TEORIJSKA I PRIMENJENA MEHANIKA 15, pp 119–125, 1989, UDK 531.26

# ON STATIONARY MOTIONS OF PLATO'S BODIES IN THE GRAVITY FIELD

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(Received 01.10.1988)

Mechanical systems consisting of material points with equal masses located at the vertices of regular polyhedrons (Plato's bodies) and connected by non-deformable weightless rode are considered. The problem of the action of higher order moments of inertia on the motion of Plato's bodies fixed at the mass centre is investigated in the central Newtonian force field, i.e., with the properties of higher order terms in the potential expansion taken into account.

Stationary motions and equilibrium positions are found and their stability is studied. A bifurcation diagram is given on the plane of constants of energy and area potentials.

An interesting fact is noted: for the considered bodies the dimension of a body element (vertex, rib, face) with which the body in stationary motions and equilibrium positions is directed towards the attrecting centre, coincides with a degree of instability.

**1.** Let us consider motion in the central gravity field of a system of material points with equal masses, located at the vertices of a regular polyhedron and connected by non-deformable weightless rods. The mass centre of the system is assumed to be fixed.

Let $O\xi\eta\zeta$ be an inertial coordinate system with the origin at the attracting centre $O$, $Gxyz$ a coordinate system with the origin at the mass centre $G$ connected rigidly with the body such that the vertex coordinates have the form [1]

$$a(1/\sqrt{2},-1/\sqrt{2},-1/\sqrt{2});\qquad a(1/\sqrt{2},1/\sqrt{2},1/\sqrt{2});$$
$$a(-1/\sqrt{2},1/\sqrt{2},-1/\sqrt{2});\qquad a(-1/\sqrt{2},-1/\sqrt{2},1/\sqrt{2})$$

for a tetrahedron;

$$a(\pm 1,\pm 1,\pm 1)$$

for a cube;

$$a(\pm 1,0,0);\quad a(0,\pm 1,0);\quad a(0,0,\pm 1)$$

for an octahedron;

$$a(\pm\tau,\pm 1,0);\quad a(0,\pm 1,\tau);\quad a(0,\pm\tau,\pm 1)$$

for an icosahedron;

$$a(\pm 1,\pm 1,\pm 1);\quad a(\pm\tau^{-1},\pm\tau,0);\quad a(\pm\tau,0,\pm\tau^{-1});\quad a(0,\pm\tau^{-1};\pm\tau)$$

for a dodecahedron. Here $a$ is a typical body dimension, $\tau=(\sqrt{5}+1)/2$. In what follows it will be assumed that $a=1$.

Distances $R_j$ from the attracting centre $O$ to the vertices of the considered bodies are determined by the relations

$$R_j=\sqrt{R^2+2(\overline{R}\cdot\overline{r}_j)+r_j^2}=R\sqrt{1+2\varepsilon(\overline{\gamma},\overline{e}_j)+\varepsilon^2} \tag{1.1}$$

where $\overline{R}=\overline{OG}$; $\overline{r}_j=\overline{GA}_j=r\overline{e}_j$; [$r=\sqrt{3/2}$ for a tetrahedron; $r=\sqrt{3}$ for a cube and a dodecahedron; $r=1$ for an octahedron; $r=\sqrt{1+\tau^2}$ for an icosahedron]; $\overline{e}_j$ and $\overline{\gamma}=(\gamma_1,\gamma_2,\gamma_3)$ are the unit vectors directed along $\overline{r}_j$ and $\overline{R}$, respectively; $A_j$ is the vertex; $j$ is the number of vertices of the respetive polyhedron, $\varepsilon=r/R$.

Force functions $U_i$ have the form

$$U_i=\sum_j\frac{fM_0M}{R_j}=\frac{fM_0M}{R}\sum_j(1+\sigma_j)^{-1/2},\quad (i=\overline{1,5})$$

where $f$ is the gravity constant, $M_0$ is the mass of the attracting body, $M$ is the mass of the entire system, $\sigma=2\varepsilon(\overline{\gamma},\overline{e}_j)+\varepsilon^2$.

Expanding the force function $U_i$ into a series in $\varepsilon$, to within terms $\varepsilon^n$ $(n\geq 3)$ we obtain $(N=fM_0M/R)$:

$U_1=(4N/\sqrt{3})(\sqrt{3}-5\varepsilon^3\gamma_1\gamma_2\gamma_3)$ for a tetrahedron;

$U_2=(4N/9)\left[18-7\varepsilon^4+35\varepsilon^4(\gamma_1^2\gamma_2^2+\gamma_2^2\gamma_3^2+\gamma_3^2\gamma_1^2)\right]$ for a cube;

$U_3=(N/4)\left[24-21\varepsilon^4+35\varepsilon^4(\gamma_1^4+\gamma_2^4+\gamma_3^4)\right]$ for an octahedron;

$U_4=(N/64)\{768+48(3+\sqrt{5})\varepsilon^2+35(7+3\sqrt{5})\varepsilon^4+128(5+\sqrt{5})\varepsilon^6[4(\gamma_1^6+\gamma_2^6+\gamma_3^6)+3(5+\sqrt{5})(\gamma_1^4\gamma_2^2+\gamma_2^4\gamma_3^2+\gamma_3^4\gamma_1^2)+3(5-\sqrt{5})(\gamma_1^2\gamma_2^4+\gamma_2^2\gamma_3^4+\gamma_3^2\gamma_1^4)]\}$ for an icosahedron;

$U_5=(5N/32)\{128+48\varepsilon^2+63\varepsilon^4+64\varepsilon^6[8(\gamma_1^6+\gamma_2^6+\gamma_3^6)+72\gamma_1^2\gamma_2^2\gamma_3^2+3(7+\sqrt{5})(\gamma_1^2\gamma_2^4+\gamma_2^2\gamma_3^4+\gamma_3^2\gamma_1^4)+3(7-\sqrt{5})(\gamma_1^4\gamma_2^2+\gamma_2^4\gamma_3^2+\gamma_3^4\gamma_1^2)]\}$ for a dodecahedron.

Since the gravity field is axially symmetric, force functions $U_i$ $(i=\overline{1,5})$ depend only on $\gamma_\nu$ $(\nu=\overline{1,3})$ and therefore all equilibrium positions of the considered bodies are indifferent to the rotation about the radius-vector $\overline{R}$.

In the considered approximation the motion equations of these bodies admit the following first integrals:

$$H_i=(1/2)J(\omega_1^2+\omega_2^2+\omega_3^2)-U_i=h=\text{const},\quad (i=\overline{1,5}),$$
$$K=J(\omega_1\gamma_1+\omega_2\gamma_2+\omega_3\gamma_3)=k=\text{const},\quad I=\gamma_1^2+\gamma_2^2+\gamma_3^2=1,$$

where $J$ is the moment of inertia (for the considered bodies the central tensor of intertia is spherical), $\overline{\omega}=(\omega_1,\omega_2,\omega_3)$ is the angular velocity vector.

Stationary motions of the considered bodies are determined on the basis of the Routh theorem [2]. According to this theorem stationary motions of the body correspond to stationary values of the energy integral if it assumed that the area integral and the geometrical integral have constant values. Thus the problem of determining stationary motions reduces to the problem of finding stationary values of the function

$$W_i = H_i - \lambda(K - k) + \tfrac{1}{2}\mu(I - 1), \quad (i = \overline{1,5}) \tag{1.2}$$

where $\lambda$ and $\mu$ are Lagrangian multipliers.

Conditions for the stationary of function (1.2) are given by a system of equations

$$\frac{\partial W_i}{\partial \lambda} = k = K = 0, \qquad \frac{\partial W_i}{\partial \mu} = I - 1 = 0,$$
$$\frac{\partial W_i}{\partial \gamma_\nu} = \frac{\partial H_i}{\partial \gamma_\nu} - \lambda\frac{\partial K}{\partial \gamma_\nu} + \frac{1}{2}\mu\frac{\partial I}{\partial \gamma_\nu} = 0, \tag{1.3}$$
$$\frac{\partial W_i}{\partial \omega_\nu} = J(\omega_\nu - \lambda\gamma_\nu) = 0.$$

The stationary motion equations (1.3) admit the following one-parameter ($\omega_\nu = \lambda\gamma_\nu$, $\lambda$ is an arbitrary parameter) families of solutions:

for a tetrahedron

$$\gamma_1 = 0, \quad \gamma_2 = 0, \quad \gamma_3 = \pm 1, \quad \mu = 0, \quad (123) \tag{1.4}$$

$$\left.\begin{array}{lll} \gamma_1 = 1/\sqrt{3}, & \gamma_2 = 1/\sqrt{3}, & \gamma_3 = 1/\sqrt{3}, \\ \gamma_1 = -1/\sqrt{3}, & \gamma_2 = -1/\sqrt{3}, & \gamma_3 = 1/\sqrt{3} \end{array}\right\} \quad \mu = J\lambda^2 - \frac{20}{3}N\varepsilon^2, \tag{1.5}$$

$$\left.\begin{array}{lll} \gamma_1 = -1/\sqrt{3}, & \gamma_2 = 1/\sqrt{3}, & \gamma_3 = 1/\sqrt{3}, \\ \gamma_1 = -1/\sqrt{3}, & \gamma_2 = -1/\sqrt{3}, & \gamma_3 = -1/\sqrt{3} \end{array}\right\} \quad \mu = J\lambda^2 + \frac{20}{3}N\varepsilon^3. \tag{1.6}$$

for a cube and a octahedron

$$\gamma_1 = 0, \qquad \gamma_2 = 0, \qquad \gamma_3 = \pm 1, \qquad \mu = J\lambda^2 + 35N\varepsilon^4 \tag{1.7}$$

$$\gamma_1 = 0, \qquad \gamma_2 = \pm\frac{1}{\sqrt{2}}, \quad \gamma_3 = \pm\frac{1}{\sqrt{2}}, \quad \mu = J\lambda^2 + 70N\varepsilon^4 \tag{1.8}$$

$$\gamma_1 = \pm\frac{1}{\sqrt{3}}, \quad \gamma_2 = \pm\frac{1}{\sqrt{3}}, \quad \gamma_3 = \pm\frac{1}{\sqrt{3}}, \quad \mu = J\lambda^2 + (35/3)N\varepsilon^4 \tag{1.9}$$

for an icosahedron

$$\gamma_1 = 0 \quad \gamma_2 = 0, \gamma_3 = \pm 1, \quad \mu = J\lambda^2 + 96(5 + \sqrt{5})N\varepsilon^6 \tag{1.10}$$

$$\gamma_1 = 0 \quad \gamma_2 = \pm\sqrt{\frac{3 - \sqrt{5}}{6}}, \quad \gamma_3 = \pm\sqrt{\frac{3 + \sqrt{5}}{6}}, \quad \mu = J\lambda^2 + \frac{272}{3}N\varepsilon^6 \tag{1.11}$$

$$\gamma_1 = 0 \quad \gamma_2 = \pm\sqrt{\frac{5 + \sqrt{5}}{10}}, \quad \gamma_3 = \pm\sqrt{\frac{5 - \sqrt{5}}{10}}, \quad \mu = J\lambda^2 + 208N\varepsilon^6. \tag{1.12}$$

for a dodecahedron

$$\gamma_1 = 0 \quad \gamma_2 = 0, \gamma_3 = \pm 1, \quad \mu = J\lambda^2 + 480N\varepsilon^6 \tag{1.13}$$

$$\gamma_1 = 0 \quad \gamma_2 = \pm\sqrt{\frac{5+\sqrt{5}}{10}}, \quad \gamma_3 = \pm\sqrt{\frac{5-\sqrt{5}}{10}}, \quad \mu = J\lambda^2 + 408N\varepsilon^6 \tag{1.14}$$

$$\gamma_1 = 0 \quad \gamma_2 = \pm\sqrt{\frac{3-\sqrt{5}}{6}}, \quad \gamma_3 = \pm\sqrt{\frac{3+\sqrt{5}}{6}}, \quad \mu = J\lambda^2 + \frac{1480}{3}N\varepsilon^6. \tag{1.15}$$

The symbol (123) denotes the circular rearrangement of indexes 123. Note that for $\lambda = 0$ these motions degenerate to an equilibrium position of the given bodies.

Using (1.1), we calculate distances $R_j$ and thereby the orientation of the considered bodies with respect to the attraction centre. There are only three types of orientation of the considered bodies. In the case of the first orientation type (solutions (1.4), (1.7), (1.10), (1.13)) the body is directed towards the attracting centre with its rib. These solutions are said to belong to type A. The radius-vector $\overline{R}$ then passes through the midpoints of this and the opposite ribs. It is obvious that the number of such solutions is equal to the number of ribs.

In the case of the second orientation type (solutions (1.5), (1.8), (1.11), (1.14)) the body is directed towards the attracting centre with its face. These solutions are of type B. It is characteristic of this orientation that the radius-vector $\overline{R}$ passes through the centre of this face and through the opposite vertex in the case of a tetrahedron and through the centres of parallel faces for the other bodies. The number of such solutions is equal to the number of faces of the given body.

In the case of the third orientation type (solutions (1.6), (1.9), (1.12), (1.15)) the body is directed towards the attracting centre with its vertex. These are solutions of type C. For this orientation the radius-vector $\overline{R}$ passes through this vertex and the centre of the opposite face in the case of a tetrahedron, and through the opposite vertex for the other bodies. The number of such solutions is equal to the number of vertices.

**2.** Stability is investigated with respect to the values $\omega_1, \omega_2, \omega_3, \gamma_1, \gamma_2, \gamma_3$. For this we calculate the second variation $\delta^2 W_i$ of the function $W$ on the linear manifolds $\delta K = 0$ and $\delta I = 0$.

For solutions (1.4), (1.7), (1.10), (1.13) these variations have respectively the form

$$\delta^2 W_1 = J\sum_{\nu=1}^{3}\Omega_\nu^2 + (2/\sqrt{3})N\varepsilon^3[(\delta\gamma_2)^2 - (\delta\gamma_3)^2],$$

$$\delta^2 W_2 = J\sum_{\nu=1}^{3}\Omega_\nu^2 + (140/9)N\varepsilon^4[4(\delta\gamma_2)^2 - (\delta\gamma_1)^2],$$

$$\delta^2 W_3 = J\sum_{\nu=1}^{3}\Omega_\nu^2 + (35/2)N\varepsilon^4[(\delta\gamma_1)^2 - 4(\delta\gamma_2)^2],$$

$$\delta^2 W_4 = J\sum_{\nu=1}^{3}\Omega_\nu^2 + 24(5+\sqrt{5})N\varepsilon^6[(\sqrt{5}+1)(\delta\gamma_1)^2 - (\sqrt{5}-1)(\delta\gamma_2)^2],$$

$$\delta^2 W_5 = J\sum_{\nu=1}^{3}\Omega_\nu^2 + 60N\varepsilon^6[(\sqrt{5}+1)(\delta\gamma_1)^2 - (\sqrt{5}-1)(\delta\gamma_2)^2].$$

From these expressions it follows that the instability degree is $\chi = 1$. Therefore stationary motions (respectively, relative equilibria) of type A are unstable.

For solutions (1.5), (1.8), (1.11), (1.14) the second variations have the form

$$\delta^2 W_1 = J\sum_{\nu=1}^{3}\Omega_\nu^2 - (4/\sqrt{3})N\varepsilon^3[3(\delta\gamma_2)^2 - (\delta\gamma_3)^2],$$

$$\delta^2 W_2 = J\sum_{\nu=1}^{3}\Omega_\nu^2 - (280/9)N\varepsilon^4[(\delta\gamma_1)^2 + (\delta\gamma_2)^2],$$

$$\delta^2 W_3 = J\sum_{\nu=1}^{3}\Omega_\nu^2 - (140/3)N\varepsilon^4[3(\delta\gamma_1)^2 + (\delta\gamma_2)^2],$$

$$\delta^2 W_4 = J\sum_{\nu=1}^{3}\Omega_\nu^2 - (16/3)N\varepsilon^6[2(\delta\gamma_1)^2 + 3(3-\sqrt{5})(\delta\gamma_2)^2],$$

$$\delta^2 W_5 = J\sum_{\nu=1}^{3}\Omega_\nu^2 - (48/5)N\varepsilon^6[2(\delta\gamma_1)^2 + (5+\sqrt{5})(\delta\gamma_2)^2].$$

Thus for stationary motions of type B the instability degree is $\chi = 2$. In this case the Routh theorem and its inversion do not allow one to conclude whether the stationary motion is stable or unstable, but from the consideration of equations of the first approximation it follows that stationary motions of type B are unstable.

For solutions (1.6), (1.9),(1.12), (1.15) the second variations have the form

$$\delta^2 W_1 = J\sum_{\nu=1}^{3}\Omega_\nu^2 - (4/\sqrt{3})N\varepsilon^3[(\delta\gamma_1)^2 + 3(\delta\gamma_2)^2],$$

$$\delta^2 W_2 = J\sum_{\nu=1}^{3}\Omega_\nu^2 + (1120/27)N\varepsilon^4[3(\delta\gamma_1)^2 + (\delta\gamma_2)^2],$$

$$\delta^2 W_3 = J\sum_{\nu=1}^{3}\Omega_\nu^2 + 35N\varepsilon^4[(\delta\gamma_1)^2 + (\delta\gamma_2)^2],$$

$$\delta^2 W_4 = J\sum_{\nu=1}^{3}\Omega_\nu^2 + (48/5)N\varepsilon^6[2(\delta\gamma_2)^2 + (5+\sqrt{5})(\delta\gamma_2)^2],$$

$$\delta^2 W_5 = J\sum_{\nu=1}^{3}\Omega_\nu^2 + (16/3)N\varepsilon^6[2(\delta\gamma_1)^2 + 3(3-\sqrt{5})(\delta\gamma_2)^2].$$

From these expressions it follows that the instability degree is $\chi = 0$ and therefore stationary motions of type C are stable. Here $\Omega_\nu = \delta\omega_\nu - \lambda\delta\gamma_\nu$, where $\delta\omega_\nu$ and $\delta\gamma_\nu$ are variations of the variables $\omega_\nu$ and $\gamma_\nu$ $(\nu = 1,2,3)$.

From the foregoing discussion we make an interesting conclusion: for Plato's bodies the dimension of a body element (vertex, rib, face), with which the bodies are directed towards the attraction centre, coincides with a degree of instability.

Our results are also valid for bodies of the mentioned type on the circular orbit. There are no centrifuga and Coriolis moments of force in the orbital coordinate system because all principal moments of body inertia are equal [3].

3. The bifurcation diagram on the plane of constants of the energy integral $h$ and the area integral $k$ consists of three parabolas $P_\alpha$ $(\alpha = 0,1,2)$ (Fig. 1) which are determined by the relations

$$h_\alpha = k^2/2J - I_i^{(\alpha)} \quad (i = \overline{1,5};\ \alpha = 0,1,2) \qquad (1.16)$$

where to the index $\alpha$ there correspond instability degrees of the considered stationary motions and equalibria, on solutions C, B and A.

These parabolas correspond to stationary motions (1.4) to (1.15) whose vertices correspond to the equilibrium positions of Plato's bodies.

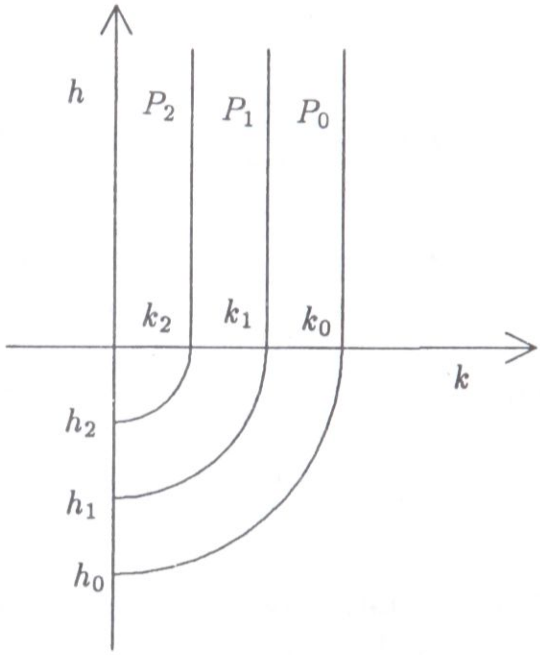


Fig. 1

Parabolas (1.16) form a bifurcation set on which the rearrangement of possible motion domains takes place [4]; these domains are determined by the relation

$$-U_i \le h \qquad (i = \overline{1,5}).$$

Note that in the given problem the altered potential energy coincides to within a constant with the potential energy.

The analysis shows that above the parabola $P_2$ the domain of possible motions if the entire sphere $S^2 = \{\gamma_1, \gamma_2\gamma_3;\ \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1\}$.

Between the parabolas $P_2$ and $P_1$ the domain of possible motions is a sphere with holes whose number is four $\left(S \setminus \bigcup_1^4 D_\nu\right)$ for a tetrahedron, six $\left(S^2 \setminus \bigcup_1^6 D_\nu\right)$ for a cube, eight $\left(S^2 \setminus \bigcup_1^8 D_\nu\right)$ for an octahedron, twenty $\left(S^2 \setminus \bigcup_1^{20} D_\nu\right)$ for an icosahedron, twelve $\left(S^2 \setminus \bigcup_1^{12} D_\nu\right)$ for a dodecahedron.

Between the parabolas $P_1$ and $P_0$ domains of possible motions are disks whose number is equal to four $\left(\bigcup_1^4 D_\nu\right)$ for a tetrahedron, eight $\left(\bigcup_1^8 D_\nu\right)$ for a cube, six

$\left(\bigcup_1^6 D_\nu\right)$ for an octahedron, twelve $\left(\bigcup_1^{12} D_\nu\right)$ for an icosahedron, twenty $\left(\bigcup_1^{20} D_\nu\right)$ for a dodecahedron.

Below the parabola $P_0$ no motion can occur.

Note that for the considered problem the results obtained are presented in greater detail in the author's papers [5,6,7].

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## О СТАЦИОНАРНЫХ ДВИЖЕНИЯХ ТЕЛ ПЛАТОНА В ПОЛЕ СИЛ ТЯГОТЕНИЯ


Рассматриваются механические системы, состоящие из материальных точек равных масс, расположенных в вершинах правильных многогранников (тел Платона) и соединенных невесомыми недеформируемыми стержнями.

Исследуются задача о влиянии моментов инерции высших порядков (т.е. с учетом свойств слагаемых высших порядков в разложении потенциала) на движение этих тел, закрепленных в центре масс в центральном ньютоновском поле сил.

Найдены стационарные движения и равновесные положения и исследована их устойчивость. Приведена бифуркационная диаграмма на плоскости констант интегралов энергии и площадей.

Отмечается любопытный факт: для рассматривамых тел размерность элемента тела (вершина, ребро, грань), которым оно обращено на стационарных движениях и равновесных положениях к притягивающему центру, совпадает со степенью неустойчивости.



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