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# ON THE BEST ENERGY MEASURE FOR THE BEST STABILITY IN FLUIDDYNAMICS*

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## Preliminaries

As it is well known, the energy method gives sufficient conditions for non linear stability in the mean — for instance in fluid dynamics — once we have chosed a *prefixed* Liapunov's functional as a measure of the perturbations [1] . . . [9] and this stability is closely connected to the chosen measure [10] [11]. Moreover variational methods gives optimum sufficient conditions for this stability, once we have chosed as a measure of the perturbations *almost prefixed* Liapunov's functional, depending on the choice of suitables positive coupling constants (maximum variational problems) [12] . . . [16].

But which is the best measure for the best stability?

Of course the measure that gives unconditional stability!

If we consider a parametric family of measures depending on *not prefixed* constants, does exist subfamilies of best measures, or at least one best measure, such that we have unconditional stability?

The aim of the present paper is the answer to this question.

We shall prove that, at least in some cases, for instance in hydrodynamics and in MHD of dusty gases, there exists the possibility of choice of *sui* table measures to have unconditional stability, Indeed we have considered dusty gases in aplane layer and in an arbitrary bounded domain, according to Saffman's mathematical model [17]. We have introduced a parametric family of energy measures and we have investigated some non linear stability problems to laminar flows in the layer and in the above arbitrary domain, in order to find best measures which assures unconditional stability. The following results are obtained: there exist a subfamily of measures such that laminar dusty gas flows in the layer, with rigid and fixed walls, are unconditionally stable to non linear — onedimensional as well as tridimensional

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— perturbations. Moreover there exists a subfamily of measures such that, in presence of a constant external magnetic field and in the isotropic case, laminar electroconducting dusty gas flows in the layer with rigid fixed and conducting walls, are unconditionally stable to non linear onedimensional perturbations, as well as to non linear tridimensional perturbations. We have unconditional stability to tridimensional perturbations, even for the MHD dusty gas flows into an arbitrary bounded domain. Although, in these tridimensional cases, the affirmation is true only for a suitable subclass of MHD flows; therefore it leaves an open problem.

## Introduction

The interest in viscous incompressible fluids with suspended particles (dusty gases) was prompted by the works of Kazakevich-Krapivin [18] and Sproull [19]. The first ones studied the aerodynamic resistance of a dusty gas flowing through a pipe. Their studies can be expressed as follows: the aerodynamic resistence of a dusty gas flowing through a pipe is less than that of a clean gas. The second one has observed that adding dust (talc for instance) to air flowing through a pipe, it can be appreciably reduced the resistance coefficient. These observations means that the pressure gradient required to maintain a given volume rate of flow is reduced by the addition of dust. For instance, when a dust concentration of 0.25 Kgm/m$^3$ was introduced, the required pressure gradient to maintain the original flow rate was reduced by 13%. Sproull noted that these experimental results present a paradox, that is the increased density of a dusty gas, as opposed to a clean rate, would require a suitable increased pressure gradient, to maintain the given flow rate, assuming constant the other parameters of the system. Sproull's interpretation of this phenomenon was that the presence of the dust implies a viscosity reduction, i.e. the viscosity of a dusty gas is less than the viscosity of the corresponding clean gas. In a paper dated 1962. [17] Saffman has pointed out the study of dusty gases on the basis of a suitable mathematical model. In his paper Saffman refused Sproull's argument concerning the viscosity reduction by observing that Sproull's explanation of this effect contradicts Einstein's formula about the viscosity of a suspension, according to which the viscosity of a dusty gas should be increased by a factor proportional to the concentration by volume of the dust particles. In his paper Saffman considers the problem of the stability by investigation the (linear) stability of plane laminar dusty gas flows and the effects of the dust particles on the critical Reynolds number from laminar to turbulent flow. The conclusion was that a dust particle in a gas has much larger inertia than the equivalent volume of gas and therefore will not partecipate as readily in the turbulent fluctuations; thus the relative motion of the dust particles with respect to the clean gas dissipate energy because of the drag and so extracting energy from the system. Hence coarse dust particles have a stabilizing effect on the dust-gas system. On the contrary he proved that fine dust destabilizes the flow. After Saffman's paper, several ones [20] . . . [30] have been published on the basis of his model. But, for that is at our knowledge, non linear stability problems have been considered only in a paper [27], in which, unfortunately, there is a mistake, as we will show in a later work.

In the present paper we study, for laminar dusty gas flows, some non linear stability problems that clearly are important, for instance, in the getting-rid of pollution problems, even in order to find the best measure of the perturbations for the best stability. The paper develops in five sect.s. The first one is devoted to the equations of the incompressible dusty gas flows, according to Saffman's mathematical model. In the second one we study non linear energy stability of laminar flows in a plane layer to onedimensional perturbations on initial data, in order to give stability conditions to two measures. In the third section we study the stability of the above flows to tridimensional perturbations. The fourth section is devoted to the study of non linear stability of an electroconducting dusty gas in the layer, in an external constant magnetic field and in isotropic case, to onedimensional perturbations. In the fifth section we investigate the stability of a general MHD isotropic dusty gas flow in an arbitrary bounded domain to tridimensional perturbations. We find, in some cases, suitables best energy measures with respect to which there is unconditional asymptotic exponential stability; including the investigation for unconditional stability of the MHD clean gas flows and, in particular, for Hartmann's flows, in the final remarks of the paper.

## 1. Saffman's mathematical model

The equations that we shall use to represent the motion of a dusty gas, following the Saffman's incompressible model [17], are:

$$(1)\quad \begin{cases} \rho\left(\dfrac{\partial \mathbf{V}}{\partial t}+\mathbf{V}\cdot\nabla\mathbf{V}\right)=-\nabla_p+\mu\Delta\mathbf{V}-KN(\mathbf{V}-\boldsymbol{v}) \\ \nabla\cdot\mathbf{V}=0 \\ mN\left(\dfrac{\partial \boldsymbol{v}}{\partial t}+\boldsymbol{v}\cdot\nabla\boldsymbol{v}\right)=KN(\mathbf{V}-\boldsymbol{v}) \\ \nabla\cdot\boldsymbol{v}=0 \end{cases}$$

In these equations $\mathbf{V}$ and $\boldsymbol{v}$ are the gas and the dust velocities respectively. $N$ is the number density of the dust particles, each of mass $m$. $K$ is the Stokes coefficient of resistance and $p$, $\rho$, $\mu$, are pressure, density, viscosity of the gas. In order to formulate the problem in a simple way, some simplifying assumptions are made, following Saffman, such as: the dust particles are uniform in size and shape and spherical of radius $a$, so in this case the Stokes drag formula gives $K=6\pi a\mu$; moreover for sufficiently small particles, the velocity of sedimentation will be small, compared with a characteristic velocity of the flow and can be neglected. Then in a particular steady state the dust particles move along the streamlines with the velocity of the gas ($\boldsymbol{v}=\mathbf{V}$) and the number density $N$ is constant along the streamlines; indeed we shall assume that $N$ is constant everywhere [17]. The following parameters will be useful for the study: $\nu=\dfrac{\mu}{\rho}$ is the kinematic viscosity of the clean gas, $r=KN/\rho$ (dimensions of frequency), $\tau=m/K$ (dimensions of time) is called relaxation time of the dust particles; $f=mN/\rho=r\tau$ (dimensionless mass concentration of the dust) and $s=1/\tau$. Finally it is supposed that the bulk concentration (by volume) of the dust $f\rho/\rho_1$, where $\rho_1$ is the density of material in the dust particles, is very small, so that the effect of the dust on the gas is equivalent to an extra force $KN(\mathbf{V}-\boldsymbol{v})$ for unit volume.

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## 2. Non linear stability of dusty gas flows in a plane layer to onedimensional perturbations

Let's consider now laminar flows, in which the streamlines are straight parallel, in a plane layer with rigid and fixed walls $y = \pm d$.

To this end we shall consider solutions of the system (1) belonging to the following class:

$$\{\mathbf{V} = V(y, t)\,\mathbf{i};\quad v = v(y, t)\,\mathbf{i};\, p\} \tag{2}$$

If we replace (2) in system (1), we obtain:

$$\begin{cases} \dfrac{\partial V}{\partial t} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x} + \nu \dfrac{\partial^2 V}{\partial y^2} - r(V - v) \\ \dfrac{\partial v}{\partial t} = s(V - v) \end{cases} \tag{3}$$

with the boundary conditions:

$$V(\pm d, t) = v(\pm d, t) = 0, \quad t \in [0, +\infty] \tag{4}$$

Of course this class is not empty. In fact to this class belongs the steady dusty gas flow (of Poiseuille type)

$$\begin{cases} V = v = -\dfrac{K}{2\rho\nu}(y^2 - d^2) \\ -K = \dfrac{\partial p}{\partial x} \end{cases} \tag{5}$$

and the equilibrium solution $\mathbf{V} = v = 0$, $p = p_0$.

Let

$$\{U(y, t)\,i;\; u(y, t)\,i;\; \pi\} \tag{6}$$

be a class of perturbations to the flows belonging to (2). From (1) . . . (6), it follows that the perturbations of this class must satisfy the dimensionless[1] system:

$$\begin{cases} \dfrac{\partial U}{\partial t} = -\dfrac{\partial \overline{\pi}}{\partial x} + \dfrac{\partial^2 U}{\partial y^2} - R_1(U - u) \\ \dfrac{\partial u}{\partial t} = R_2(U - u) \end{cases} \tag{7}$$

and the boundary conditions:

$$U(\pm 1, t) = u(\pm 1, t) = 0, \quad t \in [0, +\infty[ \tag{8}$$

In (7) the following two dimensionless numbers apprears:

$$R_1 = rd^2/\nu \text{ and } R_2 = sd^2/\nu$$

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[1] The undimensionalization used is $x = x^*d$, $t = d^2t^*/\nu$, $U = W_0U^*$, $u = W_0u^*$, the stars have been omitted. Here $W_0$ is some reference velocity.

We underline that $R_1/R_2$ is the mass concentration $f$ of the dust. Now chosen the Liapunov's functional

$$E(t)=\frac{1}{2}\int_{-1}^{1}(U^2+u^2)\,dy \tag{9}$$

as a measure of the perturbations (6), from (7) and (8), multiplying $(7)_1$ and $(7)_2$ respectively by **U** and **u**, adding and integrating in $[-1, 1]$ we find:

$$\frac{dE}{dt}=\int_{-1}^{1}\left\{-\left(\frac{\partial U}{\partial y}\right)^2-R_1U^2+(R_1+R_2)Uu-R_2u^2\right\}dy \tag{10}$$

If we introduce the functionals

$$X_1^2=\int_{-1}^{1}U^2\,dy;\qquad X_2^2=\int_{-1}^{1}u^2\,dy \tag{11}$$

from (10) and (11), by classical integral inequalities, it follows:

$$\frac{dE}{dt}\leqslant-\Phi \tag{12}$$

with

$$\Phi=(\gamma^2+R_1)X_1^2-(R_1+R_2)X_1X_2+R_2X_2^2 \tag{13}$$

in which $\gamma^2$ is the Poincare's constant for the layer. Therefore the condition

$$(R_1-R_2)^2<4\gamma^2R_2 \tag{14}$$

which make the quadratic form (13) positive definite, ensures, owing to the theorem [31], the exponential asymptotic stability in the class (2), to the perturbations (6).

Now we will show that it can be possible to improve the stability condition (14) by choosing appropriately the perturbation measure. In fact, we shall consider now the family of equivalent energy measures

$$E(t)=\frac{1}{2}\int_{-1}^{1}(c_1U^2+c_2u^2)\,dy \tag{15}$$

with, till now, $c_1$ and $c_2$ arbitrary positive constants.

Reintroducing the functionals (11), multiplying $(7)_1$—$(7)_2$ by $c_1U$ and $c_2u$, working as usual, we obtain:

$$\frac{dE}{dt}\leqslant-\Psi \tag{16}$$

with

$$\Psi = c_1 (\gamma^2 + R_1) X_1^2 - (c_1 R_1 + c_2 R_2) X_1 X_2 + c_2 R_2 X_2^2 \tag{17}$$

From (15) and (16) we have:

The flows of class (2) are asympt. exp. stable to the perturbations (6) if the condition (18) holds.

$$(c_1 R_1 - c_2 R_2)^2 < 4 c_1 c_2 \gamma^2 R_2 \tag{18}$$

If we choose now $c_1$ and $c_2$ such that $c_1/c_2 = R_2/R_1$, from (18) it follows that *the flows of the class* (2) *are absolutely unconditionally asympt. exp. stable to the perturbations* (6) *in the best energy measure*

$$E(t) = \frac{1}{2} \int_{-1}^{1} (R_2 U^2 + R_1 u^2)\, dy \tag{19}$$

We note that the same result we can easily obtain to more general perturbations

$$\{U_i(y, t)\, \mathbf{e}_i;\ u_i(y, t)\, \mathbf{e}_i;\ \pi,\ i = 1, 2, 3\}$$

because we have, for this case the perturbed equations:

$$\begin{cases} \dfrac{\partial U_1}{\partial t} = -\dfrac{\partial \pi}{\partial x} + \dfrac{\partial^2 U_1}{\partial y^2} - R_1 (U_1 - u_1) \\ \dfrac{\partial U_3}{\partial t} = -\dfrac{\partial \pi}{\partial z} + \dfrac{\partial^2 U_3}{\partial y^2} - R_1 (U_3 - u_3) \\ \dfrac{\partial u_1}{\partial t} = R_2 (U_1 - u_1) \\ \dfrac{\partial u_3}{\partial t} = R_2 (U_3 - u_3) \\ U_2(y, t) = u_2(y, t) = 0 \end{cases} \tag{20}$$

that easily can be compared with (7).

### 3. Non linear stability of dusty gas flows in a plane layer to tridimensional perturbations

We shall now extend the preceding results to the case of tridimensional perturbations. Let's consider the general class of perturbations (of course suitable periodic in the directions in which the flow goes to infinity):

$$\{U_i(x, y, z, t)\, \mathbf{e}_i;\ u_i(x, y, z, t)\, \mathbf{e}_i;\ \pi\}$$

to laminar flows of class (2). From (1) . . . (4), it results that the perturbations of this class satisfy the following dimensionless system[2]:

$$(22)\quad \begin{cases} \dfrac{\partial \mathbf{U}}{\partial t} = -R(\mathbf{U}+\mathbf{V})\cdot\nabla\mathbf{U} - R\mathbf{U}\cdot\nabla\mathbf{V} - \nabla\pi + \Delta\mathbf{U} - R_1(\mathbf{U}-\mathbf{u}) \\ \dfrac{\partial u}{\partial t} = -R(\mathbf{u}+\mathbf{v})\cdot\nabla\mathbf{u} - R\mathbf{u}\cdot\nabla\mathbf{v} + R_2(\mathbf{U}-\mathbf{u}) \\ \nabla\cdot\mathbf{U} = \nabla\cdot\mathbf{u} = 0 \end{cases}$$

where we underline that $\mathbf{V}, v, p$ belongs to the class (2), and with the boundary conditions

$$(23)\qquad \mathbf{U}(x, \pm 1, z, t) = \mathbf{u}(x, \pm 1, z, t) = 0.$$

In the equations (22) the Reynolds number $R = dW_0/\gamma$ has been introduced. Let us consider the family of energy measures

$$(24)\qquad E(t) = \frac{1}{2}\int_s (c_1\mathbf{U}^2 + c_2\mathbf{u}^2)\,dS$$

where $S$ is the "periodic cell". Multiplying $(22)_1$—$(22)_2$ respectively by $c_1U$, $c_2u$, adding and integrating in $S$, then we find:

$$(25)\qquad \frac{dE}{dt} = \int_s \{-c_1(\nabla\mathbf{U})^2 - c_1R_1\mathbf{U}^2 - c_2R_2\mathbf{u}^2 + (c_1R_1 + c_2R_2)\,\mathbf{U}\mathbf{u}\}\,dS$$

Therefore, introducing the functionals like formally (11), for our tridimensional perturbations, we obtain again the relations:

$$\frac{dE}{dt} \leqslant -\chi$$

with

$$\chi = (c_1\gamma_1^2 + c_1R_1)X_i^2 - (c_1R_1 + c_2R_2)X_1X_2 + c_2R_2X_2^2$$

that is, formally, the same relations (16)—(17); therefore also in this case we have exp. asympt. stability under the condition (18) and absolute uncond. exp. asympt. stability to the best measure (24) with $c_1/c_2 = R_2/R_1$.

### 4. Non linear stability of MHD dusty gas flows in a plane layer to onedimensional perturbations

We shall suppose now that the dusty gas flows in presence of a constant $H_0$ external magnetic field. In order to formulate the problem in a simple manner we assume that only the fluid — not the dust — is electroconducting. These flows must satisfy the following equations:

2 The undimensionalization is the same of note (1).

$$
(26)\qquad
\begin{cases}
\rho\left(\dfrac{\partial \mathbf{V}}{\partial t}+\mathbf{V}\cdot\nabla\mathbf{V}\right)=-\nabla p+\mu\Delta\mathbf{V}-KN(\mathbf{V}-\boldsymbol{v})+\mu_e \operatorname{rot}\mathbf{H}\times\mathbf{H}\\[2ex]
\dfrac{\partial \mathbf{H}}{\partial t}=\operatorname{rot}(\mathbf{V}\times\mathbf{H})+\eta_e\Delta\mathbf{H}\\[2ex]
m\left(\dfrac{\partial \boldsymbol{v}}{\partial t}+\mathbf{v}\cdot\nabla\mathbf{v}\right)=K(\mathbf{V}-\boldsymbol{v})\\[2ex]
\nabla\cdot\mathbf{V}=\nabla\boldsymbol{v}=\nabla\cdot\mathbf{H}=0
\end{cases}
$$

Let's consider a solution of the system in the plane layer with rigid, fixed and conducting boundaries, belonging to the class

$$
(27)\qquad \{\mathbf{V}=V(y\cdot t)\,\mathbf{i};\ \mathbf{v}=v(y,t)\,\mathbf{i};\ \mathbf{H}=\tilde{h}(y,t)\,\mathbf{i}+H_0 j\}
$$

Replacing (27) in (26) it follows:

$$
(28)\qquad
\begin{cases}
\dfrac{\partial V}{\partial t}=-\dfrac{1}{\rho}\dfrac{\partial p}{\partial x}+\nu\dfrac{\partial^2 V}{\partial y^2}-r(V-v)+\dfrac{\mu_e}{\rho}H_0\dfrac{\partial\tilde{h}}{\partial y}\\[2ex]
\dfrac{\partial\tilde{h}}{\partial t}=H_0\dfrac{\partial V}{\partial y}+\eta_e\dfrac{\partial^2\tilde{h}}{\partial y^2}\\[2ex]
\dfrac{\partial v}{\partial t}=s(V-v)
\end{cases}
$$

with the boundary conditions

$$
(29)\qquad V(\pm d,t)=v(\pm d,t)=0;\quad \tilde{h}(d,t)=\tilde{h}_1\quad \tilde{h}(-d,t)=\tilde{h}_2
$$

where $\tilde{h}_1$ and $\tilde{h}_2$ are given values depending on the electrical properties of the walls. We note that to this class belongs (for suitables boundary electrical conditions) the steady dusty gas flow (of Hartmann type):

$$
(30)\qquad
\begin{cases}
V=v=\dfrac{\overline{K}d^2}{M}[\operatorname{ch}M-\operatorname{ch}(My/d)]/\operatorname{sh}M\\[2ex]
\tilde{h}=\dfrac{\nu\overline{K}d^2}{M\lambda\eta_e}[\operatorname{sh}(My/d-y/d\operatorname{sh}M]/\operatorname{sh}M\\[2ex]
-\overline{K}=\dfrac{1}{\nu}\dfrac{\partial p}{\partial x};\quad \lambda=\sqrt{\mu_e\nu/(\rho\eta_e)}
\end{cases}
$$

Let's consider perturbations of the laminar class

$$
(31)\qquad \{U(y,t)\,\mathbf{i};\ u(y,t)\,\mathbf{i};\ h(y,t)\,\mathbf{i};\ \pi\}
$$

These ones, from (26), (28) and (29), must verify the dimensionless[3] system:

$$(32)\quad \begin{cases} \dfrac{\partial U}{\partial t} = -\nabla\pi + \dfrac{M^2}{R}\dfrac{\partial h}{\partial y} + \sigma\dfrac{\partial^2 U}{\partial y^2} - R_1^m (U-u) \\ \dfrac{\partial h}{\partial t} = R_1^m \dfrac{\partial U}{\partial y} + \dfrac{\partial^2 h}{\partial y^2} \\ \dfrac{\partial u}{\partial t} = R^m (U-u) \end{cases}$$

where is

$$R^m = \frac{dW_0}{\eta_e}, \quad R_1^m = \frac{rd^2}{\eta_e}, \quad R_2^m = \frac{sd^2}{\eta_e}, \quad \sigma = \frac{\nu}{\eta_e}, \quad M = H_0\, d\sqrt{\mu_e/(\rho\nu\eta_e)}$$

Boundary conditions are:

$$(33)\qquad U(\pm 1, t) = u(\pm 1, t) = h(\pm 1, t) = 0$$

Having introduced the family of measures

$$(34)\qquad E(t) = \frac{1}{2}\int_{-1}^{1} \{c_1 U^2 + c_2 h^2 + c_3 u^2\}\, dy$$

we multiply $(32)_1$ by $c_1 U$, $(32)_2$ by $c_2 h$ and $(32)_3$ by $c_3 u$. Adding and integranting in [—1, 1], we find:

$$(35)\qquad \frac{dE}{dt} = \int_{-1}^{1} \Big\{ -\sigma c_1 \left(\frac{\partial U}{\partial y}\right)^2 - c_1 R_1^m U^2 + (c_1 R_1^m + c_3 R_2^m)\, U u - c_3 R_2^m u^2 + \\ + c_1 \frac{M^2}{R} U \frac{\partial h}{dy} + c_2 R^m h \frac{\partial U}{\partial y} - c_2 \left(\frac{\partial h}{\partial y}\right)^2 \Big\}\, dy$$

Therefore, having introduced the functionals

$$(36)\qquad X_1^2 = \int_{-1}^{1} U^2\, dy; \quad X_2^2 = \int_{-1}^{1} h^2\, dy; \quad X_3^2 = \int_{-1}^{1} u^2\, dy$$

and working as usual, it follows

$$(37)\qquad \frac{dE}{dt} \leqslant -\mathcal{G}$$

with

$$(38)\qquad \mathcal{G} = c_1(\sigma\gamma^2 + R_1^m) X_1^2 + c_2\gamma^2 X_2^2 + C_3 R_2^m X_3^2 - \\ -(c_1 R_1^m + c_3 R_2^m) X_1 X_3 + \gamma\left(c_2 R^m - c_1 \frac{M^2}{R}\right) X_1 X_2.$$

[3] The undimensionalization used is: $x = d\, x^*$, $t = d^2 t^*, /\eta_e$, $U = W_0 U^*$, $u = W_0 u^*$, $h = H_0 h^*$; and the stars have been omitted.

Holding theorem [31], we have that all the flows of (27) are asympt. exp. stable under the condition

$$c_2 (c_1 R_1^m - c_3 R_2^m)^2 + \left( c_2 R^m - c_1 \frac{M^2}{R} \right)^2 < 4 \sigma \gamma^2 R_1^m c_1 c_2 c_3. \tag{39}$$

Therefore from this condition it's clear that, if we choose the positive constants $c_i$ in such a way that

$$c_2/c_1 = M^2/(RR^m); \quad c_3/c_1 = f$$

from (39) we have:
The flovs of the class (27) are abs unc. asympt. exp. stable to non linear perturbations (31) in the best measure

$$E(t) = \frac{1}{2} \int_{-1}^{1} \left\{ R^m U^2 + \frac{M^2}{R} h^2 + R^m f u^2 \right\} dy. \tag{40}$$

Also here we note that the same result we can easily obtain to more general perturbations

$$\{U_i(y, t)\, \mathbf{e}_i;\ h_i(y, t)\, e_i;\ u_i(y, t)\, \mathbf{e}_i;\ \pi\}.$$

## 5. Non linear stability of MHD dusty gas flows in a bouuded domain to tridimensional perturbations.

We shall study, now, the stability of an electroconducting dusty gas flow into an arbitrary (bounded) domain $c$ with rigid, fixed and conducting boundaries $\Sigma$, to the general tridimensional perturbations of the class

$$\{U_i(x, y, z, t)\, \mathbf{e}_i;\ h_i(x, y, z, t)\, \mathbf{e}_i;\ u_i(x, y, z, t)\, \mathbf{e}_i;\ \pi\} \tag{41}$$

In this case we have that these perturbations to a general flow $[\mathbf{V}, \mathbf{H}, \mathbf{v}, p]$ obeying to (27), such as for instance (30), must satisfy the dimensionless system:

$$\begin{cases} \dfrac{\partial \mathbf{U}}{\partial t} = -\nabla \pi - [(\mathbf{V}+\mathbf{U}) \cdot \nabla \mathbf{U} + \mathbf{U} \cdot \nabla \mathbf{V}] + \sigma \Delta \mathbf{U} - R_1^m (\mathbf{U} - \mathbf{u}) + \\ \qquad + \dfrac{M^2}{R} [\mathrm{rot}\, (\mathbf{H} \times \mathbf{h}) \times \mathbf{h} + \mathrm{rot}\, \mathbf{h} \times \mathbf{H}] \\ \dfrac{\partial \mathbf{h}}{\partial t} = R^m \{\mathrm{rot}\, [(\mathbf{V}+\mathbf{U}) \times \mathbf{h}] + \mathrm{rot}\, (\mathbf{U} \times \mathbf{H})\} + \Delta \mathbf{h} \\ \dfrac{\partial \mathbf{u}}{\partial t} = -(v + \mathbf{u}) \cdot \overline{V} \mathbf{u} - \mathbf{u} \cdot \overline{V} v + R_2^m (\mathbf{U} - \mathbf{u}) \\ \nabla \cdot \mathbf{U} = \nabla \cdot \mathbf{u} = \nabla \cdot \mathbf{h} = 0 \end{cases} \tag{42}$$

with the boundary conditions

$$\mathbf{U}(x, y, z, t) = \mathbf{h}(x, y, z, t) = \mathbf{u}(x, y, z, t) = 0, \qquad (x, y, z, t) \in \Sigma \times [0, +\infty[ \tag{43}$$

Choosen the family of measures

(44) $$E(t)=\frac{1}{2}\int_c [c_1\,\mathbf{U}^2(x,y,z,t)+c_2\,\mathbf{h}^2(x,y,z,t)+c_3\,\mathbf{u}^2(x,y,z,t)]\,dC$$

and working as usual, we find:

(45) $$\frac{dE}{dt}=\int_c\Big\{-c_1\,\mathbf{U}\cdot\mathbf{D}\cdot\mathbf{U}-c_3\,\mathbf{u}\cdot\mathbf{D}'\cdot\mathbf{u}-c_1\,\sigma(\nabla\,\mathbf{U})^2-c_1\,R_1^m\,\mathbf{U}^2-c_3\,R_2^m\,\mathbf{u}^2+$$

$$+(c_1R_1^m+c_3R_2^m)\,\mathbf{Uu}-c_2(\nabla\,\mathbf{h})^2+\frac{c_1\,M^2}{R}\,\mathrm{rot}\,\mathbf{H}\times\mathbf{h}\cdot\mathbf{U}+\frac{c_1\,M^2}{R}\,\mathrm{rot}\,\mathbf{h}\times\mathbf{h}\cdot\mathbf{U}+$$

$$+\left(c_2\,R^m-\frac{c_1\,M^2}{R}\right)\mathrm{rot}\,\mathbf{h}\cdot\mathbf{U}\times\mathbf{H}+c_2\,R^m\,\mathbf{h}\cdot\mathbf{D}\cdot\mathbf{h}+c_2\,R^m\,\mathbf{h}\cdot\nabla\,\mathbf{U}\cdot\mathbf{h}\Big\}\,dC$$

in which $D$ and $D'$ are the deformation tensors for the fluid and for the dust. From (45), if we put

(46) $$c_2/c_1=M^2/(RR^m)$$

it follows

(47) $$\frac{dE}{dt}\leqslant -G$$

with

(48) $$G=c_1\left(\sigma\gamma_1^2+R_1^m-m-\frac{QM^2}{2\,R}\right)X_1^2+\left[c_2(\gamma_1^2-m_1\,R^m)-\frac{Q\,c_1\,M^2}{2\,R}\right]X_2^2+$$

$$+c_3(R_2^m-m^1)\,X_3^2-(c_1\,R_1^m+c_3\,R_2^m)\,X_1\,X_3$$

and

$$Q=\sup_{c\times[0,\,\infty[}|\mathrm{rot}\,H|,\quad m=-\inf_{c\times[0,\,\infty[}D_{ii},\quad m'=-\inf_{c\times[0,\,\infty[}D'_{ii},\quad m_1=\sup_{c\times[0,\,\infty[}D_{ii}$$

$$X_1^2=\int_c \mathbf{U}^2\,dc,\quad X_2^2=\int_c \mathbf{h}^2\,dc,\quad X_3^2=\int_c \mathbf{u}^2\,dc.$$

From (47)—(48), thanks to the theorem given in [31], the condition assures

(49) $$(c_1\,R_1^m-c_3\,R_2^m)^2<4\,c_1\,c_3\left[\left(\sigma\gamma_1^2-m-\frac{QM^2}{2\,R}\right)(R_2^m-m^1)-m^1\,R_1^m\right]$$

assures the asympt. exp. stability for any flow belonging to the subclass of the general class

(50) $$\begin{cases} m^1<R_2^m \\ \alpha<\gamma_1^2\end{cases}\text{ with }\alpha=\min\left\{\frac{1}{\sigma}\left(m+\frac{QM^2}{2\,R}+\frac{m^1\,R_1^m}{R_2^m-m^1}\right);\ \left(m'+\frac{1}{2}\,Q\right)R^m\right\}$$

of $[V, H, v, p]$ obeying to (27).

It's easy ,at this time, to conclude that:
the choice

(51) $$c_3/c_1 = f$$

togheter with (46), gives us the (partial) uncond. asumpt. exp. stability in the above subclass, to the measure

(52) $$E(t) = \frac{1}{2} \int_c \left\{ R^m U^2 + \frac{M^2}{R} h^2 + R^m f u^2 \right\} dc.$$

1st REMARK. — We underline that:

*In the MHD isotropic case of a clean (incompressible) gas, any flow belonging to the subclass*

(53) $$\beta < \gamma_1^2 \text{ with } \beta = \min \left\{ \frac{R}{R^m} \left( m + \frac{QM^2}{2R} \right);\ \left( m^1 + \frac{Q}{2} \right) R^m \right\}$$

*is abs. uncond. asympt. exp. stable to the measure*

(54) $$E(t) = \frac{1}{2} \int_c \left\{ R^m \mathbf{U}^2 + \frac{M^2}{R} \mathbf{h}^2 \right\} dc$$

2nd REMARK. – From (30), (49), (50), we have that:

*In the plane layer. S. Hartmann's flows of the class*

(55) $$\delta < \gamma_1^2, \text{ with } \delta = \frac{1}{2} \sup_{[-1,1]} \left| \frac{d\tilde{h}}{dy} \right| \cdot \min \{ M^2/(\sigma R);\ R^m \}$$

*are abs. unc. asympt. exp. stable to the tridimensional (suitable periodic) perturbations* (41) *in the measure* (52).

3rd REMARK. — It is interesting to note that the families of measures (40) and (22) to onedimensional and tridimensional perturbations respectively, are exactly the same (formally, of course). This choice has produced, in our opinion, the costs (50) and, in particular (53)—(55), that we must have payed. It is for this that we have not called the measures (52)—(54) the best ones, as the measures (19)—(24) and (40). Perhaps measures better than (52)—(54) can exist to *total* abs. unc. stability, but til now we don't know; so this is an open problem.

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