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VARIATIONAL PRINCIPLE AND ERROR ESTIMATE FOR A NON-LINEAR HEAT CONDUCTION PROBLEM

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Abstract: The variational principle for stationary heat conduction with temperature-dependent heat generation in cylinders and spheres is constructed. On the basis of this principle approximate solutions for several problems are obtained. Also, error of such approximate solutions is estimated.

1. **Introduction.** Stationary heat conduction problems for cylinders and spheres lead to a two-point boundary value problem for the second order differential equations. These equations may be linear, or non-linear, depending on the properties of the material and, possibly, presence of heat generation terms. In particular, if the thermal properties of the material are constant, and if amount of heat generated inside the body dependens on the temperature, one obtains a non-linear boundary value problem. The typical form of the differential equation determining the stationary temperature distribution inside the body (cylinder or sphere) reads

$$\frac{1}{r^m}\frac{d}{dr}\left(r^m\frac{dT}{dr}\right)+F(T)=0; \quad 0<r<1, \tag{1.1}$$

where r is non-dimensional radial coordinate, T is non-dimensional temperature, $F(T)$ is the term representing the temperature dependent heat generation. For a cylinder, $m=1$, and for a sphere, $m=2$. We take the boundary conditions for (1.1) in the form

$$\frac{dT(0)}{dr}=0; \quad T(1)=T_0, \tag{1.2}$$

where T_0 is a given constant.

A method to integrate the boundary value problem (1.1), (1.2) numerically is presented in [1]. Recently, Wacker [5] found an exact solution of the boundary value problem in the case when the non-linear heat generation is according to the exponential law. Our intention is to study (1.1), (1.2) by the variational method developed in [3]. Thus, we shall first construct an extremum variational principle for the problem. Then, we will use this principle to obtain an approximate solution to (1.1), (1.2) by the Ritz method. Finally, we will estimate the error of such approximate solution. The error bound will be derived by a method slightly different than those presented in [3], [4]. Also, an extension of the results of [3] is made for the boundary value problem (1.1), (1.2). Namely, in this paper we shall show that the functional of the extremum variational principle attains global maximum on the solution of (1.1), (1.2). This property is of importance, especially for error estimating procedure.

2. Variational principle for boundary value problems

The boundary value problem (1.1), (1.2) is not suitable for the variational analysis that we intend to use. Therefore, we transform (1.1), (1.2) by introducing a new dependent variable t by the relation

$$t=r^{m+1},\quad m=1,2. \tag{2.1}$$

Then (1.1), (1.2) transforms to

$$\Big[(1+m)^2\, t^{\frac{2m}{1+m}}\, \dot{T}\Big]^{\cdot} + F(T)=0,\quad 0<t<1, \tag{2.2}$$

$$\Big[(1+m)\, t^{\frac{m}{1+m}}\, \dot{T}\Big]_{t=0}=0;\quad T(1)=T_0, \tag{2.3}$$

$$(.)^{\cdot}=\frac{d}{dt}(.).$$

On the physical grounds (T is the non-dimensional temperature) we are interested in the non-negative solutions ($T\geq 0$) of (2.2), (2.3). To construct a variational principle for (2.2), (2.3), we set

$$f(T)=\int_0^T F(\xi)\,\mathrm{d}\,\xi. \tag{2.4}$$

Then, it is easy to see that the following functional

$$J=\int_0^1 L(T,\ \dot{T},\ t)\,\mathrm{d}t, \tag{2.5}$$

with

$$L=\frac{(1+m)^2}{2}\, t^{\frac{2m}{1+m}}(\dot{T})^2-f(T), \tag{2.6}$$

is stationary (i. e., $\delta I=0$) on the solution of (2.2), (2.3). The functional (2.5) is called the primal functional of the problem. To construct an extremum variational principle we shall follow the method presented in [3]. Thus we define a generalized momenta by the relation

$$p=\frac{\partial L}{\partial \dot{T}}=(1+m)^2\, t^{\frac{2m}{1+m}}\, \dot{T}. \tag{2.7}$$

Solving (2.7) for $\dot{T}$, the Hamiltonian of the problem can be written as

$$H = p\dot{T} - L = \frac{p^2}{2(1+m)^2 t^{\frac{2m}{1+m}}} + f(T). \tag{2.8}$$

The canonical form of (2.2) now reads

$$\dot{T} = \frac{p}{(1+m)^2 t^{\frac{2m}{1+m}}} \tag{2.9}$$

$$\dot{p} = -F(T). \tag{2.10}$$

The functional I of the extremum variational principle formulated in [3] becomes

$$\mathrm{I}(\theta) = \int_0^1 \{L(\theta, \dot{\theta}, t) + \mathscr{L}(p, \dot{p}, t) \big|_{\substack{p = bt^a \dot{\theta} \\ \dot{p} = (bt^a \dot{\theta})^{\cdot}}}\} \, dt — b\,\dot{\theta}(1)\,\theta(1), \tag{2.11}$$

where $a = 2m/(1+m)$, $b = (1+m)^2$ and

$$\mathscr{L}(p, \dot{p}, t) = \dot{p}\, F^{-1}(-\dot{p}) + \frac{p^2}{2\,bt^a} - f(F^{-1}(-\dot{p})). \tag{2.12}$$

Also in (2.11) we used F^{-1} to denote inverse of F, that is the function for which (see (2.10))

$$T = F^{-1}(-\dot{p}). \tag{2.13}$$

Finally, the vertical bar after $\mathscr{L}$ denotes that $\mathscr{L}$ should be calculated for those p and $\dot{p}$ that are designated.

In all above formulas $\theta \in W$ is an admissible trial function. The set $W \in \overline{X}$ of admissible trial functions is defined as

$$W = \{\theta : \theta \in \overline{X},\ (bt^a \dot{\theta})^{\cdot} \leq 0 \text{ for } t \in (0,\ 1)\} \tag{2.14}$$

where

$$\overline{X} = \{T : T \in C^2(0,\ 1),\ (2.3) \text{ holds},\ T(0) = T_0\}. \tag{2.15}$$

In [3] it is shown that on the solution T of (2.2), (2.3) the functional (2.11) is stationary and has the value equal to zero, i.e.,

$$\delta I(T) = 0,\ I(T) = 0. \tag{2.16}$$

In writing explicit form of (2.11) we shall distinquish two cases:

CASE A: Let us assume that

$$F(T) = \beta e^T, \tag{2.17}$$

where $\beta > 0$ is a constant. The, (2.11) becomes

$$I_1(\theta) = \int_0^1 \left\{ bt^a\, \dot{\theta}^2 - \beta\, e^{\theta} + (bat^{a-1}\, \dot{\theta} + bt^a\, \ddot{\theta}) \ln \left[-\frac{1}{\beta} (bat^{a-1}\, \dot{\theta} + bt^a\, \ddot{\theta}) \right] \right\} \mathrm{dt} - b\, \dot{\theta}(1)\, \theta(1). \tag{2.18}$$

CASE B: In this case we choose

$$F(T)=T^n, \tag{2.19}$$

where n is a specified integer. The functional (2.11) becomes

$$I_2(\theta)=\int_0^1 \left\{ bt^a \dot{\theta}^2 - \frac{\theta^{n+1-}}{n+1}\,\frac{n}{n+1}\left[-(bat^{a-1}\dot{\theta}+bt^a\ddot{\theta})\right]^{\frac{n+1}{n}} \right\} dt - b\,\dot{\theta}(1)\,\theta(1). \tag{2.20}$$

Let θ be an arbitrary admissible trial function. Since functional (2.11) is (in both our cases A and B) Fréchet differentiable, we have the following expansion

$$I(\theta)=I(T)+\delta I(T,\zeta)+\frac{1}{2}\,\delta^2 I(\psi,\zeta), \tag{2.21}$$

where $\delta^2 I$ is the second variation calculated on the function ψ given by

$$\psi=T+\varepsilon\zeta,\ \ \zeta=\theta-T, \tag{2.22}$$

with $0<\varepsilon<1$. Observing (2.16), (2.12) becomes

$$2\,I(\theta)=\delta^2 I(\psi,\zeta). \tag{2.23}$$

Equation (2.23) is the basic relation that we will use to prove the global extremality of functionals (2.18) and (2.20). Namely, in [3] we proved (Theorem 2) that if $(\partial F/\partial T)$ does not change sign for $t\in(0,1)$, the functionals (2.18) and (2.20) have a local extremum on the exact solution of corresponding boundary value problem. Here we shall do more by showing that both (2.18) and (2.20) have a local extremum on the exact solutions of (2.2), (2.3), (2.17) and (2.2), (2.3), (2.19), respectively. To do this we consider first:

CASE A: Calculating the second variation of (2.18) and using it in (2.23), we get

$$-2\,I_1(\theta)=\int_0^1 \left\{ \beta\, e^{\Psi}(\zeta)^2 + 2\,\zeta\,[(bt^a\dot{\zeta})^{\cdot}\,] + \frac{[(bt^a\dot{\zeta})^{\cdot}\,]^2}{[-(bt^a\dot{\Psi})^{\cdot}\,]} \right\} dt, \tag{2.24}$$

where we performed partial integration on the middle term and used boundary conditions on ζ

$$[(1+m)\,t^{\frac{m}{1+m}}\,\dot{\zeta}]_{t=0}=0,\ \ \zeta(1)=0. \tag{2.25}$$

The boundary conditions (2.25) for the variation ζ follow from (2.3) and the fact that $\theta\in WC\overline{X}$. To simplify (2.24), we set

$$A_1=\min\left\{ \inf_{t\in(0,1)} \frac{1}{-(bt^a\dot{T})^{\cdot}},\ \inf_{t\in(0,1)} \frac{1}{-(bt^a\dot{\theta})^{\cdot}} \right\} \tag{2.26}$$

i

$$B_1=\min\left\{ \inf_{t\in(0,1)} \beta e^{T},\ \inf_{t\in(0,1)} \beta e^{\theta} \right\}. \tag{2.27}$$

We note that in (2.26), (2.27) T is the exact solution to the problem (2.2), (2.3), (2.17) and $\theta \in W$ is an admissable trial function. In our application θ will be an approximate solution to the boundary value problem in question and thus the second terms in (2.26), (2.27) will be easy to determine. However, to calculate the first term in (2.26), (2.27), we need information about (unknown) exact solution of the problem. Therefore for given θ we have from (2.24).

$$\int_0^1 \{B_1(\zeta)^2 + A_1[(bt^a\dot{\zeta})^{\cdot}]^2 + 2\zeta[(bt^a\dot{\zeta})^{\cdot}]\}\, dt \leqq -2I_1(\theta). \tag{2.28}$$

To simplify (2.28), we expand ζ in a generalized Fourier series

$$\zeta = \sum_{n=1}^{\infty} C_n \Phi_n \tag{2.29}$$

where C_n are constants (Fourier coefficients) and Φ_n are elements of the (complete in L_2 norm) set of eigenfunctions of the following spectral problem

$$\left[(1+m)^2 t^{\frac{2m}{1+m}} \dot{\Phi}_n\right]^{\cdot} + \lambda_n \Phi_n = 0, \quad 0<t<1 \tag{2.30}$$

$$\left[(1+m)\, t^{\frac{m}{1+m}} \dot{\Phi}_n\right]_{t=0} = 0, \quad \Phi_n(1) = 0. \tag{2.31}$$

The first eigenvalues of (2.30), (2.31) for $m=1,2$ are $\lambda_1=5.7831$ and $\lambda_1=\pi^2$, respectively.

Substituting (2.29) into (2.28), using the orthogonality property of ϕ_n and (2.30) we get (see [3] for details)

$$S_1 \|\zeta\|_{L_2}^2 \leqq -2I_1(\theta), \tag{2.32}$$

where

$$S_1(\theta) = \min_n (B_1 - 2\lambda_n + A_1\lambda_n^2) \quad \text{and} \tag{2.33}$$

$$\|\zeta\|_{L_2} = \left(\int_0^1 \zeta^2\, dt\right)^{1/2}$$

is the L_2 norm of ζ. Note that S_1 (by A_1 and B_1) depends on θ. Let $W_1 \subset W$ be the set defined by

$$W_1 = \{\theta : \theta \in W,\ S_1(\theta) > 0\}. \tag{2.34}$$

Then by considering the restriction of (2.32) to W_1 we conclude that $I_1(\theta)$ has the global maximum (equal to zero) on W_1. Inequalities (2.28) and (2.32) are the basis for estimating the error of an approximate solution θ. We leave this analysis for the next section.

CASE B: Calculating the second variation of (2.20) and using it in (2.23), we get

$$-2I_2(\theta) = \int_0^1 \left\{ n\psi^{n-1}(\zeta)^2 + \frac{1}{n}\left[-(bt^a\dot{\psi})^{\cdot}\right]^{\frac{1-n}{n}} \cdot \left[(bt^a\dot{\zeta})^{\cdot}\right]^2 + 2\zeta\left[(bt^a\dot{\zeta})^{\cdot}\right]\right\} dt. \tag{2.35}$$

Defining the constants A_2 and B_2 by

$$-A_2=\min\left\{\inf_{t\in(0,1)}\frac{1}{n}\left[-(bt^a\dot{T})^{\cdot}\right]^{\frac{1-n}{n}};\qquad \inf_{t\in(0,1)}\frac{1}{n}\left[-(bt^a\dot{\theta})^{\cdot}\right]^{\frac{1-n}{n}}\right\} \tag{2.36}$$

$$B_2=\min\{\inf_{t\in(0,1)} n\,T^{n-1};\qquad \inf_{t\in(0,1)} n\,\theta^{n-1}\} \tag{2.37}$$

and using the same procedure as in case A, we get

$$S_2\,||\,\zeta\,||_{L_2}^2\leqq-2\,I_2(\theta), \tag{2.38}$$

where

$$S_2=\min_n\,(B_2-2\,\lambda_n+A_2\,\lambda_n{}^2) \tag{2.39}$$

and λ_n, $n=1, 2, \ldots$ are, again, the eigenvalues of (2.30), (2.31). Therefore, defining the set $W_2\subset W$ as

$$W_2=\{\theta:\theta\in W,\quad S_2(\theta)>0\} \tag{2.40}$$

we conclude from (2.38) that $I_2(\theta)$ has a global maximum on W_2 for $\theta=T$, where T is the solution of (2.2), (2.3), (2.19).

3. Error estimate for boundary value problems

Results of the previous section may be used to derive a bound on the error ζ of an approximate solution θ. We first note that (2.32) and (2.38) give the following bounds

$$||\,\zeta\,||_{L_2}\leqq\left\{\frac{-2\,I_i(\theta)}{S_i}\right\}^{1/2}\qquad i=1,2 \tag{3.1}$$

on the L_2 norm of ζ. Cases A and B correspond to $i=1$ and $i=2$ respectively. We may need also an estimate of the L_∞ norm of ζ, where

$$||\,\zeta\,||_{L\infty}=\sup_{t\in(0,1)}|\,\zeta(t)\,|. \tag{3.2}$$

To get an estimate of $||\,\zeta\,||_{L\infty}$, let

$$Y=||\,(bt^a\dot{\zeta})^{\cdot}\,||_{L_2}. \tag{3.3}$$

We shall first derive a bound on Y. We consider separately cases A and B.

CASE A: By substituting (3.3) into (2.28) and using the Couchy inequality we get

$$AY^2-2\,Y\,||\,\zeta\,||_{L_2}+B\,||\,\zeta\,||_{L_2}^2+2\,I_1(\theta)\leqq 0. \tag{3.4}$$

Solving the algebraic inequality (3.4) for Y and using the fact that $Y\geqq 0$ and (3.1), we have

$$Y\leqq\lambda_k\left\{\frac{-2\,I_1(\theta)}{S_1}\right\}^{1/2}. \tag{3.5}$$

where λ_k is the eigenvalue which determines S_1, i. e.,

$$S_1 = \min_n (B_1 - 2\lambda_n + A_1 \lambda_n^2) = (B_1 - 2\lambda_k + A_1 \lambda_k^2). \tag{3.6}$$

We note that k is not equal to infinity since the eigenvalues λ_n of (2.30), (2.31) form an increasing sequence whose only point of accumulation is at infinity.

The central problem now is to derive a bound to $\|\zeta\|_{L\infty}$ in terms of Y given by (3.3). From (3.3), we have

$$Y^2 = \int_0^1 [(bt^a \dot{\zeta})^{\cdot}]^2 \, dt = \int_0^1 \left[\zeta'' + \frac{m}{r}\zeta'\right]^2 (m+1) r^m \, dr, \tag{3.7}$$

where we used (2.1) and definitions of a and b. From (3.7) it follows that

$$Y^2 = (1+m)\left[\int_0^1 r^m (\zeta'')^2 \, dr + 2m \int_0^1 (\zeta')^2 r^{m-1} \zeta' \zeta'' \, dr + \int_0^1 m^2 r^{m-2} (\zeta')^2 \, dr\right]. \tag{3.8}$$

Using partial integration and the boundary conditions on $\zeta(r)$ we get

$$Y^2 = (m+1)\left[\int_0^1 r^m (\zeta'')^2 \, dr + m \int_0^1 (\zeta')^2 r^{m-1} \, dr + m (\zeta')^2_{r=1}\right]. \tag{3.9}$$

We simplify (3.9) by the following estimates

$$\inf_{r \in (0,1)} r^m = 0; \qquad \inf_{r \in (0,1)} r^{m-2} = 1; \qquad m = 1,2. \tag{3.10}$$

With (3.10), (3.9) becomes

$$Y^2 \geqq (m+1)\,[(\zeta'(1))^2 + \|\zeta'\|^2_{L_2}]. \tag{3.11}$$

Finally we use Cauchy inequality and the boundary condition $\zeta(1)=0$ to get

$$\|\zeta\|^2_{L\infty} \leqq \|\zeta'\|^2_{L_2}. \tag{3.12}$$

Now, by using (3.12) and (3.11) in (3.5), we have

$$\left\{[\zeta'(1)]^2 + \|\zeta\|^2_{L\infty}\right\}^{1/2} \leqq \lambda \left\{\frac{-2 I_1(\theta)}{S_1(m+1)}\right\}^{1/2} = G_1(\theta). \tag{3.13}$$

Inequality (3.13) gives an estimate, of either $\|\zeta\|_{L\infty}$ or $|\zeta'(1)|$ for an approximate solution of the boundary value problem (2.2), (2.3), (2.17), in terms of the computable quantity $G_1(\theta)$.

CASE B: Following the same procedure as in *CASE A*, we have

$$Y \leqq \lambda_k \left\{-\frac{2 I_2(\theta)}{S_2}\right\}^{1/2}, \tag{3.14}$$

where S_2 is given by (2.39) and λ_k is the eigenvalue that determines S_2, i. e.,

$$S_2 = \min_n (B_2 - 2\lambda_n + A_2 \lambda_n^2) = (B_2 - 2\lambda_k + A_2 \lambda_k^2). \tag{3.15}$$

Now, combining (3.11), (3.12) and (3.14), we get

$$\left\{[\zeta'(1)]^2 + \|\zeta\|_{L_\infty}^2\right\}^{1/2} \leqq \lambda_k \left[\frac{-2 I_2(\theta)}{S_2(m+1)}\right]^{1/2} = G_2(\theta) \tag{3.16}$$

as an estimate of the error of an approximate solution to (2.2), (2.3), (2.19).

4. Numerical results

We shall illustrate the previous results by considering a few concrete boundary value problems.

4.1. Heat conduction in a cylinder with exponential heat generation. As a first example we consider (2.2) with $m=1$ and the temperature dependent heat generation as (2.17). Also, we take $T_0=0$ in the boundary conditions (2.3). Approximate solution to (2.2), (2.3), with $m=1$, we take in the form

$$\theta = \frac{C_1}{2}(6 - 8t + 2t^2) + \frac{\beta}{4}(1-t). \tag{4.1}$$

Constant C_1 is determined by substituting (4.1) into (2.18) and minimizing with respect to C_1. To estimate the error of this approximate solution we used (3.13). The constants A_1 and B_1 that are needed for determining S_1 are estimated from (2.26), (2.27) that in the present case read:

$$A_1 \leqq \min\left\{\frac{1}{\beta e^{a_2^*}}, \frac{1}{16 C_1 + \beta}\right\} \tag{4.2}$$

$$B_1 \leqq \min\{\beta\}. \tag{4.3}$$

The first terms in parenthesis of (2.26) are estimated using a similar procedure to those which is described in details in [6]. The second terms in parenthesis of (2.26), (2.27) are calculated using θ and the fact that for both θ and T the minimum is equal to zero.

The value of C_1 and correspoding estimate of the error for few values of β are given in Table 1.

Also, in the Table 1 are given values of the error criteria (see (3.13))

$$G_1 = \{[\zeta'(1)]^2 + \|\zeta\|_{L_\infty}^2\}^{1/2}$$

and the ratio $G_1/\tilde{G}_1$. The error criteria $\tilde{G}_1$ is calculated using the approximate solution (4.1) and the corresponding exact solution [5] of the problem. By the ratio $G_1/\tilde{G}_1$ we see quality of error estimate procedure presented here.

Table 1

$F(T)=\beta e^{T}, \quad T_0=0, \quad m=1$

β	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	1	1.5
C_1	0.00016	0.0006	0.0015	0.0028	0.0045	0.0067	0.0095	0.013	0.0215	0.063
$-I_1(C_1)$	2.61×10^{-4}	7.02×10^{-7}	1.494×10^{-7}	1.994×10^{-7}	6.746×10^{-7}	1.80×10^{-6}	4.192×10^{-6}	9.73×10^{-6}	3.55×10^{-5}	5.337×10^{-5}
a_2^*	0.0255	0.052	0.08	0.11	0.14	0.172	0.205	0.245	0.33	0.7
A_1	9.748	4.74	3.077	2.239	1.738	1.403	1.16	0.978	0.72	0.33
B_1	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	1	1.5
G_1	5.26×10^{-6}	3.99×10^{-4}	2.33×10^{-4}	3.23×10^{-4}	6.91×10^{-4}	1.29×10^{-3}	2.2×10^{-3}	3.85×10^{-3}	9.37×10^{-3}	4.29×10^{-2}
$\tilde{G}_1$	2.48×10^{-6}	2.98×10^{-4}	1.42×10^{-4}	8.19×10^{-5}	1.80×10^{-4}	3.20×10^{-4}	5.50×10^{-4}	1.20×10^{-3}	2.20×10^{-3}	1.15×10^{-2}
$G_1/\tilde{G}_1$	2.11	1.33	1.63	3.94	3.83	4.02	3.99	3.20	4.25	3.72

4.2 Heat conduction in a sphere with exponential heat generation. In this example we consider (2.2), (2.17), (2.3) with $m=2$ and $T_0=0$. Approximate solution we take in the form

$$\theta=\frac{C_1}{3}(7-10\,t^{2/3}+3\,t^{4/3})+\frac{\beta}{6}(1-t^{2/3}).$$

Constant C_1 is determined by minimizing (2.18) with $m=2$. Error estimating procedure is the same as in the previous case. Error bound is given bu (3.13). The results for few values of β are given in Table 2.

$F(T)=\beta\,e^T,\ T_0=0,\ m=2$ *Table 2*

β	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
C_1	0.00008	0.00035	0.0008	0.0014	0.0023	0.0083	0.0045	0.0061
$-I_1(C_1)$	4.34×10^{-8}	1.49×10^{-7}	3.596×10^{-7}	3.869×10^{-7}	1.189×10^{-6}	1.717×10^{-6}	3.206×10^{-6}	3.92×10^{-6}
a_2^*	0.017	0.035	0.053	0.0715	0.092	0.114	0.132	0.156
A_1	9.83	4.82	3.16	2.327	1.82	1.487	1.25	1.07
B_1	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
G_1	3.87×10^{-5}	0.03×10^{-4}	2.01×10^{-4}	2.459×10^{-4}	4.93×10^{-4}	6.67×10^{-4}	1.0×10^{-3}	1.21×10^{-3}

4.3 Heat conduction in a cylinder with power law heat generation. We consider now (2.2), (2.3) with $m=1$, where the temperature dependent heat generation is given by (2.19). Approximate solution to (2.2), (2.3) and (2.19), with $m=1$, we take in the form

$$\theta=T_0+\frac{(T_0)^n}{4}(1-t)+C_1(3-4\,t+t^2). \tag{4 5}$$

The constant C_1 is determined by substituting (4.5) into (2.20) with $m=1$. The constants A_2 nad B_2 are now given by

$$A_2=\min\left\{\frac{1}{n}\frac{1}{(\overset{*}{a_2})^{n-1}};\ \frac{1}{n}\frac{1}{[T_0^n+16\,C_1]^{(1-n)/n}}\right\}, \tag{4.6}$$

$$B_2=nT_0^{n-1}. \tag{4.7}$$

The constant B_2 has the value (4.7) since both T and θ satisfy $(2.3)_2$. The results of calculation for a few values of n and T_0 are given in Table 3.

$F(T)=T^n,\ m=1$ *Table 3*

n	2	3	4	5
T_0	0.6	0.6	0.7	0.7
C_1	0.00666	0.00433	0.0066	0.0026
$-I_2(C_1)$	1.29×10^{-4}	9.15×10^{-6}	2.74×10^{-5}	6.7×10^{-5}
$a_2{}^*$	0.676	0.645	0.755	0.736
A_2	0.680	0.7208	0.479	0.5839
B_2	1.2	1.08	1.372	1.2005
G_2	1.8×10^{-2}	4.73×10^{-3}	1.2×10^{-2}	1.1×10^{-2}

4.4 Heat conduction in a sphere with power law heat generation. Let us consider (2.2), (2.3) and (2.19) with $m=2$. The approximate solution to the problem we assume in the form

$$\theta=T_0+\frac{T_0{}^n}{6}(1-t^{2/3})+C_1\left(\frac{7}{3}-\frac{10}{3}t^{2/3}+t^{4/3}\right). \tag{4.8}$$

Repeating the procedure of example 4.3 we get the results given in Table 4.

$F(T)=T^n,\ m=2$ *Table 4*

n	2	3	4	5
T_0	0.6	0.6	0.7	0.7
C_1	0.0034	0.0017	0.00342	0.00171
$-I_2(C_1)$	1.911×10^{-5}	9.76×10^{-6}	6.59×10^{-6}	2.01×10^{-6}
A_2	0.748	0.801	0.5808	0.6815
B_2	1.2	1.08	1.372	1.2005
$G_2\times10^{-3}$	3.3	2.3	2.36	1.16

5. Conclusions

In this paper we developed extremum variational principles for a class of non-linear differential equations descibing stationary heat conduction in cylinders and spheres. We assumed that heat is generated inside the body according to the exponential and power laws. Thus we were lead to a non-linear two point boundary value problem. For this problem we showed global extremality of a variational functional with respect to specialy selected set of trial functions. On the basis of this variational principle approximate solutions to a number of specific problems are obtained. Error estimate of this approximate solutions are also presented. Inequalities (3.13) and (3.16) give bounds to a sum of L_∞ norm of the error and the value of the first derivative of error at $t=1$. Of course, G_1 and G_2 bound each of the above mentioned terms separatelly.

Our approximate solutions, as can be seen from the Tables 1—4, have remarkable accuracy. They are also in good agreement with the results of other authors. For example in [1] the values of temperature for $r=0\,(t=0)$ are tabulated for the problems we treated in 4.1 and 4.2. A comparison of those values (obtained by numerical integration) and our values obtained from (4.1) and (4.4) are presented in the Table 5.

$F(T)=\beta\, e^T$ *Table 5*

β	$m=1$			$m=2$		
	0.1	0.4	0.8	0.1	0.4	0.8
$T(0)$ eq. (6.4) (6.8)	0.0254	0.1084	0.239	0.01685	0.0699	0.1475
$T(0)$ ref. [1]	0.0252	0.1090	0.238	0.0168	0.0706	0.1470

Finally, we may state that the variational principle developed in [3] could be successfuly used for the stationary heat transfer problems treated in this paper.

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VARIATIONALPRINZIP UND FEHLER ABSCHÄTZUNGEN FÜR NICHT LINEARE WÄRME LEITUNG PROBLEM

Ein Variational Prinzip für stationäre Wärmegleichung mit Wärme Produktion ist formuliert. Dieses Prinzip ist angewendet für einige konkrete Probleme von stationären Wärme Leitung im Zylinder und Kugel. Gleichzeitig, Fehler Abschätzungen fur einige Näherungslösungen sind gegeben.

VARIJACIONI PRINCIP I OCENA GREŠKE ZA NELINEARNO PROVOĐENJE TOPLOTE

Konstruisan je varijacioni princip za stacionarno provođenje toplote kroz cilindar i sferu i pri temperaturno zavisnih izvora. Na osnovu ovog principa nađeno je približno rešenje za nekoliko problema. Takođe je ocenjena greška ovih približnih rešenja.

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