TEORIJSKA I PRIMENJENA MEHANIKA 15, pp 107–117, 1989, UDK 531.314

AN ALGORITHM FOR DETERMINING LAGRANGE'S FUNCTION BY USING THE VARIATIONAL PRINCIPLE WITH VANISHING PARAMETER

Dušan Stokić

(Received 21.10.1988; Revised 19.11.1989)

Introduction. Among various direct methods of variational calculations for approximate solving of mathetatical models of physical processes, an important place is taken by a vanishing parameter principle of B. Vujanović [1], [2], [3]. It was originally developed for the purpose of studying the problem of non-stationary heat conduction and it soon turned out that it can be successfully applied to the study of the boundary layer problems [4], [5], [6] as well as of those in other fields of physics [7], [8]. Although it quickly leads to highly accurate approximate solutions, it still sets the serious problem of determining Lagrange's function, that is, of that part of the action integral which is to reproduce the process equation from the stationarity conditions and probably the natural boundary conditions as well. Not only that there is no universal algorithm for determining Lagrange's function, but for the same equation many functionals can be defined and introduced into the action integral thus leading to the setting-up of the process equation from the stationarity conditions. The question still remains which of these functionals is optimal in view of obtaining the best approximate solution.

This paper presents a further elaboration of Y. T. Glazunov's ideas [8]. It presents a possibility of automatic creation of Lagrange's density for certain classes of partial differential equations used to describe many non-stationary and nonlinear processes where the applied procedure is prescribed by a vanishing parameter principle.

1. Determination of Lagrange's Functions. The stationarity condition of the action integral is

$$I = \int_t \int_V L \, dV dt, \tag{1.1}$$

where L is Lagrange's function, V is the field in which the process is being studied with standard restrictions for variations on the contour S of the field V, leading to Euler-Lagrange's equation

$$\frac{\partial L}{\partial f} - \frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{f}} + \frac{\partial^2}{\partial t^2}\frac{\partial L}{\partial \ddot{f}} + \sum_{i=1}^{3}\sum_{j=1}^{n}(-1)^j \frac{\partial^j}{(\partial x_i^j)} \frac{\partial L}{\partial\left[\dfrac{\partial^j f}{(\partial x_i)^j}\right]} = 0. \tag{1.2}$$

In the equation (1.2) $f = f(x_i, t)$ is a physical quantity to be studied, the points denote its derivatives with respect to the time t, x_i are the space coordinates, whereas n denotes the order of the highest derivative dominating in Lagrange's function, that is, the one of the function f with respect to the space coordinates.

Lagrange's function of the form

$$L = \frac{k^2}{2a(f)}\left\{a(f)\frac{\partial^2 f}{\partial t^2} + b(f)\frac{\partial f}{\partial t} - D[f(x_i,t);x_i;t]\right\}^2 e^{t/k}, \tag{1.3}$$

where k is a vanishing parameter, $D[f(x_i,t);x_i;t]$ denotes a certain differential dependence of the field $f(x_i,t)$ on the space coordinates x_i, introduced into Euler-Lagrange equation (1.2), which, carrying out the procedure prescribed by a vanishing parameter principle, leads to the equation of the form

$$a(f)\frac{\partial^2 f}{\partial t^2} + b(f)\frac{\partial f}{\partial t} - D[f(x_i,t);x_i;t] = 0. \tag{1.4}$$

The evidence for this statement is quite simple. For brevity's sake, let

$$\Phi(f) = a(f)\frac{\partial^2 f}{\partial t^2} + b(f)\frac{\partial f}{\partial t} - D[f(x_i,t);x_i;t]. \tag{1.5}$$

The derivatives domineering Euler-Lagrange's equation (1.2) are

$$\frac{\partial L}{\partial\left[\dfrac{\partial^j f}{(\partial x_i)^j}\right]} = \frac{k^2}{2}\frac{\partial}{\partial\left[\dfrac{\partial^j f}{(\partial x_i)^j}\right]}\left[\frac{1}{a(f)}\Phi^2(f)\right]e^{t/k},$$

$$\frac{\partial}{\partial t}\frac{\partial L}{\partial \dot f} = k^2\frac{\partial}{\partial t}\left[\frac{b(f)}{a(f)}\Phi(f)\right]e^{t/k} + k\left[\frac{b(f)}{a(f)}\Phi(f)\right]e^{t/k},$$

$$\frac{\partial^2}{\partial t^2}\frac{\partial L}{\partial \ddot f} = k^2\frac{\partial^2}{\partial t^2}[\Phi(f)]e^{t/k} + 2k\frac{\partial}{\partial t}[\Phi(f)]e^{t/k} + \Phi(f)e^{t/k},$$

and when they are brought into it, they lead to

$$\begin{aligned}\frac{k^2}{2}\frac{\partial}{\partial f}\left[\frac{1}{a(f)}\Phi^2(f)\right]e^{t/k} - k^2\frac{\partial}{\partial t}\left[\frac{b(f)}{a(f)}\Phi(f)\right]e^{t/k} - k\left[\frac{b(f)}{a(f)}\Phi(f)\right]e^{t/k}\\ + k^2\frac{\partial^2}{\partial t^2}[\Phi(f)]e^{t/k} + 2k\frac{\partial}{\partial t}[\Phi(f)]e^{t/k} + \Phi(f)e^{t/k}\\ + \frac{k^2}{2}\left\{\sum_{i=1}^{3}\sum_{j=1}^{n}(-1)^j\frac{\partial^j}{(\partial x_i)^j}\frac{\partial}{\partial[\partial^j f/(\partial x_i)^j]}\left[\frac{1}{a(f)}\Phi(f)\right]\right\}e^{t/k} = 0.\end{aligned}$$

By dividing the obtained equation by $\exp(t/k)$, and then by carrying out the boundary procedure that $k \to 0$ as it prescribed by a vanishing parameter principle, we obtain $\Phi(f) = 0$, that is, with regard to (1.5)

$$a(f)\frac{\partial^2 f}{\partial t^2} + b(f)\frac{\partial f}{\partial t} - D[f(x_i,t);x_i;t] = 0,$$

we obtain exactly the equation (1.4).

By applying the same procedure on Lagrange's function of the form

$$L = \frac{k}{2b(f)}\left\{ b(f)\frac{\partial f}{\partial t} - D[f(x_i,t);x_i;t]\right\}^2 e^{t/k}, \tag{1.6}$$

we come to the equation

$$b(f)\frac{\partial f}{\partial t} - D[f(x_i,t);x_i;t] = 0. \tag{1.7}$$

Therefore, we can conclude that the equations of the form (1.4) that is, of the form (1.7) are truly obtained from the action integral stationarity conditions (1.1) by applying Lagrange's functions (1.3) or (1.6), and after carrying out the procedure prescribed by a vanishing parameter principle. Thus the problem of determining Lagrange's function for the given classes of non-linear and non-stationary partial differential equations should not be posed any longer, but the question of choosing a trial function which is to reproduce a sufficiently good approximate solution still remains.

2. Approximate Solution. The approximate solution is required to satisfy both the process equation and the boundary and initial conditions as well, within the limits of the prescribed accuracy. For different processes, no matter their being described by the same equation, there are different boundary conditions and this, in general, prevents defining of an algorithm showing, in all cases, the way to attain an approximate solution of sufficient accuracy. As for the choice of a trial function, there are certain recommendations (methods) widely used today, such as Fourier's method of variables' separation, Ritz's method, Kantorovich's method of partial integration and others.

The effective use of the suggested procedure for determining approximate solutions of the processes described by the equations (1.4) or (1.7) with respective boundary and initial conditions, requires a trial function in the form that allows integration with respect to the space coordinates when Lagrange's function is formed in the above described way and then introduced into the action integral. Then the integration is performed with respect to the space V and after that the action integral is reduced to the form

$$I = \int_{t_1}^{t_2} L^*(q_i, \dot{q}_i \ddot{q}_i)dt. \tag{2.1}$$

Lagrange's function in the integral (2.1) depends on the time functions $q_i(t)$ and their derivatives which correspond to the independent coordinates of the mechanical system. The integral stationarity condition (2.1) leads to classic Lagrange's equations

$$\frac{\partial L^*}{\partial q_i} - \frac{d}{dt}\frac{\partial L^*}{\partial \dot{q}_i} + \frac{d^2}{dt^2}\frac{\partial L^*}{\partial \ddot{q}_i} = 0, \tag{2.2}$$

which reproduce a system of ordinary differential equations with respect to the unknown functions $q_i(t)$ combined with the space functions in a trial solution.

The trial function should be thus formed as to be specified with respect to the space coordinates, whereas the time functions $q_i(t)$ should remain unknown. While considering this function all known, information about the process, obtained from various sources (mostly by experiment) should be brought in, whereas in numerous situations the "smoothness" conditions are precious signposts directing us towards a good choice of functions. It is very useful to make the space coordinates functions satisfy the boundary conditions since in this case they need not be taken care of any longer. The required integration with respect to the space coordinates can be carried out by some numerical method if the form of these functions does not allow a closed-from integration.

The initial conditions are used for determining the integration constants in the general solutions of the ordinary differential equations system obtained fro Lagrange's equations (2.2).

The enclosed illustrative examples should prove the right choice of Lagrange's functions in the forms (1.3) and (1.7) for obtaining approximate solutions of the processes described by the equations (1.4) and (1.6). The common notation for function derivatives is being used in the following examples: namely, a point over a function denotes all its derivatives with respect to time, whereas the sign "′" denotes derivatives with respect to the space coordinates.

3. Application to Some Problems of Non-Linear Theory of Oscillations.

3.1. *Non-linear torsional oscillations of a cylindrical bar* [10]. The equation of these oscillations is of the form

$$\ddot{\theta} - \alpha^2[1 + \lambda(\theta')^2]\theta'' = 0, \tag{3.1.1}$$

and boundary conditions

$$\theta(0,t) = 0 \quad \text{and} \quad \theta(\pi,t) = 0. \tag{3.1.2}$$

According to (1.3) Lagrange's function is accepted as

$$L = \frac{k^2}{2}\left\{\ddot{\theta} - \alpha^2[1 + \lambda(\theta')^2]\theta''\right\}^2 e^{t/k}, \tag{3.1.3}$$

and regarding the boundary conditions (3.1.2), the trial function is of the form

$$\theta(x,t) = f(t)\sin mx, \qquad m = 1, 2, \dots . \tag{3.1.4}$$

By introducing the trial function into the Lagrangian (3.1.3) we obtain

$$L = \frac{k^2}{2}\left[\ddot{f}\sin mx + \alpha^2 m^2(1 + \lambda m^2 f^2\cos^2 mx) f \sin mx\right]^2 e^{t/k}, \tag{3.1.5}$$

and the action integral (1.1) after the integration with respect to x from 0 to π is reduced to

$$I = \int_0^t \frac{k^2\pi}{2}\left[\frac{1}{2}\ddot{f}^2 + \alpha^2 m^2 f\ddot{f}\left(1 + \frac{1}{4}\lambda m^2 f^2\right) + \frac{1}{2}\alpha^4 m^4 f^2\left(1 + \frac{1}{2}\lambda m^2 f^2 + \frac{1}{8}\lambda^2 m^4 f^4\right)\right] e^{t/k}\, dt.$$

The obtained action integral responding to the integral (2.1) will have a stationary value when Euler-Lagrange equation (2.2) is satisfied, that is, the equation in which, in this case, the Lagrangian L^* is of the form

$$L^* = \frac{k^2\pi}{2}\left[\frac{1}{2}\dot{f}^2 + \alpha^2 m^2 f\ddot{f}\left(1+\frac{1}{4}\lambda m^2 f^2\right)\right.$$
$$\left. + \frac{1}{2}\alpha^4 m^4 f^2\left(1+\frac{1}{2}\lambda m^2 f^2 + \frac{1}{8}\lambda^2 m^4 f^4\right)\right]e^{t/k}. \tag{3.1.6}$$

By determining the necessary derivatives from (3.1.6), Euler-Lagrange's equation (2.2) leads to

$$\frac{k^2\pi}{2}\left[\alpha^2 m^2 \ddot{f}\left(1+\frac{3}{4}\lambda m^2 f^2\right) + \alpha^4 m^4 f\left(1+\frac{1}{2}\lambda m^2 + \frac{3}{8}\lambda^2 m^4 f^4\right)\right]e^{t/k}$$
$$+\frac{k^2\pi}{2}\frac{d^2}{dt^2}\left[\ddot{f} + \alpha^2 m^2 f\left(1+\frac{1}{4}\lambda m^2 f^2\right)\right]e^{t/k}$$
$$+k\pi\frac{d}{dt}\left[\ddot{f} + \alpha^2 m^2 f\left(1+\frac{1}{4}\lambda m^2 f^2\right)\right]e^{t/k} \tag{3.1.7}$$
$$+\frac{\pi}{2}\left[\ddot{f} + \alpha^2 m^2 f\left(1+\frac{1}{4}\lambda m^2 f^2\right)\right]e^{t/k} = 0.$$

When the obtained equation (3.1.7) is divided by $e^{t/k}$ and when the boundary process that $k \to 0$ is carried out, as it is prescribed by the applied method, we come to

$$\ddot{f} + \alpha^2 m^2 f' + (1/4)\lambda\alpha^2 m^2 f^3 = 0. \tag{3.1.8}$$

differential equation with respect to f being identical to those obtained by Galerkin's method and the further procedure for final determination of the approximate solution is the same as in this method [10]. We will not go into details about the method here since our task is to come to the equation (3.1.8) on the basis of which the function f(t) is obtained in order to come to the approximate solution by means of the trial function (3.1.4).

3.2. *Transferse oscillations of a non-linear elastic beam* [10]. The differential equation is of the form

$$\ddot{u} + a^2 u'''' + a^2\lambda[u''''u'' + 2(u''')^2]u'' = 0, \tag{3.2.1}$$

the boundary conditions are

$$u(0) = u(\pi) = 0 \quad \text{and} \quad u''(0) = u''(\pi) = 0, \tag{3.2.2}$$

and the initial conditions are

$$u(x,0) = A \quad \text{and} \quad \dot{u}(x,0) = 0. \tag{3.2.3}$$

The trial function is taken to be in the form

$$u(x,t) = f(t)\sin mx, \tag{3.2.4}$$

because of the boundary conditions (3.2.2), whereas the initial conditions lead to

$$f(0) = Aq \quad \text{and} \quad \dot{f}(0) = 0. \tag{3.2.5}$$

Lagrange's function is of the form

$$L\frac{k^2}{2}\Big\{\ddot{u} + a^2 u'''' + a^2\lambda\big[u''''u'' + 2(u''')^2\big]u''\Big\}^2 e^{t/k}. \tag{3.2.6}$$

By determining the required derivatives from the trial function (3.2.4), Lagrange's function (3.2.6) becomes

$$L = \frac{k^2}{2}\{A\sin mx - B(2\cos^2 mx - \sin^2 mx)\sin mx\}^2 e^{t/k}, \tag{3.2.7}$$

where

$$A = \ddot{f} + a^2 m^4 f \quad \text{and} \quad B = a^2\lambda m^8 f^3. \tag{3.2.8}$$

After the integration with respect to x from 0 to π, the action integral reduces to

$$I = \int_0^t \frac{k^2\pi}{2}\left[\frac{1}{2}A^2 + \frac{1}{4}AB + \frac{5}{16}B^2\right]e^{t/k} \tag{3.2.9}$$

and has stationary value when Euler-Lagrange's equation(2.2) is satisfied, that is, the equation in which

$$L^* = \frac{k^2\pi}{2}\left[\frac{1}{2}A^2 + \frac{1}{4}AB + \frac{5}{16}B^2\right]e^{t/k}. \tag{3.2.10}$$

By determining the derivatives from (3.2.10), introduced them into (2.2), we obtain

$$\frac{k^2\pi}{2}\left[A\frac{\partial A}{\partial f} + \frac{1}{4}\left(B\frac{\partial A}{\partial f} + A\frac{\partial B}{\partial f}\right) + \frac{5}{8}B\frac{\partial B}{\partial f}\right]e^{t/k} \tag{3.2.11}$$
$$+\left[\frac{k^2\pi}{2}\frac{d^2}{dt^2}\left(A + \frac{1}{4}B\right) + k\pi\frac{d}{dt}\left(A + \frac{1}{4}B\right) + \frac{\pi}{2}\left(A + \frac{1}{4}B\right)\right]e^{t/k} = 0.$$

Dividing the obtained equation by $e^{t/k}$, and by carrying out boundary procedure that $k \to 0$, we obtain

$$A + (1/4)B = 0 \tag{3.2.12}$$

which, after introducing the values for A and B from (3.2.8) leads to

$$\ddot{f} + a^2 m^4 f + (1/4)a^2\lambda m^8 f^3 = 0,$$

ordinary differential equation with respect to f identical with the equation obtained by Galerkin's method. The further procedure for determining the function f is known and it will not be pressented here.

4. Application to Non-Stationary Heat Conduction. The general form of the equation of the heat conduction through an isotropic solid is

$$\frac{\partial}{\partial t}\big[u(T)T\big] = \operatorname{div}\big[\lambda(T)\operatorname{grad}T\big], \tag{4.1}$$

where $T = T(x_i, t)$ is a temperature field, $u(T)$ is internal energy and $\lambda(T)$ is a heat conductivity coefficient. Supposing that the velocity of the internal energy change is small, $\partial u/\partial t \to 0$, the equation (4.1) is reduced to

$$u(T)\frac{\partial T}{\partial t} = \operatorname{div}\left[\lambda(T)\operatorname{grad}T\right]. \tag{4.2}$$

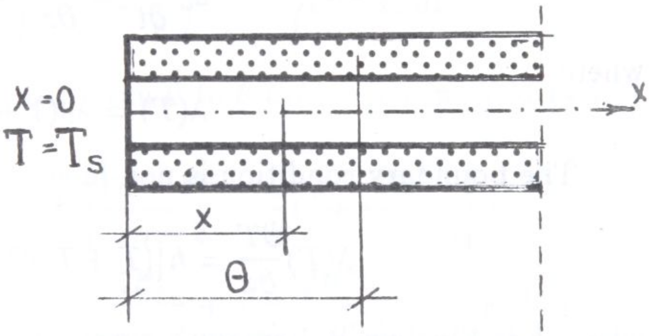
The equation (4.2) belongs to the class of the type (1.7) and therefore, according to the expression (1.6), its Lagrange's function is

$$L = \frac{k}{2u(T)}\left\{u(T)\frac{\partial T}{\partial t} - \operatorname{div}\left[\lambda(T)\operatorname{grad}T\right]\right\}^2 e^{t/k}. \tag{4.3}$$

When Euler-Lagrange equation is formed by means of the Lagrangian (4.3) and when the procedure prescribed by a vanishing parameter principle is carried out, the process equation (4.2) is obtained.

4.1. *Heat conduction through laterally isolated semiinfinite solid with the front surface kept at the constant temperature T_s.* A semiinfinite bar is laterally isolated and it has constant termo-physical characteristics. The process equation (4.2) in this case is reduced to

$$\frac{\partial T}{\partial t} - \alpha\frac{\partial^2 T}{\partial x^2} = 0,$$



therefore the action integral is of the form

$$I = \int_0^t \int_0^\theta \frac{k}{2}\left(\frac{\partial T}{\partial t} - \sigma\frac{\partial^2 T}{\partial x^2}\right)^2 \times$$

$$\times e^{t/k}\, dx\, dt, \tag{4.1.1}$$

where $\theta = \theta(t)$ is the penetration depth. The area $x > \theta$ is not within the reach of the temperature shange. The initial temperature is equal to zero and the boundary conditions are

$$T(0,t) = T_s = \text{const}, \qquad T(\theta,t) = 0, \qquad \frac{\partial T(\theta,t)}{\partial x} = 0. \tag{4.1.2}$$

By assuming that the trial function $T(x,t)$ has the form of the second order polynomial and by respecting the boundary conditions (4.1.2), the trial function is written in the form

$$T = T_s(1 - x/\theta)^2. \tag{4.1.3}$$

By introducing this function into the action integral (4.1.1) and by integration with respect to x from 0 to θ we obtain

$$I = \int_0^t \frac{k}{2}\left(\frac{\theta^3}{30}A^2 - \frac{1}{3}\theta^2 AB + B^2\theta\right)e^{t/k}\, dt \tag{4.1.4}$$

where $A = 2T_a\dot{\theta}/\theta^2$ and $B = 2\alpha T_s/\theta^2$.

The integral (4.1.4) will have a stationary value when Euler-Lagrange equation

$$\frac{\partial L^*}{\partial \theta} - \frac{d}{dt}\frac{\partial L^*}{\partial \dot{\theta}} = 0 \tag{4.1.5}$$

is satisfied, when

$$L^* = \frac{k}{2}\left(\frac{1}{30}\theta^3 A^2 - \frac{1}{3}\theta^2 AB + \theta B^2\right)e^{t/k}. \tag{4.1.6}$$

When the equation (4.1.5) is formed by means of (4.1.6), and when it is divided by $e^{t/k}$ and the boundary process $k \to 0$ is carried out, we obtain $\theta\dot{\theta} = 5\alpha$, hence, $\theta = \sqrt{10\alpha\theta}$, and the solution is identical to the one known in the references [3], [11]. By putting the obtained function $\theta = \theta(t)$ into the trial solution (4.1.3), the approximate solution is obtained.

4.2. *The case of constant internal energy and a heat conductivity coefficient lineary dependent on temperature.* The differential equation in this case is of the form

$$u_0\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left[\lambda(T)\frac{\partial T}{\partial x}\right], \tag{4.2.1}$$

where

$$\lambda(T) = \lambda_0(1 + \beta T/T_0). \tag{4.2.2}$$

The boundary condition is non-linear and of radiation form

$$\lambda(T)\frac{\partial T}{\partial x} = h\left[(T + T_0)^m - \theta_0^m\right], \quad \text{for } x = 0 \tag{4.2.3}$$

where h is Einstein-Boltzmann's constant, θ_0 is an initial apsolute temperature of the external surroundings, whereas m defines the body to be studied ($1 < m < 4$ — grey, $m = 4$ — apsolutely black body).

The trial function is taken in the form

$$T(x,t) = -(T_0 - q)(1 - x/\theta)^n, \tag{4.2.4}$$

where T_0 is an initial body temperature, $q = q(t)$ is a surface temperature and $\theta = \theta(t)$ is the penetration depth. Because of thus accepted trial function the heat conductivity coefficient (4.2.2) becomes

$$\lambda(T) = \lambda_0\left[1 - \frac{\beta}{T_0}(T_0 - q)\left(1 - \frac{x}{\theta}\right)^n\right], \tag{4.2.5}$$

whereas Lagrange's function (4.5) after introducing (4.2.4) and (4.2.5) attains the form

$$L = \frac{k}{2}\left\{\dot{q}\left(1 - \frac{x}{\theta}\right)^n - \frac{n(T_0 - q)}{\theta^2}\dot{\theta}\left(1 - \frac{x}{\theta}\right)^{n-1}x + \right. \tag{4.2.6}$$

$$+\frac{\lambda_0}{u_0}\frac{n(T_0-q)}{\theta^2}\left[(n-1)\left(1-\frac{x}{\theta}\right)^{n-2}-\frac{\beta}{T_0}(T_0-q)(2n-1)\left(1-\frac{x}{\theta}\right)^{2n-2}\right]\Bigg\}^2 e^{t/k}.$$

By introducing the notations

$$A=\frac{n(T_0-q)}{\theta^2}\dot{\theta};\quad B=\frac{\lambda_0}{u_0}\frac{n(T_0-q)}{\theta^2}\quad\text{and}\quad C=\frac{\beta}{T_0}(2n-1)(T_0-q)\tag{4.2.7}$$

the action integral

$$I=\int_{t_1}^{t_2}\int_0^{\theta}L\,dx\,dt,\tag{4.2.8}$$

where L is given by (4.2.6), becomes

$$\begin{aligned}I=\int_{t_1}^{t_2}\int_0^{\theta}\frac{k}{2}\Bigg\{&\dot{q}^2\left(1-\frac{x}{\theta}\right)^{2n}+A^2\left(1-\frac{x}{\theta}\right)^{2n-2}x^2\\&+B^2\left[(n-1)^2\left(1-\frac{x}{\theta}\right)^{2n-4}-2C(n-1)\left(1-\frac{x}{\theta}\right)^{3n-4}+C^2\left(1-\frac{x}{\theta}\right)^{4n-4}\right]\\&-2A\dot{q}\left(1-\frac{x}{\theta}\right)^{2n-1}x+2B\dot{q}\left[(n-1)\left(1-\frac{x}{\theta}\right)^{n-2}-C\left(1-\frac{x}{\theta}\right)^{2n-2}\right]\\&-2AB(n-1)\left(1-\frac{x}{\theta}\right)^{2n-3}x+2ABC\left(1-\frac{x}{\theta}\right)^{3n-3}x\Bigg\}e^{t/k}\,dx\,dt.\end{aligned}\tag{4.2.9}$$

After the integration with respect to x from 0 to θ the action integral (4.2.9) is reduced to

$$\begin{aligned}I=\int_{t_1}^{t_2}\frac{k\theta}{2}\Bigg\{&\frac{\dot{q}^2}{2n+1}+\frac{A^2\theta^2}{n(4n^2-1)}+B^2\left[\frac{(n-1)^2}{2n-3}-\frac{2}{3}C+\frac{C^2}{4n-3}\right]\\&-\frac{A\dot{q}\theta}{n(2n+1)}+2B\dot{q}\left(1-\frac{C}{2n-1}\right)-\frac{AB\theta}{2n-1}+\frac{2ABC\theta}{(3n-1)(3n-2)}e^{t/k}\,dt,\end{aligned}$$

hence Lagrange's function L is directly obtained and used to obtain Euler-Lagrange equation for the coordinate θ. That is virtually whole subintegral expression (without dt). By determining the required derivatives dominating Euler-Lagrange's equation for the function L, by carrying out the prescribed procedure and by introducing the notations (4.2.7) we finally obtain

$$\frac{2(T_0-q)}{4n^2-1}\frac{\dot{\theta}}{\theta}-\frac{\dot{q}}{n(2n+1)}-\left[\frac{1}{2n-1}+\frac{\beta}{T_0}\frac{2(2n-1)(T_0-q)}{(3n-1)(3n-2)}\right]\frac{\lambda_0}{u_0}\frac{n(T_0-q)}{\theta^2}=0,$$

an ordinary differential equation with respect to unknown time functions θ and q. In order to solve the set-up problem it is necessary to find another equation or some connection between the unknown functions.

The Lardner's suggestion has been accepted according to which the connection between functions $\theta(t)$ and $q(t)$ is made by means of the boundary condition (4.2.3). By using (4.2.4) and (4.2.5) this connection is

$$\theta=\frac{\lambda_0 n(T_0-q)}{h}\cdot\frac{1-(\beta/T_0)(T_0-q)}{q^m-v_0^m}.\tag{4.2.11}$$

When it has been used and after non-dimensional quantities have been introduced

$$z = q/T_0 \quad \text{and} \quad z_0 = v_0/T_0$$

the differential equation (4.2.10) becomes

$$-\frac{1}{2n+1} + \frac{2n}{4n^2-1}\left[\frac{2(1-z)-1}{1-(1-z)} - \frac{m(1-z)z^{m-1}}{z^m - z_0^m}\right]$$
$$= \frac{(z^m - z_0^m)^2}{(1-z)[1-\beta(1-z)]^2}\left[\frac{1}{2n-1} + 2\frac{(2n-1)(1-z)}{(3n-1)(3n-2)}\right]\frac{d\tau}{dz}. \qquad (4.2.12)$$

where $\tau = \left(h^2 T_0^{2(m-1)}/u_0 v_0\right)t$.

Studying the same problem with specified values: $\beta = 0$, $n = 2$, $m = 4$ and $z_0 = 0$, Rafalsky and Zyskowsky [13] have used Biot's method and obtained the approximate solution

$$(1 - 88\tau)z^8 + 24.5z^2 - 57z + 31.5 = 0. \qquad (4.2.13)$$

When the same parameter values introduced into the equation (4.2.12) and after the integration is performed by considering condition $q(0) = T_0$ the aproximate solution is obtained in the form

$$(1 - 70\tau)z^8 + 21z^2 - 50z + 28 = 0,$$

which is identical to the solution obtained in the reference [14].

All these examples show:

1. that the equations of the form (1.4) and (1.7) are obtained by using Lagrange's functions of the form (1.3) and (1.6) from the action integral stationary conditions and by applying the procedure prescribed by a vanishing parameter principle;

2. that the approximate solution of the non-stationary and non-linear processes is relatively simple. Although the given cases are one-dimensional, due to the computer's ability to perform a numerical integration of complex integrals, the described procedure can be used to slove two- and three-dimensional spatial processes. The greatest trouble practically lies in the integration with respect to the field V, but it can be eliminated when computers are being used.

REFERENCES

[1] Vujanović B.: *An approach to linear and nonlinear heat-transfer problem using a Lagrangian*, AIAA J. 1, Vol. 9, No. 1, 1971.

[2] Vujanović B., Đukić Đ.: *On one variational principle of Hamilton's type for nonlinear heat-transfer problem*, Int. J. Heat Mass Transfer, 1972.

[3] Vujanović B.: *Metodi optimizacije*, RU "Radivoj Ćirpanov", Novi Sad 1980.

[4] Vujanović B., Đukić Đ.: *Variational principle for the theory of laminar boundary layers in incompressible fluida*, Publications de l'Institut Mathématique, T. 11 (25), Beograd 1971.

[5] Vujanović B., Đukić Đ., Pavlović M.: *A variational principle for the laminar boundary layer theory*, Bollettino U.M.I. (4) 7, 1973.

[6] Boričić Z., Nikodijević D.: *The research of the flow in the plane magnetohydrodynamic boundary layer around the body of the arbitrary shape*, Proceedings of the sixth conference on fluid machinery, Budapest, Akadémiai kiadó, Vol. 1. 1979.

[7] Borčić Z., Nikodijević D.: *Primena metode varijacije na izučavanje osnosimetričnog MHD graničnog sloja na obrtnim telima pri strujanju fluida promenljive provodnosti*, Zbornik radova Matematičkog instituta, N.S. 4 (12), Beograd, 1984.

[8] Glazunov Y. T.: *Variacionij princip tipa Gamiljtona nelinejnih javlenij vzaimosvjazannogo perenosa*, Inženerno-fizičeskij žurnnal, Tom XXXIX, No. 3, 1980.

[9] Schechter R. S.: *The Variational Methods in Engineering*, Mc Graw-Hill, Series in Chemical engineering, New York, 1967.

[10] Kauderer H.: *Nichtlineare Mechanik*, Springer, Berlin, 1958.

[11] Carlslaw H. S., Jaeger J. C.: *Conduction of Heat in Solids*, Sec. Ed. Oxford at the Clarendon Press, 1959.

[12] Lardner T. J.: *Biot's Variational Principle in Heat Conduction*, AIAA J. Vol. 1, 1963.

[13] Rafalsky P., Zyskowsky M.: *Lagrangian Approach to the Non-Linear Boundary Heat-Transfer Problem*, AIAA J. Vol. 6, 1968.

[14] Stokić D. B.: *Radiation of solids with thermally changeable conductivity*, Publications de l'Institut Mathématique, T. 16 (30) 1973, Beograd.

[15] Stokić D. B.: *Primena varijacione metode pri proučavanju prenošenja toplote zračenjem kod čvrstih tela sa termički promenljivim svojstvima*, doktorski rad, Prirodno-matematički fakultet, Beograd 1975.

ОДИН АЛГОРИТМ ДЛЯ ОПРЕДЕЛЕНИЯ ФУНКЦИИ ЛАГРАНЖА ПРИ ИСПОЛЬЗОВАНИИ ВАРИАЦИОННОГО ПРИНЦИПА С ИСЧЕЗАЮЩИМ ПАРАМЕТРОМ

При использовании прямых методов вариационного расчета для приближенного решения дифференциальных уравнений в частных производных, с помощью которых описаны многие физические процессы, определение функции Лагранжа представлает серьезную трудность.

В статье рассматривается возможность прямой продукции функции Лагранжа из равенства процесса, когда для приближенного решения уравнений процессов применяют вариационный принцип с исчезающим параметром Б. Вуяновича. С помощью четырех иллюстративных примеров обнаружен способ употребления данного примера.

JEDAN ALGORITAM ZA ODREĐIVANJE LAGRANŽEVE FUNKCIJE PRI KORIŠĆENJU VARIJACIONOG PRINCIPA SA IŠČEZAVAJUĆIM PARAMETROM

Pri korišćenju direktnih metoda varijacionog računa za aproksimativno rešavanje parcijalnih diferencijalnih jednačina kojima su opisani mnogi fizički procesi, ozbiljnu teškoću predstavlja odredjivanje Lagranževe funkcije.

U ovom radu je pokazana mogućnost direktne produkcije Lagranževe funkcije iz jednačine procesa, kada se za aproksimativno rešavanje jednačina procesa upotrebi varijacioni princip sa iščezavajućim parametrom B. Vujanovića. Kroz četiri ilustrativna primera pokazan je način upotrebe prikazane metode.

Dušan B. Stokić
Mašinski fakultet u Nišu
Beogradska 14, 18000 Niš