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SHEAR FLOW IN COMPOSITE BEAM STRUCTURES

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In the scope of the engineering beam theory we shall determine the shear flow expression assuming that the external load lies in the plane of symmetry of the composite beam (see Appendix) which has the uniform bending stiffness. The following permanent influences are introduced: dead load (G), prestressing by forces (P), shrinkage (S) and movement of supports (C). The shear flow expression will be developed under supposition that along the beam the cross section resultants depend linearly on the concrete relaxation function [1], [2].

First we shall consider a statically determinate structure and a primary system ($X_{\lambda H}=0$, $X_{\lambda H}$=redundant force) of a statically indeterminate structure. The axial force and the bending moment are as follows:

$$N_{H\Phi}=N_{1H}1^*+N_{2H}R^*,\qquad M_{H\Phi}=M_{1H}1^*+M_{2H}R^*,\qquad H=G,P,S,C, \tag{1}$$

N_{kH} and M_{kH} ($k=1, 2$) being functions on z only (z=coordinate along the beam axis). The normal stress $\sigma_{jH\Phi}$ in an arbitrary point of the part j of a composite cross section, due to the introduced influences, is given in Ref. [2]. This expression includes all assumptions concerning the rheological properties of materials given in the Appendix. It will be written in the form which is more convenient for evaluating the shear flow expression:

$$\sigma_{jH\Phi}=\nu_j\sum_{a=1}^{3}\sum_{h=1}^{2}\left[\sum_{k=1}^{2}a_{jahk}E_u\varepsilon_{hkH}+d_{jahH}\right]A^*_{ah}, \tag{2}$$

$j=c, p, n, m$; $H=G, P, S, C$; A^*_{ah} being functions as follows:

$$A^*_{1h}=1^*,\quad A^*_{2h}=R^*,\quad A^*_{3h}=B^*_h,\quad h=1,2; \tag{3}$$

and:

$$\varepsilon_{hkH}=\eta_{hkH}+y\varkappa_{hkH}, \tag{4}$$

y=coordinate along the axis of symmetry of the cross section;

$$E_u\eta_{hkH}=\delta\overline{\gamma}_{3-h}\frac{N_{kH}}{A_i}+(-1)^{3-h}\overline{\gamma}_{12}\frac{M_{kH}}{S_i},\qquad E_u\varkappa_{hkH}=(-1)^{3-h}\overline{\gamma}_{12}\frac{N_{kH}}{S_i}+\delta\overline{\gamma}_h\frac{M_{kH}}{J_i}, \tag{5}$$

$$h=1,2;\quad k=1,2;\quad H=G,P,S,C;$$

$$a_{j1h1} = \frac{1-\rho_j}{1-\gamma_h}, \qquad a_{j3h1} = -\frac{\gamma_h - \rho_j}{1-\gamma_h},$$
$$a_{j2h2} = \frac{\rho_j}{\gamma_h}, \qquad a_{j3h2} = \frac{\gamma_h - \rho_j}{\gamma_h}; \tag{6}$$

$$d_{c1hS} = -d_{c2hS} = -\frac{1}{2} E_u r,$$
$$d_{p1hP} = \frac{1}{2}(1-\rho_p)\left(\frac{P}{A_{pr}} - \overset{0}{\sigma}_{uP\Phi}\right), \qquad d_{p2hP} = \frac{1}{2}\rho_p\left(\frac{P}{A_{p\cdot}} - \overset{0}{\sigma}_{uP\Phi}\right), \tag{7}$$

$h=1, 2$. All other coefficients a_{jahk} and d_{jahH} are equal to zero. The term $\overset{0}{\sigma}_{uP\Phi}$ in d_{pahP} is given as follows:

$$\overset{0}{\sigma}_{uP\Phi} = \frac{N^0_{P\Phi}}{A_i} + \frac{M^0_{P\Phi}}{J_i} y, \tag{8}$$

in which:

$$N^0_{P\Phi} = -P + \frac{A_{pr}}{A_{oi}} N^0_{oP} + \frac{A_{pr} y_{op}}{J_{oi}} M^0_{oP},$$
$$M^0_{P\Phi} = y_p N^0_{P\Phi} + \frac{I_{pr}}{J_{oi}} M^0_{oP}, \tag{9}$$

N^0_{oP} and M^0_{oP} being the cross section resultants at time $t=t_{0-}$ due to prestressing by forces (influence $H=P$) affecting the beam at $t=t_{0-}$ too [4], [3].

Applying Jourawsky's hypothesis we determine the shear flow q, representing the total longitudinal force transmitted across the plane determined by $y=$const. per unit lenght along the beam. We establish the equilibrium of a segment of the beam which is obtained by isolating the part of the beam element dz below $y=$ $=$const., so that:

$$q_H = q_H(z, y, t, t_0) = \sum_j \int_{\hat{A}_j} \frac{\partial \sigma_{jH}}{\partial z} dA, \tag{10}$$

$H=G, P, S, C$; where $\hat{A}_j=$area of the part j belonging to the part of the cross section separated by $y=$const. We use also the known relations:

$$n_{H\Phi} = -\frac{\partial N_{H\Phi}}{\partial z} = n_{1H} 1^* + n_{2H} R^*, \qquad n_{kH} = -\frac{dN_{kH}}{dz},$$
$$T_{H\Phi} = \frac{\partial M_{H\Phi}}{\partial z} = T_{1H} 1^* + T_{2H} R^*, \qquad T_{kH} = \frac{dM_{kH}}{dz}, \tag{11}$$

$k=1, 2$; $H=G, P, S, C$; $n_{H\Phi}=$distributed forces having the z axis direction; $T_{H\Phi}=$ $=$shear force, and:

$$\overset{0}{n}_{P\Phi} = -\frac{dN^0_{P\Phi}}{dz}, \qquad T^0_{P\Phi} = \frac{dM^0_{P\Phi}}{dz}. \tag{12}$$

When we perform the operations indicated in Eq. (10) we arrive at the shear flow expression:

$$q_{H\Phi}=\sum_{j}\sum_{a=1}^{3}\sum_{h=1}^{2}\left[\sum_{k=1}^{2} a_{jahk}(b_{jh}n_{kH}+c_{jh}T_{kH})+f_{jahH}\right]A^{*}_{ah}, \tag{13}$$

$j=c, p, n, m$; $H=G, P, S, C$. The coefficients b_{jh} and c_{jh} are as follows:

$$b_{jh}=\delta\overline{\gamma}_{3-h}\frac{\hat{A}_{jr}}{A_i}+(-1)^{3-h}\overline{\gamma}_{12}\frac{\hat{S}_{jr}}{S_i},$$
$$c_{jh}=(-1)^{3-h}\overline{\gamma}_{12}\frac{\hat{A}_{jr}}{S_i}+\delta\overline{\gamma}_h\frac{\hat{S}_{jr}}{J_i}, \tag{14}$$

i.e. when $\hat{A}_{jr}$ is substituted for N_{kH} and $\hat{S}_{jr}$ for M_{kH} we obtain b_{jh} from $E_u\eta_{hkH}$ and c_{jh} from $E_u\varkappa_{hkH}$, Eq. (5);

$$\hat{A}_{jr}=\nu_j\hat{A}_j, \qquad \hat{S}_{jr}=\nu_j\hat{S}_j, \tag{15}$$

$j=c, p, n, m$; $\hat{S}_j$=first moment of $\hat{A}_j$. Coefficients f_{jahH} are as follows:

$$f_{p1hP}=-\frac{1}{2}(1-\rho_p)g_P, \qquad f_{p2hP}=-\frac{1}{2}\rho_p g_P, \qquad h=1, 2,$$
$$g_P=\frac{\hat{A}_{pr}}{A_i}n^0_{P\Phi}+\frac{\hat{S}_{pr}}{J_i}T^0_{P\Phi}, \tag{16}$$

all other coefficients f_{jahH} are equal to zero.

In a statically determinate structure or primary system ($X_{\lambda H}=0$) the cross section resultants vary according to the law given by Eq. (1) in the following way: for dead load ($H=G$):

$$N_{1G}\neq 0, \qquad N_{2G}=0,$$
$$M_{1G}\neq 0, \qquad M_{2G}=0; \tag{17}$$

for prestressing by forces ($H=P$) if prestressing steel is affected by force P at $t=t_{0-}$ and immediately after that, at $t=t_0$, it becomes a part of a composite cross section:

$$N_{1P}=(1-\rho_p)N^0_{P\Phi}, \qquad N_{2P}=\rho_p N^0_{P\Phi}$$
$$M_{1P}=(1-\rho_p)M^0_{P\Phi}, \qquad M_{2P}=\rho_p M^0_{P\Phi}; \tag{18}$$

$N^0_{P\Phi}$ and $M^0_{P\Phi}$ are determined by Eq. (9);

for shrinkage ($H=S$), under assumption given by Eq. (A.4):

$$N_{1S}=-N_{2S}=E_u rA_{cr},$$
$$M_{1S}=-M_{2S}=E_u rS_{cr}; \tag{19}$$

for movement of supports ($H=C$):

$$N_{1C}=N_{2C}=0,\qquad M_{1C}=M_{2C}=0. \tag{20}$$

Now we shall consider a statically indeterminate structure with n redundant forces. The cross section resultants vary according to the introduced law, Eq. (1), when the redundant forces depend linearly on the concrete relaxation function:

$$X_{\lambda H}=X^0_{\lambda H}+\Delta X_{\lambda H}\,(1^*-R^*), \tag{21}$$

$\lambda=1, 2, \ldots, n$; $H=G, P, S, C$; where $X^0_{\lambda H}=X_{\lambda H}(t_0, t_0)$ and $\Delta X_{\lambda\ H}$ are time independent quantities. This assumption is usual in the engineering practice [5]. The cross section resultants are represented as follows:

$$N_H=N_{H\Phi}+\sum_{\lambda=1}^{n} N_\lambda X_{\lambda H},\qquad M_H=M_{H\Phi}+\sum_{\lambda=1}^{n} M_\lambda X_{\lambda H}, \tag{22}$$

$H=G, P, S, C$; N_λ and M_λ being the cross section resultants in the primary system due to $X_{\lambda H}=1^*$ ($\lambda=1, 2, \ldots, n$); $N_{H\Phi}$ and $M_{H\Phi}$ are given by Eq. (1).

On the basis of Eq. (22) the normal stress expressions are as follows:

$$\sigma_{jH}=\sigma_{jH\Phi}+\sigma_{jHX}, \tag{23}$$

$j=c, p, n, m$; $H=G, P, S, C$; where $\sigma_{jH\Phi}$ is given by Eq. (2). The terms σ_{jHX}, representing the part of the normal stress in a primary system due to $X_{\lambda H}$, are developed in Ref. [2]. Here it will be given in another form:

$$\sigma_{jHX}=\nu_j\sum_{\lambda=1}^{n}\sum_{a=1}^{3}\sum_{h=1}^{2}\left[\sum_{k=1}^{2} a_{jahk}\,E_{t.}\,\varepsilon_{h\lambda}\,x_{kH}+d_{jah\lambda}X^0_{\lambda H}\right]A^*_{ah}, \tag{24}$$

$j=c, p, n, m$; $H=G, P, S, C$. We obtain the coefficients $\varepsilon_{h\lambda}$ when the subscript λ is formally substituted for subscripts kH in Eqs. (4) and (5);

$$x_{kH}=\delta_{1k}X^0_{\lambda H}+(-1)^{k-1}\Delta X_{\lambda H},\qquad \delta_{1k}=\begin{cases}1 & \text{for } k=1,\\ 0 & \text{for } k=2,\end{cases} \tag{25}$$

$k=1, 2$; $\lambda=1, 2, \ldots, n$; $H=G, P, S, C$. The coefficients $d_{jan\lambda}$ are as follows:

$$d_{p1h\lambda}=-\frac{1}{2}(1-\rho_p)\,\sigma^0_{u\lambda},\qquad d_{p2h\lambda}=-\frac{1}{2}\rho_p\,\sigma^0_{u\lambda}, \tag{26}$$

$h=1, 2$; $\lambda=1, 2, \ldots, n$. The quantities $\sigma^0_{u\lambda}$ are obtained when N_λ and M_λ are substituted for $N^0_{P\Phi}$ and $M^0_{P\Phi}$, respectively, in Eq. (8). All other coefficients $d_{jah\lambda}$ are equal to zero.

Similarly, the shear flow is represented in the form:

$$q_H=q_{H\Phi}+q_{HX}, \tag{27}$$

$H=G, P, S, C$; where $q_{H\Phi}$ is given by Eq. (13). Applying Eqs. (10) and (21)—(24) we arrive at:

$$q_{HX}=\sum_{\lambda=1}^{n}\sum_{j}^{3}\sum_{a=1}^{2}\sum_{h=1}\left[\sum_{k=1}^{2} a_{jahk}(b_{jh}n_{\lambda}+c_{jh}T_{\lambda})x_{kH}+f_{jah\lambda}\overset{0}{X}_{\lambda H}\right]A^{*}_{ah}, \tag{28}$$

$j=c, p, n, m$; $H=G, P, S, C$. Here:

$$n_{\lambda}=-\frac{dN_{\lambda}}{dz}, \qquad T_{\lambda}=\frac{dM_{\lambda}}{dz}, \tag{29}$$

$\lambda=1, 2, \ldots, n$; and the coefficients $f_{jah\lambda}$ are as follows:

$$f_{p1h\lambda}=-\frac{1}{2}(1-\rho_p)g_{\lambda}, \qquad f_{p2h\lambda}=-\frac{1}{2}\rho_p g_{\lambda}, \tag{30}$$

$h=1, 2$; $\lambda=1, 2, \ldots, n$; g_{λ} we obtain when n_{λ} and T_{λ} are substituted for $\overset{0}{n}_{P\Phi}$ and $\overset{0}{T}_{P\Phi}$, respectively, from g_P, Eq. (16). All other coefficients $f_{jah\lambda}$ are equal to zero.

The shear flow q_H, Eqs. (27), (13) and (28) as well as the normal stress σ_{jH} Eqs. (23), (2) and (24) are linear combinations of the concrete relaxation function R^* and the basic functions B^*_h $(h=1, 2)$. By experiment we obtain the concrete creep functions F^*. In Ref. [1] it is shown that functions R^* and B^*_h $(h=1, 2)$ can be determined directly from F^*: we create Volterra's integral equation of the second kind in which the kernel is expressed through the concrete creep function and in which the parameter γ_h appears. For $\gamma_h=0$ the solution gives the concrete relaxation function; for two different values of γ_h, γ_1 and γ_2, being the individual values of the matrix of the cross section geometry, the solution provides the basic functions $B_1{}^*$ and $B_2{}^*$, respectively. It is obvious that the shear flow expresssion, developed here, can be applied to any given concrete creep function F^*.

For the concrete creep function F^* of any given form the values of the functions B^*_h $(h=1, 2)$ and R^* can be, once for ever, calculated for a series of values t_0, $t-t_0$ and γ_h $(0\leq\gamma_h\leq 1)$ so that the corresponding values of σ_{jH} and q_H can be easily obtained [6].

We developed the shear flow expressions (13) and (28) under supposition that the beam has an uniform bending stiffness. Now we shall show how we apply these expressions when the bending stiffness vary along the beam axis. As it is usual in the engineering practice we adopte that such a beam has constant cross sections in a finite number of intervals. Keeping in mind that the basic functions B^*_h $(h=1, 2)$ depend on the coordinate z through the coefficients γ_h $(h=1, 2)$ and that the coefficients in Eqs. (13) and (28) depend on z through the cross section geometry, too, we conclude that for each of the intervals Eqs. (13) and (28) are valid and that for two different intervals the above mentioned quantities change their values because the cross section geometry is changed.

Finally, it is necessary to emphasize that the shear flow expression for statically determinate structures as well as for primary systems $(X_{\lambda H}=0)$ is accurate in the framework of the introduced mathematical description of rheological properties of the materials coacting in the composite cross section. For statically indeterminate structures the expression is accurate in the above mentioned framework adding the assumption concerning the redundant forces time-dependance given by

Eq. (21). In this case for any given pair (t, t_0) the redundant forces are the solutions of the system of n algebraic equations, while the exact redundant forces are the solutions of the system of n inhomogeneous integral equations [3].

Appendix

Here are some necessary information for better understanding of the evaluation of the shear flow expression. For more detailed explanations the, quoted references should be used.

1. In a composite cross section coact: concrete (c) as an aging linear visco-elastic material, prestressing steel (p) whose relaxation property is taken into account, steel parts (n) and reinforcing steel (m) being elastic materials. Using the earlier results [7] it is adopted that the nondimensional prestressing steel relaxation function in the interval (t_0, t) depends linearly on the nondimensional concrete relaxation function.

The uniaxial stress-strain relation for the j^{th} material due to the influence H may be written in the unique form using linear integral operators:

$$\sigma_{jH}=\nu_j\left[\tilde{Q}_j'\,(E_u\,\varepsilon_H-\delta_{HS}\,\delta_{jc}\,E_u\,\varepsilon_{cS})+\delta_{HP}\,\delta_{jp}\left(\frac{P}{A_{pr}}-E_u\,\varepsilon_P^0\right)Q_p^*\right], \tag{A.1}$$

$$\delta_{qs}=\begin{cases}1 \text{ for } q=s\\ 0 \text{ for } q\neq s\end{cases}, \qquad q=H, j;\quad s=S, c, p; \qquad j=c, p, n, m;\quad H=G, P, S, C.$$

$\tilde{Q}_j'$ = relaxation operator of the j^{th} material:

$$\tilde{Q}_j'=(1-\rho_j)\,\tilde{1}'+\rho_j\,\tilde{R}',\quad 0\leqslant\rho_j\leqslant 1,\quad j=c, p, n, m, \tag{A.2}$$

$$\rho_c=1,\quad 0<\rho_p<1,\quad \rho_n=\rho_m=0; \tag{A.3}$$

$\tilde{1}'$ = unity operator; $Q_j^*=\tilde{Q}_j'\,1^*=Q^*(t, t_0)$=nondimensional relaxation function of the j^{th} material; $1^*=1^*(t, t_0)$=Heaviside function; $\tilde{R}'$=concrete relaxation operator; $R^*=R^*(t, t_0)$=nondimensional concrete relaxation function; t=time; t_0=time of loading. $\nu_j=E_j/E_u$; $E_c=E_c(t_0)$; E_u=compaired elastic modulus.

In the engineering practice the usual assumption is that in the observed interval (t_0, t) the shrinkage strain ε_{cS} depends linearly on the concrete creep function F^*:

$$\varepsilon_{cS}=r\,(F^*-1^*), \tag{A.4}$$

r = const. for any pair (t, t_0):

$$r=\frac{\varepsilon_S}{F^*(t, t_0)-1}, \tag{A.5}$$

ε_S=measured value for a given pair (t, t_0).

In Eq. (A.1) P=prestressing force; $A_{pr}=\nu_p A_p$; A_p=area of the prestressing steel in the composite cross section; ε_P^0=strain in the arbitrary point of the cross section at $t=t_0$ due to the prestressing force P.

2. In Ref. [4] it is shown that a) the elements γ_{hl} $(h, l=1, 2)$ of the symmetric matrix $||\gamma_{hl}||_{2,2}$ describe the reduced cross section geometry; b) the individual values of this matrix are γ_1 and γ_2, $0 \leq \gamma_2 < \gamma_1 \leqslant 1$; c) two basic functions $B_h^* = B_h^*(z, t, t_0)$ $(h=1,2)$ correspond to any given composite cross section. They depend on the rheological properties of all materials coacting in the composite cross section as well as on the cross section geometry; d) Volterra's integral equation of the second kind in which the parameter γ_h appears can be formed. For three different values of this parameter $\gamma_h = \gamma_1$, $\gamma_h = \gamma_2$ and $\gamma_h = 0$ the solution of this equation gives the corresponding basic functions B_1^*, B_2^* and the concrete relaxation function R^*. In Ref. [2] and [6] the known numerical procedure [8] is used for obtaining this functions corresponding to CEB—FIP 1978 and ACI concrete creep functions.

3. The following notations avre used:

$$\delta\bar{\gamma}_h = \frac{\delta\gamma_h}{\Delta\gamma}, \quad h=1, 2; \quad \bar{\gamma}_{12} = \frac{\gamma_{12}}{\Delta\gamma},$$

$$\delta\gamma_1 = \gamma_1 - \gamma_{11}, \quad \delta\gamma_2 = \gamma_1 - \gamma_{22}, \quad \Delta\gamma = \gamma_1 - \gamma_2. \tag{A.6}$$

BIBLIOGRAPHIE

[1] Lazić J. D., Lazić V. B.: "Generalized Age-Adjusted Effective Modulus Method for Creep in Composite Structures. Part I — Theory", Cement and Concrete Research, 14, 819—832, (1984).

[2] Lazić J. D., Lazić V. B.: "Generalized Age-Adjusted Efffective Modulus Method for Creep in Composite Structures. Part II — Application", Cement and Concrete Research, 15, 1—12, (1985).

[3] Lazić J. D., Lazić V. B.: "General Theory of Composite and Prestressed Structures", The Serbian Academy of Sciences and Arts, Monographs, Vol DXLII, Section for Technical Sciences, No 22, Beograd, (1982).

[4] Lazić V. B.: "Stresses and Deflections in Composite Linear Girders", Izgradnja, 1, 5—17, (1985).

[5] Sattler K.: "Theorie der Verbundkonstruktionen", Band I: Theorie, Wilhelm Ernst &. Sohn Berlin, pp.: 88—90, 93—100, 7—8, 181—182, (1959).

[6] Lazić V. B., Lazić J. D.: In F. H. Wittmann (Ed.), "Fundamental Research on Creep and Shrinkage of Concrete", Martinus Nijhoff Publ., The Hague/Boston/London, pp.: 413—423, (1982).

[7] Lazić J. D., Lazić V. B.: "Les contraintes et les pertes de précontrainte dans les structures précontraintes" Ann. de l'ITBTP, TMC 211, 34—55, (1977).

[8] Bažant Z. P.: "Prediction of Concrete Creep Effects Using Age—Adjusted Effective Modulus Method", Am. Concrete Inst. J. 69, 1972, pp. 212—217.

FLUX DE CISAILLEMENT DANS LES STRUCTURES MIXTES

On étude les structures mixtes et précontraintes en prenant en compte la relaxation de l'acier de précontrainte.

On établit l'expression du flux de cisaillement à partir de l'expression connue de la contrainte normale. On a démontrée que le flux de cisaillement depend linéarment des functions de base et de la fonction de relaxation du béton. Ces fonctions sont obtenues de la résolution d'une seule équation intégrale de Volterra de la deuxiéme espéce contenant le paramètre dependant des caractéristiques géométriques réduites de la section.

TOK SMICANJA KOD SPREGNUTIH LINIJSKIH NOSAČA

Razmatra se spregnuti ili prethodno napregnut linijski nosač pri čemu je uzeta u obzir osobina relaksacije čelika za prethodno naprezanje.

Na osnovu poznatog izraza za normalni napon izveden je izraz za tok smicanja. Pokazano je da tok smicanja predstavlja linearnu kombinaciju osnovnih funkcija preseka i funkcije relaksacije betona. Ove funkcije se dobijaju iz rešenja samo jedne Volteraove integralne jednačine druge vrste u kojoj se javlja parametar zavisan od redukovanih geometrijskih karakteristika spregnutog preseka.

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