) = t_{ij},$$

$$u_{(\alpha)i} = v_{(\alpha)i} - v_k, \quad \sum_\alpha \rho_{(\alpha)}\big(e_{(\alpha)} + \tfrac{1}{2}u^2_{(\alpha)}\big) = \rho e, \tag{79}$$

$$\sum_\alpha \big(q_{(\alpha)j} - t_{(\alpha)ij}u_{(\alpha)j} + q_{(\alpha)}\big(e_{(\alpha)} + \tfrac{1}{2}u^2_{(\alpha)}\big)u_{(\alpha)j}\big) = q_j,$$

$$\sum_\alpha \rho_{(\alpha)}f_{(\alpha)i} = \rho f_i, \quad \sum_\alpha b_{(\alpha)i} = 0, \quad \sum_\alpha \big(b_{(\alpha)i}v_{(\alpha)i} + u_{(\alpha)i}\big) = 0,$$

$$\sum_\alpha \big(\rho_{(\alpha)}h_{(\alpha)} + \rho_{(\alpha)}f_i f_{(\alpha)i}u_{(\alpha)i}\big) = \rho h,$$

balance laws (78) take the form

$$\frac{d\rho}{dt} + \rho v_{i,i} = 0, \quad \rho\frac{dv_i}{dt} = t_{ij,j} + \rho f_i, \quad \rho\frac{d\rho}{dt} = t_{ij}v_{i,j} + q_{i,i} + \rho h, \tag{80}$$

which represent the well-known balance laws of a single component material. In that case, the interface balance laws (23), (46) and (64) are reduced to the balance laws obtained by Daher and Maugin [6] investigated a single component material containing the interface:

$$\frac{d\hat\rho}{dt} + \hat\rho\hat v_{i,i} + [\![m]\!] = 0, \quad \hat\rho\frac{d\hat v_i}{dt} + [\![\rho(v_i - \hat v_i)(v_j - \nu_j) - t_{ij}]\!] N_j = \hat t_{ij} + \hat\rho\hat f_i,$$

$$\hat\rho\frac{d\hat e}{dt} + \left[\!\!\left[\rho\Big\{(e - \hat e) + \frac{1}{2}(v_i - \hat v_i)^2\Big\}(v_j - \nu_j) - t_{ij}(v_i - \hat v_i) + q_j \right]\!\!\right] N_j$$

$$= \hat\sigma_{ij}\hat v_{i,j} + \hat\rho\hat h - \hat q_{i,i}. \tag{81}$$

3) In the case of the immovable interface not exposed to thermal influences, the following is obtained

$$\rho_{(\alpha)}\frac{dv_{(\alpha)i}}{dt} = t_{(\alpha)ij,j} + \rho_{(\alpha)}f_{(\alpha)i} + b_{(\alpha)i} + \lambda_{,i}\varphi_{(\alpha)} \qquad \text{in } \mathcal{V}^+ \cup \mathcal{V}^-$$

$$T_{(\alpha)i} = t_{(\alpha)ij}n_j \qquad \text{on } \partial\mathcal{V}^+ \cup \partial\mathcal{V}^-$$

$$\mathcal{F}^{\pm}_{(\alpha)i} = t^{\pm}_{(\alpha)ij} N_j, \qquad \text{on } S^{\pm}$$
$$[\![m_{(\alpha)} v_{(\alpha)i} - \mathcal{F}_{(\alpha)i}]\!] = 0, \qquad \text{across } S.$$

If here the assumptions stated in 2) are utilized, expresions are consistent with the ones obtained in [6], by considering the dynamic-mechanical example.

4) In the case of the immovable interface in the immovable immiscible mixture and if the thermal effects are neglected, it follows that

$$\begin{aligned} &0 = t_{(\alpha)ij,j} + \rho_{(\alpha)} f_{(\alpha)i} + b_{(\alpha)i} + \lambda_{,i}\varphi_{(\alpha)}, && \text{in } \partial\mathcal{V}^+ \cup \partial\mathcal{V}^-; \\ &T_{(\alpha)i} = t_{(\alpha)ij} n_j, && \text{on } \partial\mathcal{V}^+ \cup \partial\mathcal{V}^-; \\ &\mathcal{T}^{\pm}_{(\alpha)i} = t^{\pm}_{(\alpha)ij} N_j, && \text{on } S^{\pm}; \\ &[\![\mathcal{T}_{(\alpha)i}]\!] = 0 && \text{across } S. \end{aligned} \tag{83}$$

If the assumptions given in 2) are used in this case, the relations are consistent to those arrived at [6] in the quasi-static mechanical example.

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ÜBER DAS PRINZIP DER VIRTUELLEN ARBEIT IN DER THEORIE DER UNMISCHBAREN MISCHUNGEN DIE EINE ZWISCHENFLÄCHE ENTHALTEN

In dieser Arbeit ist der materiele Körper, der die Zwischenfläche enthält, durch Anwendung des Virtuelarbeitprinzips betrachtet worden. Der Körper ist eine unmischbare Mischung mit Eingenschaften, dass die einzelne Bestandteile der Mischung durch die ganze Zeit physikalisch getrennt sind. Die thermomechanische Zwischenflächeneigenschaften sind vollkommen unterschiedlich von den Eigenschaften des Umgebungsmaterials. Die Anwesenheit des Materials ausserhalb der Zwichenfläche wird in den Bilanzrelationen der Zwischenfläche über die Sprunge der dreidimenzionelen Felder betrachtet.

O PRINCIPU VIRTUALNOG RADA U TEORIJI "NEMEŠLJIVIH" MEŠAVINA KOJE SADRŽE MEĐUPOVRŠ

U radu je korišćenjem principa virtualnog rada razmatrano telo koje sadrži međupovrš. Telo je „nemešljiva“ mešavina sa svojstvima da pojedini sastojci mešavine sve vreme ostaju fizički razdvojeni. Termomehanička svojstva međupovrši se kompletno razklikuju od svojstava sastojaka okolnog materijala. Prisustvo materijala izvan međupovrši je uzeto u obzir u jednačinama balansa međupovrši preko članova skoka trodimenzionalnih polja.

Dr Zoran Golubović
Faculty of Mechanical Engineering
University of Belgrade
27. marta 80
11000 Beograd, Yugoslavia

Dr Predrag Cvetković
Faculty of Transport Engineering
Univeristy of Belgrade
Vojvode Stepe 305
11000 Beograd, Yugoslavia