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STABILITY OF THE MOMENTS OF DOUBLE PARAMETRIC RANDOM EXCITATION OF A DAMPED MATHIEU OSCILLATOR

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The effect of double parametric excitation on the stability of the moments of a damped Mathieu oscillator is investigated. Stability conditions of the first and second moments of the response are obtained in the case when the frequency of the harmonic excitation lies in the neighbourhood of twice the natural frequency of the oscillator. It has been found that the presence of additional random excitations can cause either a stabilizing or destabilizing effect depending on the values of certain parameters of random disturbances. In particular, it is shown that the first moments of the response of the first order, when both excitations are of a white noise type, are only effected by the parametric excitations of dampness. The second moments of the response are effected by both the excitations having a destabilizing effect. For the case of exponentially correlated noise, stability conditions for the first and second moments of the response are obtained.

1. Introduction

It is known that the instability of a Mathieu Oscillator is defined when the ratio of the frequency of the forced excitation is twice the natural frequency of the oscillator and if it is in the neighbourhlood of the values $1/p$, where, p is a positive integer. The most important of these corresponds, to $p=1$ and is known as the region of fundamental resonance.

When the harmonic excitation is replaced by stationary stohastic excitation, conditions for stochastic stability have been obtained by [1], [2] and [3] in their works showing that, to a first approximation, stability depends only on the value of the excitation spectrum in the neighbourhood of twice the system natural frequency. In [4] are defined conditions for the stability of the Mathieu oscillator when the random excitation was added by a harmonic one and they have found that the stability conditions depend only on the spectral frequency of the oscillator.

This paper investigates the conditions of stability of the moments of the response of a similar system subjected to double random parametric excitation when the frequency of the harmonic excitation lies in the region of fundamental parametric resonance. Conditions are defined for the stability of the first and second moments of the response and they are valid for the first approximation. It has been found that they depend on the random excitation only through their spectral values at zero frequency and at twice the natural frequency of the oscillator.

2. Formulation

The damped Mathieu oscillator subjected to a parametric random dampness and parametric random excitation can be described by a following stohastic differential equation of the form

$$\ddot{x}+\omega_0(2\zeta+\varepsilon^{1/2}G_{(t)})\dot{x}+\omega_0^2(1+h\sin\nu t+\varepsilon^{1/2}F_{(t)})x=0, \qquad (1)$$

where ζ, ω_0 represent, respectively, the dampness ratio and the undamped natural frequency of the oscillator, h denotes amplitude, ν is the frequency of the harmonic excitation. $F(t)$ and $G(t)$ are non-correlated random processes with zero mean value, $\varepsilon\ll 1$ is a small parameter. Let us consider the case of the fundamental parametric resonance, i.e. when $\nu\approx 2\omega_0$ and setting

$$\omega_0^2=\nu/4+\varepsilon\cdot\Delta, \qquad (2)$$

where $\Delta\cdot\varepsilon$ denotes the amount of detuning, equation (1) may be written as

$$\ddot{x}+\frac{\nu^2}{4}x=-\varepsilon[2\zeta\omega_0\dot{x}+(\Delta+\omega_0^2 h\sin\nu t)x]-\varepsilon^{1/2}\omega_0 G_{(t)}\dot{x}-\varepsilon^{1/2}\omega_0^2 F_{(t)}x \qquad (3)$$

Transforming to new variables Z_1, Z_2 by the relations

$$x=Z_1\cos\left(\frac{\nu}{2}t\right)+Z_2\sin\left(\frac{\nu}{2}t\right), \qquad \dot{x}=-\frac{Z_1\nu}{2}\sin\left(\frac{\nu}{2}t\right)+\frac{Z_2\nu}{2}\cos\left(\frac{\nu}{2}t\right), \qquad (4)$$

equation (3) may be replaced by the pair of first order equations,

$$\dot{Z}_1=-\varepsilon\left[2\zeta\omega_0\left(Z_1\sin^2\frac{\nu t}{2}-\frac{Z_2}{2}\sin\nu t\right)-\left(\frac{2\cdot\Delta}{\nu}+\omega_0 h\sin\nu t\right)\left(\frac{Z_1}{2}\sin\nu t+Z_2\sin^2\frac{\nu t}{2}\right)\right]+$$

$$+\varepsilon^{1/2}\omega_0 G_{(t)}\left(-Z_1\sin^2\frac{\nu t}{2}-\frac{1}{2}Z_2\sin\nu t\right)+\varepsilon^{1/2}\omega_0 F_{(t)}\left(\frac{Z_1}{2}\sin\nu t+Z_2\sin^2\frac{\nu t}{2}\right), \qquad (4a)$$

$$\dot{Z}_2=\varepsilon\left[2\zeta\omega_0\left(\frac{1}{2}Z_1\sin\nu t-Z_2\cos^2\frac{2\nu t}{2}\right)-\left(\frac{2\Delta}{\nu}+\omega_0 h\sin\nu t\right)\left(Z_1\cos^2\frac{\nu t}{2}+\frac{Z_2}{2}\sin\nu t\right)\right]-$$

$$-\varepsilon^{1/2}\omega_0 G_{(t)}\left(-\frac{Z_1}{2}\sin\nu t+Z_2\cos^2\frac{\nu t}{2}\right)-\varepsilon^{1/2}\omega_0 F_{(t)}\left(Z_1\cos^2\frac{\nu t}{2}+\frac{Z_2}{2}\sin\nu t\right) \qquad (4b)$$

If the relaxation time of the Z_1, Z_2 processes is much larger than the correlation time of the stohastic non-corrrelated processes $G(t)$ and $F(t)$, the solutions of equations may be approximated over a time interval of order $0(\varepsilon^{-1})$ by a MARKOV vektor process with probability one. The procedure for this approximation has been given by [1] and [2] and when applied to equation (4), leads to the following homogenuous Ito equations:

$$dZ_1 = \varepsilon\left[\omega_0\left(\frac{h}{4} - \zeta^*\right)Z_1 + \left(\frac{\Delta}{\nu} - a\right)Z_2\right]dt + \varepsilon^{1/2}\left[\sigma_{11(z)}\,dw_1 + \sigma_{12(z)}\,dw_2\right] \quad (5a)$$

$$dZ_2 = \varepsilon\left[-\left(\frac{\Delta}{\nu} - a\right)Z_1 - \omega_0\left(\frac{h}{4} + \zeta^*\right)Z_2\right]dt + \varepsilon^{1/2}\left[\sigma_{21(z)}\,dw_1 + \sigma_{22(z)}\,dw_2\right] \quad (5b)$$

$$\zeta^* = \zeta + \frac{\omega_0}{8}\left(\Phi_{F(0)} - \Phi_{F(2\omega_0)} - \Phi_{G(0)} - \Phi_{G(2\omega_0)}\right),\ a = \frac{\omega_0^2}{8}\left(\psi_{F(2\omega_0)} + \psi_{G(2\omega_0)}\right)$$

$$[\sigma\sigma^*]_{11} = \frac{\omega_0^2}{8}\left[2\Phi_{G(0)} + \Phi_{G(2\omega_0)} + \Phi_{F(2\omega_0)}\right]Z_1^2 + \frac{\omega_0^2}{8}\left[2\Phi_{F(0)} + \Phi_{F(2\omega_0)} + \Phi_{G(2\omega_0)}\right]Z_2^2$$

$$[\sigma\sigma^*]_{21} = [\sigma\sigma^*]_{12} = \left(\Phi_{G(0)} - \Phi_{F(0)}\right)\frac{\omega_0^2}{4}Z_1 Z_2$$

$$[\sigma\sigma^*]_{22} = \frac{\omega_0^2}{8}\left[2\Phi_{F(0)} + \Phi_{F(2\omega_0)} + \Phi_{G(2\omega_0)}\right]Z_1^2 + \frac{\omega_0^2}{2}\left[2\Phi_{G(0)} + \Phi_{G(2\omega_0)} + \Phi_{F(2\omega_0)}\right]Z_2^2$$

$$\Phi_{F(\omega)} + i\psi_{F(\omega)} = 2\int_0^\infty E[F_{(t)}F_{(t+\tau)}]\,e^{i\omega\tau}\,dt,\quad \Phi_{G(\omega)} + i\psi_{G(\omega)} = 2\int_0^\infty E[G_{(t)}G_{(t+\tau)}]\,e^{i\omega\tau}\,d\tau$$

where $E[\cdot]$ denote the expected value of the ensemble in brackets.

3. Stability conditions

The differential equations governing the mean values z_1 and z_2 can be obtained by taking the expectations of both sides of equations (5).

$$\frac{d}{dt}\begin{bmatrix} <Z_1> \\ <Z_2> \end{bmatrix} = \begin{bmatrix} \omega_0\left(\frac{h}{4} - \zeta^*\right) & \left(\frac{\Delta}{\nu} - a\right) \\ -\left(\frac{\Delta}{\nu} - a\right) & -\omega_0\left(\frac{h}{4} + \zeta^*\right) \end{bmatrix} \cdot \begin{bmatrix} <Z_1> \\ <Z_2> \end{bmatrix} \quad (6)$$

Here, the conditions for stability in the first moments are,

$$\zeta^* > 0,\quad \omega_0^2\zeta^{*2} + \left(\frac{\Delta}{\nu} - a\right)^2 > \omega_0^2\frac{h^2}{16} \quad (7)$$

which are equivalent to,

$$\zeta > \frac{\omega_0}{8}\left(\Phi_F(2\omega_0) + \Phi_G(2\omega_0) + \Phi_{G(0)} - \Phi_{F(0)}\right); \quad (8a)$$

$$4\left\{\varepsilon^2\left[\zeta - \frac{\omega_0}{8}\left(\Phi_F(2\omega_0) - \Phi_{F(0)} + \Phi_G(2\omega_0) + \Phi_{G(0)}\right)\right]^2 + \left[1 - \frac{\nu}{2\omega_0} - \varepsilon\frac{\omega_0}{8}\left(\psi_F(2\omega_0) + \psi_F(2\omega_0)\right)\right]^2\right\}^{1/2} > \varepsilon h. \quad (8b)$$

In the case when both the excitations are of a white noise type, conditions (8) reduce to

$$\zeta > \frac{\omega_0}{4}\Phi_{G0}$$

$$4\left\{\varepsilon^2\left[\zeta - \frac{\omega_0}{4}\Phi_{G0}\right]^2 + \left[1 - \frac{\nu}{2\omega_0}\right]^2\right\}^{1/2} > \varepsilon h, \tag{9}$$

$$\Phi_{F(\omega)} = \Phi_{F0} = \text{const}, \quad \Phi_{G(\omega)} = \Phi_{G0} = \text{const}, \quad \psi_{G(\omega)} = \psi_{F(\omega)} = 0.$$

We may remark that apart from the stability conditions in the first moments in [4], conditions for stability (9) depend on the random parametric excitation The differential equations corresponding to the second moments can be found by applying Ito's differential rule to the variables $Z^2{}_1$, Z_1Z_2, $Z_2{}^2$. This procedure leads to

$$\frac{d}{dt}\begin{bmatrix} <Z_1^2> \\ <Z_1 Z_2> \\ <Z_2^2> \end{bmatrix} = 2\varepsilon \begin{bmatrix} d-C & A & B \\ -\frac{A}{2} & -\omega_0(B+C) & \frac{A}{2} \\ B & -A & -(d+C) \end{bmatrix} \cdot \begin{bmatrix} <Z_1^2> \\ <Z_1 Z_2> \\ <Z_2^2> \end{bmatrix} \tag{10}$$

where

$$d = \frac{\omega_0 h}{4}, \quad c = \frac{\omega_0}{8}\Phi_{F(0)}, \quad b = \frac{\omega_0}{16}\Phi_F(2\omega_0), \quad g = \frac{\omega_0}{4}\Phi_{G(0)}, \quad k = \frac{\omega_0}{16}\Phi_G(2\omega_0),$$

$$A = \frac{\Delta}{\nu} - a, \quad B = \omega_0(c+b+k), \quad C = \omega_0(\zeta + c - 3b - g - 3k).$$

From equations (10) the stability conditions in the second moments (i.e. mean square stability) are found to be,

$$\zeta > 4b + g + 4k, \tag{11a}$$

$$\frac{\zeta - 4b - g - 4k}{\zeta - 2b + 2c - g - 2k}\left[\left(\frac{\Delta}{\nu} - a\right)^2 + \omega_0^2(\zeta - 2b + 2c - g - 2k)^2\right] > \frac{\omega_0^2 h^2}{16} \tag{11b}$$

In the case when both exictations are of a white noise type, the conditions (11) reduce to,

$$\zeta > \frac{\omega_0}{4}(\Phi_{F0} + 2\Phi_{G0}), \tag{12a}$$

$$4\left\{\left(1 - \frac{\nu}{2\omega_0}\right)^2 + \varepsilon^2\left(\zeta + \frac{\omega_0}{8}\Phi_{F0} - \frac{3\omega_0}{8}\Phi_{G0}\right)^2\right\}^{1/2}\left[\frac{8\zeta - 2\omega_0(\Phi_{F0} + 2\Phi_{G0})}{8\zeta + \omega_0(\Phi_{F0} - 3\Phi_{G0})}\right]^{1/2} > \varepsilon h. \tag{12b}$$

As an example of non-white random parametric excitations, suppose that $F(t)$ and $G(t)$ are exponentially correlated so that

$$K_F(\tau) = <F_{(t)}F_{(t+\tau)}> = \sigma_F^2 \exp(-\alpha_F|\tau|), \quad K_G(\tau) = <G_{(t)}G_{(t+\tau)}> = \sigma_G^2 \exp(-\alpha_G|\tau|).$$

Then,

$$\Phi_F(\omega)=\frac{2\sigma_F^2\alpha_F}{\alpha_F^2+\omega^2},\quad \Phi_G=\frac{2\sigma_G^2\alpha_G}{\alpha_G^2+\omega^2},\quad \psi_F(\omega)=\frac{2\sigma_F^2\omega}{\alpha_F^2+\omega^2},\quad \psi_G(\omega)=\frac{2\sigma_G^2\omega}{\alpha_G^2+\omega^2},$$

and the stabillity conditions (8) and (11) become

$$\zeta>\frac{1}{8}\left[\frac{1+2K_G^2}{K_G(1+K_G^2)}\sigma_G^2-\frac{\sigma_F^2}{K_F(1+K_F^2)}\right], \tag{13a}$$

$$\varepsilon^2\left\{\zeta+\frac{1}{8}\left[\frac{\sigma_F^2}{K_F(1+K_F^2)}-\frac{1+2K_G^2}{K_G(1+K_G^2)}\sigma_G^2\right]\right\}^2+$$
$$+\left\{1-\frac{\nu}{2\omega_0}-\frac{\varepsilon}{8}\left[\frac{\sigma_F^2}{1+K_F^2}+\frac{\sigma_G^2}{1+K_G^2}\right]\right\}^2>\frac{\varepsilon^2h^2}{16} \tag{13b}$$

for the first moments, and

$$\zeta>\frac{1}{4}\left[\frac{K_F}{1+K_F^2}\sigma_F^2+\frac{1+2K_G^2}{K_G(1+K_G^2)}\sigma_G^2\right], \tag{14a}$$

$$D\left\{\left[1-\frac{\nu}{2\omega_0}-\frac{\varepsilon}{8}\left(\frac{\sigma_F^2}{1+K_F^2}+\frac{\sigma_G^2}{1+K_G^2}\right)\right]^2+\varepsilon^2\left[\zeta+\frac{(2+K_F^2)\sigma_F^2}{8K_F(1+K_F^2)}-\right.\right.$$
$$\left.\left.-\frac{(1+3K_G^2)}{8K_G(1+K_G^2)}\sigma_G^2\right]^2\right\}>\frac{\varepsilon^2h^2}{16}, \tag{14b}$$

for the second moments, where

$$K_F=\frac{\alpha_F}{2\omega_0},\quad K_G=\frac{\alpha_G}{2\omega_0},\quad D=\frac{\zeta-\frac{1}{4}\left(\frac{K_F}{1+K_F^2}\sigma_F^2+\frac{1+2K_G^2}{K_G(1+K_G^2)}\sigma_G^2\right)}{\zeta+\frac{1}{4}\left(\frac{2+K_F^2}{K_F^2(1+K_F^2)}\sigma_F^2-\frac{1+3K_G^2}{K_G(1+K_G^2)}\sigma_G^2\right)}$$

The relaxation time of the system given by the equations (5) is of the order of $O(\varepsilon^{-1})$, while the correlation times of the processes $G(t)$ and $F(t)$ are of the order of $O(\alpha_G^{-1})$ and $O(\alpha_F^{-1})$. The rezults obtained above are valid for $\alpha_F\gg\varepsilon$, $\alpha_G\gg\varepsilon$. Comparing them to the results obtained in the paper [4], we can see that the excitation $G(t)$ reduces the region of stability of the moments of the Mathieu oscillator.

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DER ZWEIPARAMETRISCH ZUFALLIG ANGEREGTE GEDAMPETE MATHIEUS OSZILLATOR

In der vorliegenden Arbeit werden die Stabilitatsbedingungen des Ansprechmomments von System, beeinflusst durch zweiparametrische Zufallanregung, untersucht, wenn die Frequenz der harmonischen Anregung im Bereich der parametrischen Fundamentalresonanz liegt. Es sind die Stabilitatsbedingungen fur des erste und das zweite Ansprechmoment erreicht, die fur die erste Approximation gelten. Es ist festgestellt, dass die Stabilitatsbedingungen von Spektralwerten der Zufallanregungen der Null-und der zweifaschen Eigenfrequenz des Oszillators abhangen.

DVO PARAMETARSKI SLUČAJNO POBUĐEN PRIGUŠEN MATHIEU-ov OSCILATOR

U ovom radu istražuju se uslovi stabilnosti momenata odgovora sistema podvrgnutog dvo-parametarskoj slučajnoj pobudi kada frekvencija harmonijske pobude leži u oblasti fundamentalne parametarske rezonancije. Dobijeni su uslovi stabilnosti za prvi i drugi moment odgovora, koji važe za prvu aproksimaciju. Nađeno je da uslovi stabilnosti zavise od spektralnih vrednosti slučajnih pobuda nulte i dvostruke sopstvene frekvencije oscilatora.

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