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CONSERVATION LAWS IN THERMOELSTIC MEMBRANE THEORY

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Introduction

Conservation laws (or path-independent integrals) for linear and non-linear materials have been considered by various authors [1—4]. This is probably due to the close relation of the laws to the pathi-ndependent integrals widely used in fracture mechanics [1, 3].

Path-independent integrals in shell theory have been considered by several authors [5 – 6]. Shells are finite bodies, and hence the boundary conditions, at the shell faces should be explicitly satisfied. Intrinsic geometry of a shell imposes the necessity of using curvilinear coordinates, and, related to them, the special feature of Riemannian geometry.

Path-independent integrals in shells have been considered by Berger and Radenković [5] and Berger [6]. However, it appears that they have not placed any restrictions an the geomotry of the shells and based on the ocnsiderations on invariance in this study, such integrals are not path independent in general. Lo [7] examined path-independent integrals for cilindrical shell and shell of revolution. He exepcts that path-independent integrals do not exist in general for shells except those which enjoy a higher degree of symmetry.

Recently, a new method for the study of conservation laws has been proposed and used by Kienzler and Golebiewska-Herrmann [8] in the context higher-order shell theories, for finding various conservation laws. The starting point in the approach taken in [8] wos to establish the most general variational principle for shell. The conditions for possibble conservation laws can be derived from this principle.

It seems to be worthwile to study conservation laws in the context of thermoelastic membrane theories.

In this paper our intention is to derive conservation laws of J-type integra using invariant characterstics of variational principles.

Variational principle

Variational principles unite various fields of natural science in a special way, and as such are often used in many fields of physical and technical sciences. They have an invariant property and a serries of invariant characteristics in relation to infinitesimal transformation group so that they enable the derivation of the corresponding conservation laws.

Classical variational principles have been used to derive equilibrium eequa tions from stationary conditions of the functional.

Let ξ_α $(\alpha=1, 2)$ be the Gaussian coordinates of the middle surface of a shell, $u_j(\xi)(j=1, 2, 3)$ and $\psi(\xi)$ be arbitrary vector and scalar fields, respectively, defined and twice continuosly differentiable on S. All fields and their behaviour are then expressed in terms of these variables and derivatives with respect to them.

Throughout the paper, the usual summation convention is used. Repeated Latin indices represent summation over the range (1, 2, 3) and repeated Greek indices, over the range (1, 2). A double vertical line denotes covariant differentiation with respect to ξ_α.

Now we define a functional I on the class of given fields $X=X(\xi, u(\xi), \psi(\xi))$ by the formula

$$I(u, \psi)=\int_S L(Y)\, d\, S, \tag{2,1}$$

where $L=L(Y)$ is a real scalar function defined and diferentiable for all values of its arguments

$$Y=Y(u_j, u_{j||\alpha}, \psi, \xi_\alpha)$$

and dS is the element of area on the middle surface.

By this, variational principle is mathematically expresed such that integral variation (2.1) is equal to zero

$$\overline{\delta} I=0 \tag{2.2}$$

under the hypotesis that the considered system on the boundary is specified.

The symbol $\overline{\delta}$ denotes a variation when the bounderies are fixed, whereas δ will be used for more general variations, with varying boundaries.

Under this condition, variational equation (2.2) is equivalent to Euler-Lagrange equations

$$\frac{\partial L}{\partial u_j}-\frac{\partial L}{\partial u_{j||\alpha}}{}_{||\alpha}=0, \tag{2.3}$$

When this problem is correctly formulated, besides differential equation of motion (2.3) it also neccessarly gives the number of boundary conditions. It is impossible, however, to obtain at the same time conservation laws because the class of variation is too restricted. In addition to the local variation $\overline{\delta}$ "convective variations" δ related to the variation of the independent variables ξ_α, have to be admitted. This leads to a variational principle with varying boundaries [8].

For the action integral (2.1), we introduce the small transformations of dependent and independent variables as:

$$\bar{\xi}_\alpha = \xi_\alpha + \delta\, \xi_\alpha \tag{2.4}$$

and

$$\bar{u}_j = u_j(\xi) + \delta\, u_j$$

$$\bar{\psi} = \psi(\xi) + \delta\, \psi \tag{2.5}$$

where $\delta\xi_\alpha$, δu_j and $\delta\,\psi$ represent the variations of ξ_α, u_j and ψ, respectively. Or in an expanded form:

$$\bar{\xi}_\alpha = \xi_\alpha + \alpha_\alpha\, \eta + 0(\eta^2)$$

$$\bar{u}_j = u_j + \beta_j\, \eta + 0\,(\eta^2) \tag{2.5a}$$

$$\bar{\psi} = \psi = \gamma\, \eta + 0\,(\eta^2)$$

where the quantities $\delta\xi_\alpha = \alpha_\alpha$, $\delta\, u_j = \beta_j$, $\delta\,\psi = \gamma$ etc. are taken to be of infinitesimal order and η small parameter. With these transformations, the action integral (2.1) changes into

$$\bar{I}\,(\bar{u}, \bar{\psi}) = \int_S \bar{L}(\bar{Y})\, d\bar{S} \tag{2.6}$$

Making use of expansion tehnique, we may calculate:

$$\bar{u}_j\,(\bar{\xi}) = \bar{u}_j\,(\xi_\alpha + \delta\, \xi_\alpha) = \bar{u}_j\,(\xi) + u_{j||\alpha}\, \delta\, \xi_\alpha + 0\,(\delta^2)$$

$$\bar{\psi}\,(\bar{\xi}) = \bar{\psi}\,(\xi_\alpha + \delta\, \xi_\alpha) = \bar{\psi}\,(\xi) + \psi_{||\alpha}\, \delta\, \xi_\alpha + 0\,(\delta^2)$$

Using all the above expression $\bar{\xi}_\alpha$, $\bar{u}_j\,(\bar{\xi})$ and $\bar{\psi}\,(\bar{\xi})$, we can derive variation:

$$\delta u_j = \bar{\delta}\, u_j + u_{j||\alpha}\, \delta\, \xi_\alpha + 0\,(\delta^2)$$

$$\delta\, \psi = \bar{\delta}\, \psi + \psi_{||\alpha}\, \delta\, \xi_\alpha + 0\,(\delta^2) \tag{2.7}$$

$$\delta\, u_{j||\alpha} = \bar{\delta}\, u_{j||\alpha} + u_{j||\alpha\beta}\, \delta\xi_\beta + 0\,(\delta^2)$$

with $u_{j||\alpha}$ and $\psi_{||\alpha}$ representing the covariant derivatives of u_j and ψ with respect to ξ_α, and $\bar{\delta}$ means a variational operator only due to the transformation of variables u_j, ψ and $u_{j||\alpha}$ themselves:

$$\bar{\delta}\, u_j = \bar{u}_j\,(\xi) - u_j\,(\xi)$$

$$\bar{\delta}\, \psi = \bar{\psi}\,(\xi) - \psi\,(\xi)$$

$$\bar{\delta}\, u_{j\,||\,\alpha} = \bar{u}_{j\,||\,\alpha}(\xi) - u_{j\,||\,\alpha}\,(\xi).$$

In addition, the new element of area dS on the middle surface $d\bar{S}$ becomes

$$d\bar{S} = [1 + \delta\xi_{\alpha \,||\, \alpha}]\, dS. \tag{2.8}$$

Substituting (2.7) and (2.8) into (2.6), after some algebraic manipulation, we obtain

$$\delta I = \bar{I} - I = \tag{2.9}$$

$$\int_S \left[\left(L\, \delta\xi_\alpha + \frac{\partial L}{\partial u_{j\,||\,\alpha}} \bar{\delta} u_{j\,||\,\alpha} \right)_{||\alpha} + \frac{\partial L}{\partial \psi} \bar{\delta}\psi + \left(\frac{\partial L}{\partial u_j} - \frac{\partial L}{\partial u_{j\,||\,\alpha}} \bigg|\bigg|_\alpha \right) \bar{\delta} u_j \right] dS.$$

In fact, (2.9) represent a basic expression for variation action δI. Substituting $\delta\xi_\alpha$, δu_j and $\delta\psi$ by their values from (2.5), we obtain

$$\delta I = \eta \int_S \left[\left(L\, \alpha_\alpha + \frac{\partial L}{\partial u_{j\,||\,\alpha}} p_j \right)_{||\alpha} + \frac{\partial L}{\partial \psi} q - \left(\frac{\partial L}{\partial u_{j\,||\,\alpha\,||\,\alpha}} - \frac{\partial L}{\partial u_j} \right) p_j \right] dS, \tag{2.10}$$

where the vector p_j and scalar q are expresed by:

$$p_j = \beta_j - u_{j\,||\,\alpha}\, \alpha_\alpha$$

$$q = \gamma - \psi_{||\,\alpha}\, \delta_\alpha.$$

From (2.10) we come to the conclusion: if the fields (u_j, ψ) satisfy the corresponding Euler-Lagrange equations (2.3), the functional (2.1) remains infinitesimally invariant at (u_j, ψ) under the small transformations (2.5a) if $\delta I = 0$.

The requirement of the stationary value of the functional I gives

$$\int_S \left[L\, \alpha_\alpha + \frac{\partial L}{\partial u_{j\,||\,\alpha}} p_j \right]_{||\alpha} dS + \int_S \frac{\partial L}{\partial \psi} q\, dS = 0 \tag{2.10a}$$

Equations (2.10a) which we call the equation of variational invariance, is the mathematical version of the celebrated Neother's theorem.

If we apply the Green theorem to the first term of (2.10a), we obtain a new integral

$$\oint_C \left[L\alpha_\alpha + \frac{\partial L}{\partial u_{j\,||\,\alpha}} p_j \right] \eta_\alpha\, dl + \int_S \frac{\partial L}{\partial \psi} q\, dS = 0, \tag{2.11}$$

where S is the simply connected region on the middle surface bounded by a smooth closed curve C, and n_α is the unit normal (in S) to C.

We now proceed to write the integral form of the conservation law (2.11) which corresponds to the particular transformations (2.5a).

By taking all of the arbitrary variation (2.4, 2.5) except one in turn, we obtain the corresponding conservation laws.

As a special case we consider first:

$$\alpha_\alpha = 0, \quad \beta_j \neq 0, \quad \gamma = 0$$

$$p_j = \beta_j \quad q = 0.$$

Then we introduce the family of transformations

$$\bar{\xi}_\alpha = \xi_\alpha$$

$$\bar{u}_j = u_j + \beta_j \eta$$

$$\bar{\psi} = \psi,$$

which represents rigid body translation (infinitesimal).
The corresponding conservation law when there are no body forces, i.e. all $F=0$, (2.11) now reads

$$\int_C \frac{\partial L}{\partial u_{j||\alpha}} n_\alpha \, d\, l = 0,$$

And if

$$\alpha_\alpha = 0, \beta_j = 0, \gamma \neq 0$$

$$p_j = 0, q = \gamma$$

we consider a family of transformations

$$\bar{\xi}_\alpha = \xi_\alpha$$

$$\bar{u}_j = u_j$$

$$\bar{\psi} = \psi + \gamma \eta$$

which represents translation of amount η of scalar function ψ, then eqn (2.11) gives to the rather known trivial statement

$$\int_S \frac{\partial L}{\partial \psi} d S = 0.$$

A new class of transformations:

$$\alpha_\alpha \neq 0, \quad \beta_j = 0 \quad \gamma = 0$$

$$p_j = -u_{j||\alpha} \alpha_\alpha, \; q = -\psi_{||\alpha} \alpha_\alpha$$

and the coresponding family of transformations

$$\bar{\xi}_\alpha = \xi_\alpha + \alpha_\alpha \eta$$

$$\bar{u}_j = u_j$$

$$\bar{\psi} = \psi$$

which represent a family of coordinate translations and leada us to conservation law which are of a special interest for us

Then the conservation law reads

$$\oint_c \left[L\, \delta_{\alpha\beta} - \frac{\partial L}{\partial u_{j\|\beta}}\, u_{j\|\alpha} \right] n_\beta\, dl - \int_S \frac{\partial L}{\partial \psi}\, \psi_{\|\alpha}\, dS = 0. \tag{2.12}$$

This is the integral we are very familiar with, whose first member is the *J*-integral [1] of fracture mechanics.

Nonlinear membrane theory

The starting point of the study is the linear elastic shell theory of Budiansky and Sanders [9] and nonlinear shell theory of Budiansky [10]. Only the essentials of the theory are given here and details can be found in [9, 10].

The displacement vector U^i of a material point in the middle surface of the shell is given by

$$U^i = u^\alpha\, x^i_{|\alpha} + w\, N^i \tag{3.1}$$

where u^α, w are the surface and surface-normal components of the displacement vector.

The membrane strain mesaures $E_{\alpha\beta}$ given in terms of these components (u_α, w) is

$$E_{\alpha\beta} = \frac{1}{2}\,(u_{\alpha|\beta} + u_{\beta|\alpha}) + b_{\alpha\beta}\, w \tag{3.2}$$

while the rotation is $\Phi_\alpha = -w_{,\alpha} + b_\alpha^{\ \gamma} u_\gamma$ and $b_\alpha^{\ \gamma}$ is the (mixed) curvature tensor of the middle surface

We restate here briefly the essential relationships of a nonlinear membrane theory. Details can be found in [11]. The finite strain measure $\hat{E}_{\alpha\beta}$ given by

$$\hat{E}_{\alpha\beta} = E_{\alpha\beta} + \frac{1}{2}\,(d_\alpha^{\ \gamma}\, d_{\gamma\beta} + \Phi_\alpha\, \Phi_\beta) \tag{3.3}$$

where the linear part of the stretching strain $E_{\alpha\beta}$ is given by (3.2), Φ_α is he same rotation as before, and

$$d_{\alpha\beta} = u_{\alpha|\beta} + b_{\alpha\beta}\, w. \tag{3.4}$$

The Kirchoff stress-resultant $n^{\alpha\beta}$ is related to the membrane stress-resultant by

$$n^{\alpha\beta} = \sqrt{\frac{\bar{g}}{g}}\, N^{\alpha\beta} \tag{3.5}$$

where $\bar{g} = \det(\bar{g})$.

For nonlinear membrane theory the equilibrium equantions, with no pressure loads present, are [11]:

$$[(g_{\gamma\beta} + d_{\gamma\beta})\, n^{\alpha\beta}]_{|\alpha} - b_{\gamma\alpha}\, \Phi_\beta\, n^{\alpha\beta} = 0$$

$$-(\Phi_\beta\, n^{\alpha\beta})_{|\alpha} - b_\alpha{}^\gamma (g_{\gamma\beta} + d_{\gamma\beta})\, n^{\alpha\beta} = 0. \tag{3.6}$$

The above equations are exact and are derivable from the Principle of Virtual Work applied to the deformed shell.

If we indetify the vector field u_j as $[u_\alpha, w]$ and the scalar field ψ as θ which represents temperature, in relations (2.1) and (2.12) we obtain expressions adapted for thermoelasticity membrane.

Defining now a Lagrangian density (with negative sign) by the relation

$$-L(u_\alpha, u_{\alpha||\beta}, w, w_{||\alpha}, \theta, \xi_\alpha) = W(\hat{E}_{\alpha\beta}, \theta) \tag{3.7}$$

where the free energy density per unit area is denoted by W, it may be verified that

$$\frac{\partial L}{\partial u_\gamma} = b_{\gamma\alpha}\, \Phi_\beta\, n^{\alpha\beta}, \qquad \frac{\partial L}{\partial u_{\alpha||\beta}} = (g_{\alpha\gamma} + d_{\alpha\gamma})\, n^{\alpha\beta}$$

$$\frac{\partial L}{\partial w} = n^{\alpha\beta}(g_{\gamma\beta} + d_{\gamma\beta})\, b_\alpha{}^\gamma; \qquad \frac{\partial L}{\partial w_{||\beta}} = n^{\alpha\beta}\, \Phi_\alpha; \qquad \frac{\partial L}{\partial \theta} = -S_e \tag{3.8}$$

where Se is the entropy.

The equilibrium equations can be directly derived from L as Euler-Lagrange equations:

$$\frac{\partial L}{\partial u_{\alpha||\beta||\beta}} - \frac{\partial L}{\partial u_\alpha} = 0, \qquad \frac{\partial U}{\partial w_{||\alpha||\alpha}} - \frac{\partial L}{\partial w} = 0, \tag{3.9}$$

i.e. the relations (3.6) coincide with Euler-Lagrange eqns (3.9).

Using the above expression for L (3.7) we can rewrite (2.12) to derive the conservation law for nonlinear membrane theory

$$\oint_C \left[W\, \delta_{\alpha\beta} + \frac{\partial W}{\partial u_{\alpha||\gamma}}\, u_{\gamma||\beta} + \frac{\partial W}{\partial w_{||\alpha}}\, \omega_{||\beta} \right] n_\beta\, dl - \int_S \frac{\partial W}{\partial \theta}\, \theta_{||\alpha}\, dS = 0 \tag{3.10}$$

The conservation law (3.10), using the above expression for L, $u_{\alpha||\beta}$, L, $w_{||\alpha}$ and L, θ is given by

$$J = \oint_C \left[W\, n_\lambda - T^\gamma\, u_{\gamma||\lambda} - Q\, w_{||\lambda} \right] dl - \int_S S_e\, \theta_{||\lambda}\, dS = 0 \tag{3.11}$$

where W is the free energy per undeformed area of the middle surface such that it is related to the Kirchoff stress resultant by

$$n^{\alpha\beta} = \frac{1}{2}\left(\frac{\partial W}{\partial E_{\alpha\beta}} + \frac{\partial W}{\partial E_{\beta\alpha}} \right) \tag{3.12}$$

and along any curve C with normal n (in 1) in the underformed state, the edge force per unit underformed lenght Q^i is

$$Q^i = T^\alpha X^i_{|\alpha} + Q N^i \tag{3.13}$$

where

$$T_\alpha = (g_{\alpha\beta} + d_{\alpha\beta})\, n^{\gamma\beta}\, n_\gamma$$

$$Q = -\Phi_\beta\, n^{\alpha\beta}\, n_\alpha.$$

Equations (3.11) represent conservation law for nonlinear thermoelastic membrane theory which is believed to be new. This eqn. reduces to the conservation law obtained in [7] for nonlinear membrane theory when the Lagrangian density L in (3.11) is independent on the temperature without the integral $\int_S S_e \theta,_\gamma dS$.

Apart from its inherent theoretical interest, the conservation law made explicit is of practical importance in connection with the direct asymptotic analysis of geometrically induced singular stress concentrations, such as those occasioned by cracks and notches. For example, the stress intensity factor can be determined by calculating the J-line integral over an arbitrary path C surrounding the crack tip and by calculating the area integral over the area (A) enclosed by this path. Such applications may be found in [12].

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DIE ERHALTUNGSGESETZE IN DER THERMO — ELASTISCHEN THEORIE DER MEMBRANEN

In dieser Arbeit werden die Invariantheitsbedingungen der Lagrangeschen Funktion (oder des Aktionsintegrals) im Bezug auf die angenommen Transformationsgruppen, die zu den entsprechenden Erthalungsgesetzen führen, erörtert.

Die gewonnenen Erhaltungsgesetze in der allgemeinen From wurder auf die Probleme der thermoelastischen Membranen angewandt und daraus die entsprechenden Gesetze abgeleitet.

Es ist zu bemerken, dass eines der gewonnenen Gesetze die Verallgemeinerung des bekannten J-Integrals in der thermoelastischen Membranentheorie darstellt. Auch wird darauf angewiesen, dass in der Temperaturabwesentheit dasselbe Gesetz auf das von der Bahn unabhängige Integral, das Lo [7] fur die elastischen Memaranen erhielt, abgeleitet wird.

ЗАКОНИ ОДРЖАЊА У ТЕРМОЕЛАСТИЧНОЈ ТЕОРИЈИ МЕМБРАНА

У раду су разматрани услови инваријантности Лагранжијана (или акционог интеграла) у односу на прихватљиве групе трансформација које доводе до одговарајућих закона одржања.

Добијени закони одржања у општој форми примењени су на проблем термоеластичних мембрана и из њих изведени одговарајући закони. Евидентно је да један од добијених закона представља генерализацију познатог J-интеграла у теорији термоеластичних мембрана. Исто тако узакано је да се у отсуству температуре исти закон своди на интеграл независан од путање добијен од стране Ло [7] за случај еластичне мембране.

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