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SOLUTION OF SEN-DUNN VACUUM MODEL

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In this paper we consider the scalar tensor theory of gravitation (proposed by Sen and Dunn [1]) in a modified Riemannian manifold in which both scalar and tensor fields have intrinsic geometrical significance. The scalar field in this theory is characterised by the function $X^\circ = X^\circ(X^i)$ where X^i are the co-ordinates in the four dimensional Lyra Manifold and the tensor field is identified by the fundamental metric tensor g_{ij} of the manifold.

In this theory we assume that the spacetime is described by a metric with two degrees of freedom whose form in hyperbolic canonical co-ordinate system [2] is

$$ds^2 = e^{2A-2B}(dt^2 - dr^2) - r^2 e^{-2B} d\theta^2 - e^{2B}(C d\theta + dz)^2 \tag{1}$$

where $A = A(r, t)$, $B = B(r, t)$ and $C + C(r, t)$.
B and C represent first and second degree of freedom respectively. Here co-ordinates r, θ, z and t correspond to X^1, X^2 and X^3 and X^4 respectively.

In Einstein theory it is shown [3], [4] that massive scalar field can not be a source of gravitation in the spacetime described by (1). Hence the Einstein's field equations

$$R_{ij} - \frac{1}{2} R g_{ij} = -\frac{K}{4\pi}\left(V_i V_j - \frac{1}{2} g_{ij} V_k V^k\right)$$

and Klein-Gordon equation g^{ij}; $V_{ij} = 0$ (hereafterwards suffix after the field variable indicates partial differentiations and semicolon denotes covariant differentiation) for massless scalar field V in this theory for metric (1) can be written explicitly in the following form:

$$\overset{1}{\Delta} B - \frac{e^{4B}}{2r^2}(C_1{}^2 - C_4{}^2) = 0, \tag{2}$$

$$\overset{2}{\Delta} C + 4(B_1 C_1 = B_4 C_4) = 0, \tag{3}$$

$$A_1 = r(B_1{}^2 + B_4{}^2) + \frac{e^{4B}}{4r}(C_1{}^2 + C_4{}^2) + \frac{Kr}{8\pi}(V_1{}^2 + V_4{}^2), \tag{4}$$

$$A_4 = 2rB_1 B_4 + \frac{e^{4B}}{2r} C_1 C_4 + \frac{Kr}{4\pi} V_1 V_4, \tag{5}$$

$$A_{11} - A_{44} + B_1^2 - B_4^2 - \frac{e^{4B}}{4r^2}(C_1^2 - C_4^2) - \frac{Kr}{8\pi}(V_1^2 - V_4^2) \tag{6}$$

and $\overset{1}{\Delta} V = 0$ (7)

Where $\overset{1}{\Delta} \equiv \frac{\partial^2}{\partial \gamma^2} - \frac{\partial^2}{\partial t^2} + \frac{1}{\gamma}\frac{\partial}{\partial \gamma}$ and $\overset{2}{\Delta} \equiv \frac{\partial^2}{\partial \gamma^2} - \frac{\partial^2}{\partial t^2} - \frac{1}{\gamma}\frac{\partial}{\partial \gamma}$ (8)

Hereafterwards the suffixes 1 and 4 after the field variables indicate partial derivatives with respect to r and t respectively. Equations (2) and (3) which determine B and C are identical to those of empty spacetime for the metric (1). Equation (6) being obtainable from the rest of the field equations is redudant. Now, A can be obtained from the equations (4) and (5) as

$$A = \int \Big[\Big\{ r(B_1^2 + B_4^2) + \frac{e^{4B}}{4r}(C_1^2 + C_2^4) + \frac{Kr}{8\pi}(V_1^2 + V_4^2) \Big\} dr + $$

$$+ (2\, rB_1 B_4 + \frac{e^{4B}}{2r} C_1 C_4 + \frac{Kr}{4\pi} V_1 V_4)\, dt \Big]. \tag{9}$$

Thus the Einstein's cylindrical anisotropic inhomogeneous microscopic model in general is governed by the equations (2), (3), (7) and (9). However, for any solution V of the equation (7), one can generate the corresponding solution of Einstein massless scalar field equations. In particular, considering $V = a J_0(kr) \cos kt$ (a and k are constants and J_0 is the Bessel function of zeroeth order) as solution of equation (7), one get from (2), (3) and (9)

$$C = C(r-t),\ B = \frac{1}{4} \ln r + b \text{ and}$$

$$A = \frac{1}{16} \ln r - \frac{1}{2} e^{4B} \int C'^2 dr + \frac{Ka^2 kr}{16\pi} J'_0(kr) \cos 2kt$$

$$+ \frac{Ka^2 K^2 r^2}{16\pi} [\{J'_0(kr)\}^2 - J_0(kr) J''_0(kr)]. \tag{10}$$

The field equations given by Sen-Dunn [1] for combined scalar tensor fields are

$$R_{ij} - \frac{1}{2} R g_{ij} = \frac{W}{(X^0)^2} \Big[X^0{}_{,i} X^0{}_{,j} - \frac{1}{2} g_{ij} X^0{}_{,k} X^{0,k} \Big] - 8\pi G T_{ij} \tag{11}$$

where $W = 3/2$, T_{ij} is the energy momentum tensor of the fields, R_{ij} and R are respectively the usual Ricci tensor and Riemanncurvature scalar. In the matter free region these field equations reduce to

$$R_{ij} - \frac{1}{2} R g_{ij} = \frac{W}{(X^0)^2} \Big[X^0{}_{,i} X^0{}_{,j} - \frac{1}{2} g_{ij} X^0{}_{,k} X^{0,k} \Big]. \tag{12}$$

The explicit form of these field equations (12) in the spacetime (1) can be given in the following forms:

$$\overset{1}{\Delta} B - \frac{e^{4B}}{2r^2}(C_1^2 - C_4^2) = 0, \tag{13}$$

$$\overset{2}{\Delta} C + 4(B_1 C_1 - B_4 C_4) = 0, \tag{14}$$

$$A_1 = r(B_1^2 + B_4^2) + \frac{e^{4B}}{4r}(C_1^2 + C_4^2) - \frac{Wr}{2}(H_1^2 + H_4^2). \tag{15}$$

$$A_4 = 2rB_1 B_4 + \frac{e^{4B}}{2r} C_1 C_4 - wr H_1 H_4 \tag{16}$$

Where $X^0 = e^H$. (17)

The condition of integrability for 'A' is satisfied if $\overset{1}{\Delta} H = 0$.

Thus the Sen-Dunn Vacuum model is completely governed by the equations (13) to (17).

In view of the structure of the field equations of both models one can assume that the Sen-Dunn Scalar field is a function of Einstein's scalar field as

$$H = f(V). \tag{18}$$

With the help of equation (18), equation (17) yields

$$f'' g^{ij} V_i V_j + f' g^{ij} V_{;ij} = 0. \tag{19}$$

Klein Gordon equation for the spacetime (1) is of the form

$$g^{ij} V_{;ij} = 0$$

and the scalar field is assumed to be nonnull (i.e., $g^{ij} V_i V_j = 0$) as the null massless scalar field does not survive in the spacetime (1). Then the equation (19) reduces to

$$f'' = 0 \tag{20}$$

Integrating equation (20) we obtain

$$H = mV + n \tag{21}$$

Where m and n are arbitrary constants. The arbitrary constant m **can** be easily obtained by comparing the field equations of both models. Thus, we get

$$H \rightarrow iV \text{ defined by } H = i\sqrt{\frac{4\pi w}{K}}\, V + n, \quad i = \sqrt{-1}. \tag{22}$$

With the help of (22), one can generate solution of vacuum Sen-Dunn scalar tensor field equations from those of Einstein's massless nonnull scalar field equations and vice-versa.

Now using (22) in (2), (3) and (9), one can get the solution of vacuum Sen-Dunn model as

$$C=C(r-t), \quad B=\frac{1}{4}\ln r+b \text{ and}$$

$$A=\frac{1}{16}\ln r-\frac{1}{2}e^{4b}\int C'^2\,du-\frac{W}{4}a^2{}_{kr}J_0'(kr)\cos(2kt)$$

$$-\frac{W}{4}a^2k^2r^2[\{J_0'(kr)\}^2-J_0(kr)J_0''(kr)]- \tag{23}$$

Where $u=t-r$.

Similarly one can get the corresponding solution for the case $C=C(t+r)$. It may be verified that the solution (23) can be obtained by solving directly the field equations (13)—(17).

The immediate use of the transformation (22) is in obtaining the solution of Sen and Dunn Scalar tensor theory from any massless nonnull scalar solution of Einstein's theory. The null scalar field in Einstein's theory for the spacetime (1) does not survive and it is either of the form $V=V(r-t)$=constant or $V=V(r+t)$= =constant. In Sen-Dunn theory the null scalar field also behaves alike. Thus null scalar field in both theories does not survive for the spacetime (1). Moreover, it may be easily verified from the field equations (2)—(7) that if $\mathring{A}$, $\mathring{B}$, $\mathring{C}$ and $\mathring{V}$ is a solution of nonnull massless scalar field in Einstein's theory then $A=\mathring{A}+f(t-r)$, $\mathring{B}$, $\mathring{C}$ and $\mathring{V}$ (or $A=\mathring{A}+g(t+r)$, $\mathring{B}$, C and $\mathring{V}$) is also a solution which may be transformed by (22) to the corresponding solution of the Sen and Dunn's theory where f and g are arbitrary functions of their arguments represent monocromatic out going and in coming waves respectively. Besides the generation of solutions one may also use the transformation obtained in this paper in discussing the singularities of the two theories as the transformation being linear is singularity preserving. Thus the solution in both theories will have same singularities.

One may hope that the consequence of this paper will lead to a deeper understanding of the relation between Einstein's theory of gravitation and scalar-tensor theory of Sen and Dunn.

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