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VAN DER POL'S OSCILLATOR EXCITED BY NARROW-BAND NOISE

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1. Introduction

A number of authors have investigated the response characteristics of a hardening-type oscillator excited by a wide-band process either from analysis of the associated Fokker-Planck equation or from the moments of the response [1—4]. The interesting feature is that the response amplitude is singlevalued. One usually associates multivalued response phenomena with this type of equation. It is because of the wide-band spectrum, or lack of correlation of the excitation that the system is unable to extract energy from the input process (or lose energy) which is required for the occurrence of jumps. Lyon, et al. [3], demonstrated analytically and experimentally that jumps can occur when the oscillator is subjected to narrow-band noise. The analytical work is based on a linearization method for which a necessary condition is that the size of the fluctuations must be restricted. Also the results are based on finding roots of a polynomial by approximate methods and is more qualitative than quantitative. Lennox [5] has demonstrated the existence of jumps for the hardening-type oscillator and derived an expression for the probability of jumps.

In this paper a quasi-static approach is applied [6]. It is essentially the opposite approach to that of the stochastic of Markov method with its associated Fokker-Planck equation. Since the input spectrum is narrow, the fluctuations inherent in the input amplitude occur at a more slower rate than the fluctuations in the response amplitude and phase. That is, if τ_{cor} is the correlation time of the stationary excitation $\xi(t)$ given by

$$\tau_{kor} = \frac{\int_0^\infty |K_\xi(\tau)|\, d\tau}{K(0)} \tag{1}$$

where

$$K_\xi(\tau) = E[\xi(t)\,\xi(t+\tau)] \tag{2}$$

and, for simplicity, it is assumed that

$$E[\xi(t)] = 0 \tag{3}$$

then the quasi-static method can be applied whenever the inequality

$$\tau_{cor} \gg T_{rel} \tag{4}$$

is satistified where T_{rel} is the greater of the relaxation time constants describing the time it takes for either the amplitude or phase to change appreciably.

2. Analysis

The system is assumed to be described by the stochastic differential equation

$$\ddot{y} + (\alpha + \beta y^2)\dot{y} + \omega_0{}^2 y = \xi(t) \tag{5}$$

where $y(t)$ is the response of interest, α, β are parameters ($\alpha > 0$, $\beta < 0$), ω_0 is the undamped natural frequency of the system and $\xi(t)$ is a Gaussian stationary narrow--band stochastic process whose properties are completely described by equations (2) and (3).

The narrow-band perturbation can be expressed in the form

$$\xi(t) = h(t) \sin[\nu t + \psi(t)]$$

where the amplitude $h(t)$ and the phase $\psi(t)$ are slowly varying functions in time compared to the oscillatory term $\sin(\nu t)$. The spectrum of $\xi(t)$ will be appreciably different from zero only in the narrow band near the center frequency $\nu \gg \Delta\omega$.

$$|\omega - \nu| \leqslant \frac{\Delta\omega}{2} \tag{6}$$

If $\Delta\omega$ is the smallest such bandwidth then τ_{cor} of the process $\xi(t)$ will be relatively large, i.e.,

$$\tau_{cor} \sim \frac{1}{\Delta\omega}$$

Equation (5) is transformed to the corresponding equations of "standard form" describing the amplitude $a(t)$ and phase $\Phi(t)$ of the response $y(t)$ by requiring that

$$\begin{aligned} y(t) &= a(t)\cos\Phi, \qquad \Phi = \nu t + \varphi(t) \\ \dot{y}(t) &= -\nu a(t) \sin\Phi \end{aligned} \tag{7}$$

with the result

$$\dot{a} = \frac{a(\omega_0{}^2 - \nu^2)}{2\nu} \sin 2\Phi - \alpha a \sin^2\Phi - \beta a^3 \cos^2\Phi \sin^2\Phi - g_1(t) \tag{8a}$$

$$\dot{\Phi} = \frac{\omega_0{}^2 - \nu^2}{\nu} \cos^2\Phi - \frac{\alpha}{2} \sin 2\Phi - \beta a^2 \cos^3\Phi \sin\Phi - g_2(t) \tag{8b}$$

where

$$g_1(t) = \frac{h(t)}{\nu} \sin\Phi \sin(\nu t + \psi) \tag{9a}$$

$$g_2(t) = \frac{h(t)}{a\nu} \cos\Phi \sin(\nu t + \psi) \tag{9b}$$

are the fluctuation terms. These equations are exact and are now simplified byusing the method based on the "averaging principle" of Bogoliubov and Mitropolsky [7] for the oscillatory terms and by using the method described by Stratonovich [6] for the fluctuational terms. The first approximation leads to the following equations:

$$\dot{a} = -\frac{\alpha}{2}a - \frac{\beta a^3}{8} - \frac{h(t)}{2\nu}\cos(\varphi - \psi) \tag{10a}$$

$$\dot{\Phi} = \frac{\omega_0^2 - \nu^2}{2\nu} + \frac{h(t)}{2a\nu}\sin(\varphi - \psi) \tag{10b}$$

It should be noted that the small parameter, which is usually associated with this type of approximation is not explicitly included in the analysis. Thus the approximation is valid only under certain conditions. We could require, for example $1/\nu < 1$. Physically, however, if the bandwidth of the excitation is small then the response will be close to sinusoidal and both the amplitude and phase will be slowly varying functions of time.

The relaxation time of the amplitude $a(t)$ is of the order $1/\alpha$ while the relaxation time of the phase $\varphi(t)$ can be estimated as

$$T_{\mathrm{rel}\,\Phi} = \left\langle \frac{2a\nu}{h(t)} \right\rangle \tag{11}$$

Thus, in the present case, the conditions for applying the quasi-static method take the form

$$\tau_{\mathrm{cor}} \gg \frac{1}{\alpha} \qquad \tau_{\mathrm{cor}} \gg \left\langle \frac{2a\nu}{h(t)} \right\rangle \tag{12}$$

If these conditions are satisfied, then the amplitude $a(t)$ and the phase $\varphi(t)$ manage to take "quasi-static" values so that both $\dot{a}$ and $\dot{\varphi}$ can be assumed to be zero. Thus,

$$\begin{aligned} h(t)\sin(\varphi - \psi) &= a(\nu^2 - \omega_0^2) \\ h(t)\cos(\varphi - \psi) &= -\alpha a \nu - \frac{\beta\nu a^3}{4} \end{aligned} \tag{13}$$

Equations (13) determine the amplitude and phase az zeromemory functions of the excitation, i.e., as functions of $h(t)$ and $\psi(t)$ at the same instant of time. Statistical characteristics of both the amplitude and phase can be found by using methods of nonlinear transformations. To obtain more exact results, higher-order approximations can be used.

On squaring both sides and adding them, an inputoutput relationship is obtained:

$$h^2(t) = a^2\left[(\nu^2 - \omega_0^2)^2 + \nu^2\left(\alpha + \frac{\beta}{4}a^2\right)^2\right] \tag{14}$$

In Fig. 1, h^2 is plotted as a function of a^2 fo specific parameters.

3. Stability response

As in the deterministic case certain values of a will be unstable. Equations (13) and (14) represent the necessary but not sufficient conditions for the existence of stable values of a and φ. Sufficient conditions are obtained by letting δa and $\delta\varphi$ be small deviations of a and φ from their stable values so that the following linearized variational equations for δa and $\delta\varphi$ are obtained:

$$\delta \dot{a} = a_{11}\,\delta a + a_{12}\,\delta\varphi$$

$$\delta \dot{\varphi} = a_{21}\,\delta a + a_{22}\,\delta\varphi \qquad (15)$$

where

$$a_{11} = -\frac{\alpha}{2} - \frac{3\beta}{8}a^2$$

$$a_{12} = \frac{a}{2\nu}(\nu^2 - \omega_0^2)$$

$$a_{21} = -\frac{\nu^2 - \omega_0^2}{2a\nu}$$

$$a_{22} = -\frac{\alpha}{2} - \frac{\beta}{8}a^2 \qquad (16)$$

It is obvious that the necessary and sufficient conditions for asymptotic stability

$$a_{11} + a_{22} = -\alpha - \frac{\beta}{2}z < 0$$

$$a_{11}a_{22} - a_{12}a_{21} = \frac{3\beta^2 z^2}{64} + \frac{\alpha\beta}{4}z + \frac{\alpha^2}{4} + \frac{(\nu^2 - \omega_0^2)^2}{4\nu^2} > 0 \qquad (17)$$

where $z = a^2$. Note that the first condition is always fulfilled for $z > -2\alpha/\beta$. The second condition can be expressed as

$$\frac{1}{4\nu^2}\frac{df(z)}{dz} > 0 \qquad (18)$$

If equation (14) is rewritten in the form

$$f(z) = z\left[(\nu^2 - \omega_0^2)^2 + \nu^2\left(\alpha + \frac{\beta}{4}z\right)^2\right] \qquad (19)$$

where $f(z) = h^2(t)$ is always nonnegative.

The function $f(z)$, as plotted in Fig. 1 has a maximum point at $z = z_1$ and minimum point at $z = z_2$ where

$$z_{1/2} = -\frac{8\alpha}{3\beta} \pm \frac{4\alpha}{3\beta}\sqrt{1 - \frac{3}{\alpha^2\nu^2}(\nu^2 - \omega_0^2)} \qquad (20)$$

for $\alpha^2\nu^2 - 3(\nu^2 - \omega_0^2)^2 > 0$. It is found that the range of possible stable values of $z = a^2$ to be determined by the positive derivative of $f(z)$ and condition $z > -2\alpha/\beta$, lie in the region

$$z_3 < z < \infty \tag{21}$$

This corresponds to the region

$$h_1^2 < h^2(t) < \infty \tag{22}$$

where h_1 can easily be determined by equation (19) at $z = z_1$. Note that the occurrence of the multivalued responses of the system is not possible in this case.

4. Probability of stability region

Consider the case where $\xi(t)$ is a narrow-band Gaussian process with zero mean and variance

$$\sigma_0^2 = \langle f^2(t) \rangle = \langle \frac{1}{2} h^2(t) \rangle \tag{23}$$

given that the initial phase is completely random, that is, uniformly distributed in $(0, 2\pi)$

$$w_\Phi(\varphi) = \frac{1}{2\pi} \qquad 0 \leqslant \varphi \leqslant 2\pi \tag{24}$$

Then the amplitude $h(t)$ the process $\xi(t)$ will be Rayleigh distributed

$$w_h(h)\,dh = \frac{h}{\sigma_0^2} \exp\left(-\frac{h^2}{2\sigma_0^2}\right) dh = -d\left[\exp\left(-\frac{h}{2\sigma_0^2}\right)\right], \quad 0 \leqslant h \leqslant \infty \tag{25}$$

with mean value

$$\langle h \rangle = \int_0^h h w_h(h)\, dh = \sigma_0 \sqrt{\frac{\pi}{2}} \tag{26}$$

mean square value

$$\langle h^2 \rangle = 2\sigma_0^2 \tag{27}$$

and variance

$$\sigma_h^2 = \left(2 - \frac{\pi}{2}\right) \sigma_0^2 \tag{28}$$

Referring to Fig. 1, it is found that the stability region $h_1 \leq h(t) \leq \propto$ occurs with probability

$$P = -\int_{h_1}^{\infty} d\left[\exp\left(-\frac{h^2}{2\sigma_0^2}\right)\right] = \exp\left(-\frac{h_1}{2\sigma_0^2}\right) \tag{29}$$

Thus, for a given system $(\alpha, \beta, \omega_0)$ and given narrow-band excitation (ν, σ_0^2), the probability of stability region will occur as it is given by equations (29), (22) and (19).

In the region $z>z_3$ the relation between $f(z)$ and z is the single-valued and usual method of probability transformation to be employed. Thus the density function for the response amplitude a is obtained

$$w_a(a)=\frac{a}{\sigma_0^2}\left[\frac{3\,\nu^2\,\beta^2\,z^2}{16}+\alpha\beta\nu^2\,z+\alpha^2\,\nu^2+(\nu^2-\omega_0^2)^2\right]\cdot$$

$$\cdot\exp\left\{-\frac{a^2}{2\,\sigma_0^2}\left[(\nu^2-\omega_0^2)^2+\nu^2\left(\alpha+\frac{\beta}{4}\,a^2\right)^2\right]\right\} \qquad (30)$$

which holds for $a\geq a_3$. Also, since the occurrence of response amplitudes on the branch BQC has been verified to be unstable, it is assumed that

$$w_a(a)=0, \qquad a<a_3 \qquad (31)$$

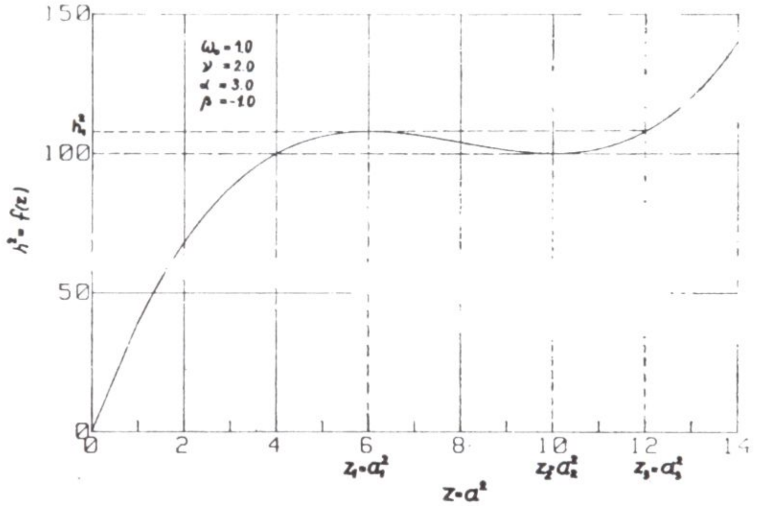

Fig. 1 Equation (14) ($z=a^2$)

5. Conclusions

Van der Pol oscillator is excitet by a stationary narrow-band noise. The analysis is based on the concept of quasi-tatic amplitude and phase values which is exactly opposite to the stochastic of Markov approach, in effect, replaces a system with memory with one wihout memory. It is demonstrated that the occurrence of jumps is not possible in this system and an expression for probability of stability region occurrence of the response amplitude is derived.

REFERENCES

[1] Lyon, R., On the Vibration Statistics of a Randomly Excited Hard-Spring Oscillator, J. Acoust. Soc. Amer. 32 (1960) 716—719.

[2] Lyon, R., On the Vibration Statistics of a Randomly Excited Hard-Spring Oscillator II, J. Acoust. Soc. Amer. 33 (1960) 1395—1403.

[3] Lyon, R., Neckl, M., and Hazelgrove, C. B. Response of Hard-Spring Oscillator to Narrow-Band Excitation, J. Acoust. Soc. Amer. 33 (1961) 1404—1411.

[4] Caughey, T. K., Response of Van Der Pol's Oscillator to Random Excitation, J. Appl., Mech. 26, No3, (1959) 345—348.

[5] Lennox, W. C., and Kuak, Y. C., Narrow-Band Excitation of a Nonlinear Oscillator, J. App. Mech. (1976) 340—344.

[6] Stratonovich, R. L., Topics in the Random Noise, Vol. II, New York, Gordon and Breach, 1976.

[7] Bogoliubov, N. N., and Mitropolsky, Y. A., Asymptotic Methods in the Theory of Nonlinear Oscillations, New York, Gordon and Breach, 1961.

[8] Ibrahim, A. R., Parametric Random Vibration, John Wiley & Sons Inc. (1985).

VAN DER FOLSCHE OSZILLATOR DURCH BANDRAUSCHEN

Van der polsche Oszillator ist durch den stationären Engbandrausch angeregt. Die Analyse beruht am Konzept des quasistatischen Amplituden und Phasenwertes, welcher der stochastischen oder Markschen mit Fokker-Plankschen Gleichung verbundenen Approximation vollkommen entgegengesetzt ist. Diese Appromxiation ersetzt eigentlich das System mit Speicher durch das System ohne Speicher. Es wurde gezeigt, dass in diesem System kein Sprung möglich ist, und die Formel zur Erscheinungswarscheinlichkeit eines stabilen Geibiets der Antwortamplituden wurde ausgeleitet.

USKOPOJASNO POBUĐEN VAN DER POL-ov OSCILATOR

Van der Pol-ov oscilator pobuđen je stacionarnim uskopojasnim šumom. Analiza je bazirana na konceptu kvazistatičke vrednosti amplitude i faze koja je potpuno suprotna stohastičkoj ili Markovoj aproksimaciji povezanoj sa Fokker--Planckovom jednačinom. Ovom aproksimacijom ustvari sistem sa memorijom zamenjuje se sistemom bez memorije. Pokazano je da u ovom sistemu nije moguća pojava skoka i izvedena je formula za verovatnoću slučaja stabilne oblasti amplitude odgovara.

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