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# ADIABATIC AIR FLOW WITH REAL FRICTION FACTOR

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## Introduction

One dimensional adiabatic gas flow in the constant diameters pipes can be described with the following differential equation:

$$\lambda \frac{dx}{D} = \frac{2}{k} \frac{1-M^2}{M^3\left(1+\frac{K-1}{2}M^2\right)} dM, \tag{1}$$

where: $\lambda$ is the friction factor, $D$ diameter of the pipe, $M$ Mach number and $k$ adiabatic exponent. The famous conclusion can be derived from this equation: velocity of the flow proposed above tends to the velocity of sound. This practically means that we have accelerating or decelerating flow for the velocities with $M<1$ and $M>1$ respectively. It is only needed that the pipe has appropriate lenght, depending on starting Mach number. Solution of the equation (1) is known in literature, but it is obtained with the assumption that $\lambda=\lambda_0=$const. (this yields $T=T_0=$const.). According to this we have:

$$\lambda_0 \frac{L}{D} = \frac{2}{k}\{F(M_2)-F(M_1)\}, \tag{2}$$

where $F(M)$ is the function defined bellow:

$$F(M) = \frac{k+1}{4} \ln \frac{1+\frac{k-1}{2}M^2}{M^2} - \frac{1}{2M^2} \tag{3}$$

In the solution of the equation (2) $L$ is lenght of the pipe, $M_1$ and $M_2$ are starting and final Mach number respectively. The length of the pipe needed to reach the final mach number equal to one ($M_2=1$) with starting Mach number $M_1$ will be:

$$\lambda_0 \frac{L^*}{D} = \frac{2}{k}\{F(1)-F(M_1)\}, \tag{4}$$

with the values for air flow, for example, $k=1,4$; $F(1)=-0,391$. However solution can not be accepted for all regimes of flow, because $\lambda$ will not always be constant. Really, only for extra large values of $Re$ number $\lambda$ is constant disregardin $T \neq T_0$ (in this region friction factor depends on relative roughness of the pipe only).

## Formulation of the problem

Actually friction factor is dependent on $Re$ number, that means on Mach number. The entire scale of $Re$ numbers we can divide in a few regions according to relations between friction factor and $Re$ number:

1

$$\lambda = \frac{C}{Re^n} \tag{5}$$

where we have values $n=1$, $C=64$ for laminar flow (defined with $Re<2320$) and $n=0,25$, $C=0,3164$ for turbulent flow in hydraulically smooth pipes [defined with $4000 < Re < \left(62,7 \frac{D}{\delta}\right)^{1,143}$; the criterion for a pipe being smooth is $\delta < \delta_{gs}$]. Here $\delta$ is absolute roughness of the pipe, $\delta_{gs}$ is the thickness of the boundary layer defined with the famous formula given by Prandtl $\left(\frac{\delta_{gs}}{D} = \frac{62,7}{Re^{0,875}}\right)$, and

2

$$\lambda = 0,1\left(1,46\frac{\delta}{D} + \frac{100}{Re}\right)^{0,25} \tag{6}$$

which is valuable for the flow in hydraulically rough pipes $\left[\text{defined with } Re < \left(62,7\frac{D}{\delta}\right)^{1,143}\right]$. The region with $Re$ numbers between 2320 and 4000 can also be treated analytically, but it has no practical importance.

If $T_0$ and $T$ are temperatures of a gas with the velocity equal to zero and some other velocity respectively, we have following equations:

$$\mu = \mu_0\left(\frac{T}{T_0}\right)^m, \qquad T_0 = T\left(1 + \frac{k-1}{2}M^2\right). \tag{7}$$

The first equation denotes dynamical viscosity as a function of temperature (for air flow $m=0,75$) and the second one shows relation between temperature and corresponding Mach number. From the equations (7) and the continuity equation ($\rho v = \dot{m}/A = \text{const.}$) we can obtain $Re$ number as a function of temperature or Mach number:

$$Re = \frac{v D \rho}{\mu} = \frac{\dot{m}}{A}\frac{D}{\mu_0}\left(\frac{T_0}{T}\right)^m = \frac{\dot{m} D}{A \mu_0}\left(1 + \frac{k-1}{2}M^2\right)^m. \tag{8}$$

Using equations (5) and (6) we finally obtain the expressions for the regimes of flow mentioned above:

$$\lambda = \frac{\lambda_{01}}{\left(1 + \frac{k-1}{2} M^2\right)^{mn}}; \qquad \lambda_{01} = \left(\frac{\dot{m}}{A}\right)^{-n} \frac{C \mu_0^{\,n}}{D^n} \tag{5'}$$

$$\lambda = \lambda_{02} \left[1 + \alpha \left(1 + \frac{k-1}{2} M^2\right)^{-m}\right]^{0,25}; \tag{6'}$$

$$\lambda_{02} = 0.11 \left(\frac{\delta}{D}\right)^{0,25}; \quad \alpha = 68,5 \left(\frac{\dot{m}}{A}\right)^{-1} \frac{\mu_0}{\delta}$$

Obviously, in the equations (5′) and (6′) $\lambda_{01}$ and $\lambda_{02}$ are related to the case with the constant temperature ($T = T_0$), the same as solutions (2) and (3). This means that the factor $\left(1 + \frac{k-1}{2} M^2\right)^m$ takes into account correction of the values $\lambda_{01}$ and $\lambda_{02}$ caused by friction factor $\lambda$ being dependent on Mach number.

## Solution of the problem

Putting the expressions (5′) and (6′) in the equation (1), one can obtain two differential equations for mentioned two regimes of flow:

$$\lambda_{01} \frac{dx}{D} = \frac{2}{k} \frac{1 - M^2}{M^3 \beta^{1-mn}} \, \mathrm{d}M \tag{9}$$

$$\lambda_{02} \frac{dx}{D} = \frac{2}{k} \frac{1 - M^2}{M^3 \beta} \left(1 + \frac{\alpha}{\beta^m}\right)^{-0,25} \mathrm{d}M \tag{10}$$

where we have used the abbreviation:

$$\beta \overset{def}{=} 1 + \frac{k-1}{2} M^2$$

Solving the equation (9), we actually integrate the differential binom and obtain rational functions. For example, if we consider air flow ($m = 0,75$) the solution will be:

$$\lambda_{01} \frac{L}{D} = \frac{2}{k} \{F(M_2) - F(M_1)\} \tag{11}$$

with:

$$F(m) = -\frac{\beta^{0,75}}{2M^2} - \frac{k+7}{8} \operatorname{arc\,tg} \beta^{0,25} + \frac{k+15}{16} \operatorname{arc\,ctg} h \, \beta^{0,25} \tag{12}$$

for laminar flow ($n=1$, $C=64$), and

$$F(M)=-\frac{\beta^{0,2}}{2M^2}-\frac{2k+3}{10}\ln(\beta^{0,2}-1)+\frac{2k+3}{10}\cos\frac{\pi}{5}\cdot\ln\left(1+2\beta^{0,2}\cos\frac{\pi}{5}+\beta^{0,4}\right)+$$

$$+\frac{3k+3}{10}\cos\frac{3\pi}{5}\cdot ln\left(1+2\beta^{0,2}\cos\frac{3\pi}{5}+\beta^{0,4}\right)+ \tag{13}$$

$$+\frac{2k+3}{5}\sin\frac{\pi}{5}\cdot\mathrm{arc\,tg}\frac{\beta^{0,2}+\cos\frac{\pi}{5}}{\sin\frac{\pi}{5}}+\frac{2k+3}{5}\sin\frac{3\pi}{5}\cdot\mathrm{arc\,tg}\frac{\beta^{0,2}+\cos\frac{3\pi}{5}}{\sin\frac{3\pi}{5}},$$

for turbulent flow in hydraulically smooth pipes. It is useful to mention that the values $\lambda_{01}$ differ for these two cases because the consctants $n$ and $C$ are difeerent too. It is impossible to find exact solution of the equation (10) neither for air flow nor for other gasses. To get the approximate solution we shall use the Newton's binomial expansion:

$$(1+\alpha\beta^{-0,75})^{-\frac{1}{4}}=\sum_{i=0}^{\infty}\binom{-\frac{1}{4}}{i}\alpha^i\,\beta^{-3i/4}. \tag{14}$$

But this expansion will be useful for us only if $\alpha\beta^{-0'75}$ is essentially less than one (at least $\alpha\beta^{-0,75}\leqslant 0,5$). So, let's examine this variable. In the investigated regime of flow we have:

$$\frac{\delta_{gs}}{D}=\frac{62,7}{Re^{0,875}}\leqslant\frac{\delta}{D},\quad \alpha\beta^{-0,75}=\frac{68,5}{Re}\frac{D}{\delta},$$

and after obvious transformations we derive:

$$\alpha\beta^{-0,75}\leqslant\frac{68,5}{62,7}Re^{-0,125}\leqslant 0,3874.$$

The last value was obtained by putting $Re=4000$. Actually this is an extremal case and $Re$ is usually much greater ($\alpha\beta^{-0'75}$ smaller). It will be enough to take only three first terms of the expansion (14). The first term gives the correction less than 9,7%, the second less than 2,3% and the third one less than 0,7%, all compared to unity. The equation (10) will therefore become:

$$\lambda_{02}\frac{dx}{D}=\frac{2}{k}\frac{1-M^2}{M^3\beta}\cdot\frac{1}{2}\left(1-\frac{1}{4}\alpha\beta^{-0\cdot75}+\frac{5}{32}\alpha^2\beta^{-1,5}\right)dM. \tag{10'}$$

Taking into consideration the expression for $\beta$, the solution of the equation (10') is found:

$$\lambda_{02}\frac{L}{D}=\frac{2}{k}\{F(M_2)-F(M_1)\}, \tag{15}$$

where $F(M)$ denotes the following function:

$$F(M)=\frac{1}{2}F_0(M)-\frac{1}{8}\alpha F_1(M)+\frac{5}{64}\alpha^2 F_2(M), \qquad (16)$$

$$F_0(M)=\frac{k+1}{2}\ln\frac{\beta}{\beta-1}-\frac{k-1}{2}\frac{1}{\beta-1},$$

$$F_1(M)=-\frac{k-1}{1}\frac{\beta^{0,25}}{\beta-1}-\frac{4}{3}\frac{k+1}{2}\beta^{-0,75}+\frac{7k+1}{4}\operatorname{arc\,tg}\beta^{0,25}-\frac{7k+1}{8}\ln\frac{\beta^{0,25}-1}{\beta^{0,25}+1},$$

$$F_2(M)=-\frac{2k}{\beta^{0,5}}-\frac{2}{3}\frac{k+1}{3}\beta^{-1,5}-\frac{k-1}{2}\frac{\beta^{0,5}}{\beta-1}-\frac{5k-1}{4}\ln\frac{\beta^{0,5}-1}{\beta^{0,5}+1},$$

The functions $F(M)$ according to equations (3), (12), (13) and (16) have been calculated and given in the following table:

| M | F(M) | | | |
|---|---|---|---|---|
| | (3) | (12) | (13) | (16) |
| 0.01 | —4994,474 | —4994,315 | —4990,727 | —4620,923 |
| 0.02 | —1245,306 | —1245,025 | —1241,522 | —1153,375 |
| 0.03 | —551,348 | —550,995 | —547,549 | —510,180 |
| 0.04 | —308,637 | —308,235 | —304,827 | —285,338 |
| 0.05 | —196,405 | —195,963 | —192,586 | —181,317 |
| 0.06 | —135,512 | —135,039 | —131,686 | —124,875 |
| 0.07 | —98,849 | —98,349 | —95,016 | —90,904 |
| 0.08 | —75,093 | —74,570 | —71,255 | —68,896 |
| 0.09 | —58,838 | —58,294 | —54,995 | —53,828 |
| 0,10 | —47,236 | —46,674 | —43,389 | —43,079 |
| 0,20 | —10,564 | —9,883 | —9,690 | —9,091 |
| 0,30 | —4,100 | —3,353 | —3,211 | —3,097 |
| 0,40 | —2,006 | —1,216 | 1,892 | —1,152 |
| 0,50 | —1,139 | —0,317 | 2,766 | —0,344 |
| 0,60 | —0,734 | 0,109 | 3,176 | 0,036 |
| 0,70 | —0,536 | 0,323 | 3,377 | 0,224 |
| 0,80 | —0,441 | 0,429 | 3,474 | 0,317 |
| 0,90 | —0,401 | 0,476 | 3,515 | 0,359 |
| 1,00 | —0,391 | 0,489 | 3,526 | 0,374 |
| 1,10 | —0,397 | 0,483 | 3,519 | 0,372 |
| 1,20 | —0,414 | 0,465 | 3,501 | 0,362 |
| 1,30 | —0,436 | 0,439 | 3,478 | 0,346 |
| 1,40 | —0,460 | 0,410 | 3,452 | 0,328 |
| 1,50 | —0,486 | 0,378 | 3,425 | 0,309 |
| 1,60 | —0,511 | 0,345 | 3,397 | 0,290 |
| 1,70 | —0,536 | 0,311 | 3,370 | 0,272 |
| 1,80 | —0,560 | 0,278 | 3,344 | 0,254 |
| 1,90 | —0,583 | 0,245 | 3,319 | 0,237 |

For example, if we want to calculate the values

$$\lambda \frac{L^*}{D} = \frac{2}{k} [F(1) - F(M)]$$

according to derived equations it will be ($M$=0,1):

$$\text{(2) and (3):} \quad \lambda_0 \frac{L^*}{D} = \frac{2}{1,4} (-0,391 + 47,236) = 66,9 = A \,,$$

$$\text{(11) and (12):} \quad \lambda_{01} \frac{L^*}{D} = \frac{2}{1,4} (0,4890 + 46,674) = 67,4 = B \,,$$

$$\text{(11) and (13):} \quad \lambda_{01} \frac{L^*}{D} = \frac{2}{1,4} (3,5260 + 43,389) = 67,0 = C \,,$$

$$\text{(15) and (16):} \quad \lambda_{02} \frac{L^*}{D} = \frac{2}{1,4} (0,3740 + 43,079) = 62,1 = D .$$

The differences between these values are:

$$d(BA) = 0,74\% \,, \quad d(CA) = 0,15\% \,, \quad d(AD) = 7,2\% \,.$$

It is clear that we shall have the other differences for different Mach numbers and for different regimes of flow. However, the calculated values show that the differences are greater for greater Mach numbers. That is in connection with quite different influence of Mach number on friction factor in different regimes of flow.

## REFERENCES

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## ADIABATISCHE LUFTSTROMUNG MIT DEM SACHILCHEN REIBUNGSFAKTOR

In der Fachliteratur wird vorläufig eine Formel benutzt, die bei der adiabatischen Gasströmung die Rohrparameter in die Verbindung mit dem Reibungsfaktor, gleich wie mit den Machnummern an Enden des Rohrs setzt. Diese Formel ist durch die Integration entsprechender Differentialgleichung hergeleitet mit dem Ansatz eines durch Reibungsfaktor eine Funktion der Machnummer stellt vor, und ebenso durch das Rohr variiert. Unsere Arbeit vernachlässigt nicht, und eine Lösung entsprechender Gleichung für die laminare und turbulente Strömung gibt. Weil in diesen zwei Fällen ausgrechnete Rohrlänge bedeutend unterschiedlich ist, darf man nicht einen konstanten Reibungsfaktor annehmen.

# ADIJABATSKO STRUJANJE VAZDUHA SA REALNIM KOEFICIENTOM TRENJA

U literaturi je poznata formula koja pri adijabatskom strujanju gasa daje vezu između koeficijenta trenja, prečnika i dužine cevi, i vrednosti Machovih brojeva na njenim krajevima. Ova formula dobijena je integracijom odgovarajuće diferencijalne jednačine uz pretpostavku da se koeficijent trenja ne menja duž cevi. Međutim, koeficijent trenja zavisi od Machovog broja koji se menja duž cevi pa se samim tim menja i koeficijent trenja. U ovom radu se dolazi do rešenja pomenute diferencijalne jednačine kako za laminarno tako i za turbulentno strujanje, uzimajući u obzir i promenu koeficijenta trenja duž cevi. Izračunate dužine cevi po dobijenoj i po postojećoj formuli znatno se razlikuju na osnovu čega se može zaključiti da se pretpostavka o konstantnosti koeficijenta trenja ne može unapred prihvatiti.

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