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THERMAL STRESSES IN A DISK OF VARIABLE THICKNESS

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This paper deals with the solution of differential equations for the determination of thermal stresses (radial and circumferential) in thin disks of variable thickness. In general, a numerical procedure is required in order to determine both the temperature and stress distributions in the considered body. A simple numerical case is presented in order to demonstrate the applicability of the proposed procedure, and results compared with constant thickness disk stresses.

1. Introduction. Equilibrium equation of a disk with constant thickness is known in the following form

$$\frac{d\sigma_R}{dr} + \frac{\sigma_R - \sigma_\phi}{r} = 0 \tag{1}$$

or by eliminating the circumferential stresses, bearing in mind thermally induced forces

$$\frac{d^2\sigma_R}{dr^2} + \frac{3}{r}\frac{d\sigma_R}{dr} + \frac{\alpha E}{r}\frac{dT}{dr} = 0. \tag{2}$$

If also the variability of the disk thickness $t(r)$ is taken into account, the radial stress equation turns to be

$$\frac{d^2\sigma_R}{dr^2} + \left(\frac{3}{r} + \frac{1}{t}\frac{dt}{dr}\right)\frac{d\sigma_R}{dr} + \left[\frac{1}{t}\frac{d^2t}{dr^2} + \frac{(2+\nu)}{rt}\frac{dt}{dr} - \left(\frac{1}{t}\frac{dt}{dr}\right)^2\right]\sigma_R + \frac{\alpha E}{r}\frac{dT}{dr} = 0 \tag{3}$$

with boundary conditions

$$\sigma_R(r=a) = -p_a, \qquad \sigma_R(r=b) = -p_b. \tag{4}$$

The solution of the above equation (3) can be sought with known $t(r)$ and temperature $T(r)$, but except for $t = \text{const.}$, analytical solutions are not known, and numerical means have to be employed. There is an exception of the hyperbolically shaped disk (Stodola [1]).

2. Temperature. Heat conduction equation may be simplified to (where $\Theta = T - T_0$)

$$t\frac{d^2\Theta}{dr^2} + \left(\frac{t}{r} + \frac{dt}{dr}\right)\frac{d\Theta}{dr} - \frac{2\alpha}{\lambda}\Theta = 0 \tag{5}$$

with its boundary conditions

$$\Theta(r=a)=\Theta_a, \qquad \left.\frac{d\Theta}{dr}\right|_{(r=b)}=0. \tag{6}$$

The solution of the above equation (5) shall be written here for two simple forms of the disk.

(a) Constant thickness ($t=$ const.)

A modified Bessel equation is obtained ($m=2\alpha/\lambda t_a$)

$$r^2\frac{d^2\Theta}{dr^2}+r\frac{d\Theta}{dr}-m^2r^2\Theta=0 \tag{7}$$

with its general solution

$$\Theta=C_1I_0(mr)+C_2K_0(mr) \tag{8}$$

the particular solution of which, considering boundary conditions (6) is

$$\Theta=\Theta_a\frac{K_1(mb)I_0(mr)+I_1(mb)K_0(mr)}{K_1(mb)I_0(ma)+I_1(mb)K_0(ma)}. \tag{9}$$

(b) Hyperbolic shape ($t=t_a a/r$)

Differential equation is obtained here ($M^2=m^2/a$)

$$\frac{d^2\Theta}{dr^2}-M^2r\Theta=0 \tag{10}$$

which is also the modified Bessel equation, and has its general solution

$$\Theta=\sqrt{r}\Big[C_1I_{1/3}\big(\tfrac{2}{3}Mr^{2/3}\big)+C_2I_{-1/3}\big(\tfrac{2}{3}Mr^{2/3}\big)\Big] \tag{11}$$

By including boundary conditions (6) the particular solution reads

$$\Theta=\Theta_a\sqrt{\frac{r}{a}}\frac{I_{2/3}\big(\frac{2}{3}Mb^{3/2}\big)I_{1/3}\big(\frac{2}{3}Mr^{3/2}\big)-I_{-2/3}\big(\frac{2}{3}Mb^{3/2}\big)I_{-1/3}\big(\frac{2}{3}Mr^{3/2}\big)}{I_{2/3}\big(\frac{2}{3}Mb^{3/2}\big)I_{1/3}\big(\frac{2}{3}Ma^{3/2}\big)-I_{-2/3}\big(\frac{2}{3}Mb^{3/2}\big)I_{-1/3}\big(\frac{2}{3}Ma^{3/2}\big)}. \tag{12}$$

For the trapezoidal and triangular profiles analytical solutions do not exist, as far as we are aware of.

3. Stresses. Analytical solution of equation (3) for $t=$ const., has a known form, which can be fond in references (e.g. Timoshenko-Goodier [2]).

$$\sigma_R=\frac{\alpha E}{r^2}\left[\frac{r^2-a^2}{b^2-a^2}\int_a^b T(r)r\,dr-\int_a^r T(r)r\,dr\right]$$
$$-p_a+\frac{(p_a-p_b)b^2(1-a^2/r^2)}{(b^2-a^2)} \tag{13}$$

$$\sigma_\phi = \frac{\alpha E}{r}\left[\frac{r^2+a^2}{b^2-a^2}\int_a^b T(r)r\,dr + \int_a^r T(r)r\,dr - Tr^2\right] - p_a + \frac{(p_a-p_b)b^2(1+a^2/r^2)}{(b^2-a^2)}. \tag{14}$$

With linear (trapezoidal and triangular) profile the thickness is given by

$$t = kr + c, \qquad \frac{dt}{dr} = k, \qquad \frac{d^2t}{dr^2} = 0 \tag{15}$$

so that the equation (3) gets the following formulation

$$\frac{d^2\sigma_R}{dr^2} + \left(\frac{3}{r} + \frac{k}{kr+c}\right)\frac{d\sigma_R}{dr} + \left[\frac{(2+\nu)k}{r(kr+c)}\right]\sigma_R + \frac{\alpha E}{r}\frac{dT}{dr} = 0 \tag{16}$$

and the solution of it cannot be found analytically, so that numerical tools have to be used. If the hyperbolic profile is considered, where the thickness and its derivatives are

$$t = \frac{t_a a}{r}, \qquad \frac{dt}{dr} = -\frac{t_a a}{r^2}, \qquad \frac{d^2t}{dr^2} = -\frac{2t_a a}{r^3} \tag{17}$$

the differential equation for the radial stress is obtained

$$\frac{d^2\sigma_R}{dr^2} + \frac{2}{r}\frac{d\sigma_R}{dr} - \frac{(1+\nu)}{r^2}\sigma_R + \frac{\alpha E}{r}\frac{dT}{dr} = 0. \tag{18}$$

The homogeneous part of it is due to Euler, and may be solved by r^w, yielding the characteristic polynomial

$$w^2 + w - (1+\nu) = 0 \tag{19}$$

with its roots

$$w = \tfrac{1}{2}(-1 \pm \sqrt{5+4\nu}) \tag{20}$$

and the general solution is ($\nu = 0.3$)

$$\sigma_R = C_1 r^{0.745} + C_2 r^{-1.745} \tag{21}$$

where integration constants are to be determined from boundary conditions (4). Particular solution for the nonhomogeneous term in eq. (18) may also be calculated by variation of constants method, but it is not presented here (see ref. [8]). With $t =$ const. the corresponding general solution of the homogeneous part reads

$$\sigma_R = C_1 + C_2 r^{-2} \tag{22}$$

demonstrating the difference between (21) and (22).

Circumferential (hoop) stress σ_ϕ, which has its largest values at the boundaries, can be obtained by the formula

$$\sigma_\phi = r\frac{d\sigma_R}{dr} + \sigma_R\left(1 + \frac{r}{t}\frac{dt}{dr}\right). \tag{23}$$

4. Numerical case

Radii	$a = 10\,\mathrm{mm}$, $b = 100\,\mathrm{mm}$
thickness	$t_a = 5\,\mathrm{mm}$, $t_b = 3\,\mathrm{mm}$, $t(r) = (47 - 2r/10)/9$
temperature	$T_a = 980\,\mathrm{K}$, $T_b = 800\,\mathrm{K}$, $T(r) = 1000 - 2r$ (approximation)
pressure	$p_a = p_b = 0$
elasticity	$E = 210\,\mathrm{GPa}$, $\alpha = 12\,\mathrm{ppm/K}$, $\nu = 0.3$

Radial stress, (compared between trapezoidal and constant profiles) is given in Figure 1.

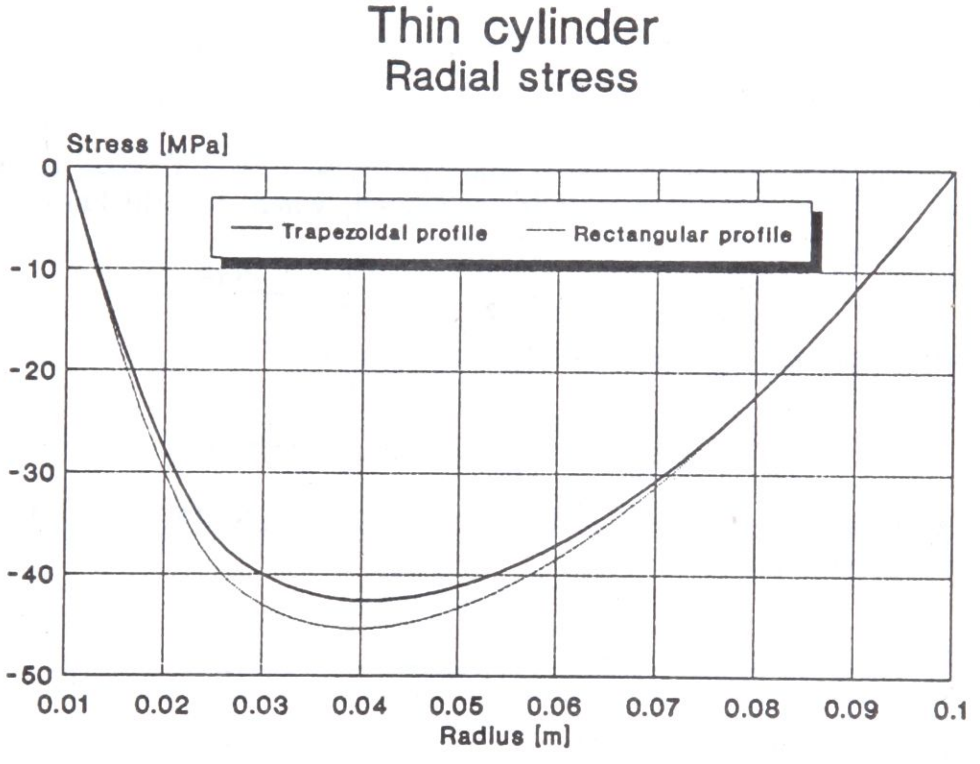

Figure 1.

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THERMISCHE SPANNUNGEN IN DISKSCHEIBEN VERÄNDERLICHER DICKE

Im Beitrag wird die Lösung von Differentialgleichungen zur Bestimmung thermischer Spannungen (radial und tangential) in dünnen Diskscheiben veränderlicher Dicke behandelt.

Im allgemeinen braucht man numerische Berechnung für die Temperatur als auch Spannungen im behandelten Körper. Dabei ist ein einfaches Beispiel für das Trapezprofil ausgefürht, mit dem man die Verwendbarkeit des vorgestellten Verfahrens beweisen kann.

Die Resultate sind mit schon bekannten Spannungswerten von Diskscheiben mit konstanter Dicke verglichen.

TERMIČNE NAPETOSTI V DISKU SPREMENLJIVE DEBELINE

V prispevku obravnavamo reševanje diferencialnih enačb za določitev toplotnih napetosti (radialnih in obročnih) v tankem kolutu, ki ima spremenljivo debelino. V splošnem potrebujemo numerično reševanje tako temperature kot tudi napetostnih porazdelitev v obravnavanem telesu. Podan je preprost zgled trapeznega profila, ki naj dokaže uporabnost predlaganega postopka, rezultati pa so primerjani z znanimi vrednostmi napetosti diska nespremenljive debeline.

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