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BOUNDARY ELEMENTS FOR LARGE PLASTIC DISPLACEMENT PROBLEMS

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1. Introduction

Rapid development of the computer technology in the last three decades enabled an essential progress of numerical means for the approximate solution of engineering problems, described by differential equations. The most commonly used methods are:

— the finite difference method (FDM),

— the finite element method (FEM), and

— the boundary element method (BEM).

With finite differences the governing differential equations within the continuum are modelled by difference relations of variables at a finite number of nodes in the body. This produces an algebraic system of equations, with a narrow band summetrical matrix.

The finite element method represents today the most widely used numerical tool. With this approach, the domain has to be divided into small elements, in which variables are interpolated by polynomials. Using suitable variational principles, the minimum energy distribution is obtained. The FEM also produces an algebraic system of equations, having a symmetric narrow band. This method is very popular for engineering applications, but it also contains some drawbacks.

A third option is given by the boundary element method. It uses a transformation of the problem to a corresponding integral equation for the boundary. This asks for the disretization on the surface of the domain only, what represents an essential reduction of input data required with respect to other listed methods. The resulting algebraic system of equations is smaller, but the matrix is fully populated and asymmetric. This weakness is usually outbalanced by several advantages, like simple modelling and good results for infinite and semifinite domain problems, simple data preparation, and essentially reduced matrix size.

In this paper the boundary element method is used for the elasto-plastic evaluation of solid continua, exhibiting large displacements (metal plate bending problem).

2. Elastoplastic formulation

2.1. Elasticity

At each point of the interior of a continuum, equilibrium equation applies, which can be written as follows

$$\sigma_{jk,j} + b_k = 0 \tag{1}$$

where σ_{jk} is a symmetric stress tensor ($\sigma_{jk}=\sigma_{kj}$) and b_k are body forces per unit volume. Equilibrium is also applied at all boundary points, where

$$p_i = \sigma_{ij} n_j \tag{2}$$

with p_i being components of the stress vector on the boundary, and n_j representing components of the external surface normal. Deformation of an elastic continuum is described by displacements u_i. Assuming small deformation theory, Cauchy symmetric tensor is defined by.

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{3}$$

If heat loads apply, there will also be thermal strains ε_{ij}^{th}, additive to elastic ones, producing total strain

$$\varepsilon_{ij} = \varepsilon_{ij}^{e} + \varepsilon_{ij}^{th}, \qquad \varepsilon_{ij}^{th} = \delta_{ij}\,\alpha\,\theta \tag{4}$$

where α is temperature expansion coefficient and θ is temperature difference. Compatibility of displacements in a simply connected domain is ensured by the relation

$$\varepsilon_{ij,\,mn} + \varepsilon_{mn,\,ij} - \varepsilon_{im,\,jn} - \varepsilon_{jn,\,im} = 0 \tag{5}$$

Constitutive relationship of stresses and strains within the elastic regime is given by (what is known as the Hookean law)

$$\sigma_{ij} = C_{ijmn}\varepsilon_{mn}^{e} = \left\{\frac{2\,G\,\nu}{1-2\,\nu}\,\delta_{ij}\,\delta_{mn} + G\,(\delta_{im}\,\delta_{jn} + \delta_{in}\,\delta_{jm})\right\}\varepsilon_{mn}^{e} \tag{6}$$

Combining eq. (6) with (4), Duhamel-Neumann equations are obtained

$$\sigma_{ij} = 2\,G\left\{\varepsilon_{ij} + \frac{1}{1-2\,\nu}\,\delta_{ij}\,[\nu\,\varepsilon_{kk} - (1+\nu)\,\alpha\,\theta]\right\} \tag{7}$$

Substituting eq. (3) in (7), and bearing in mind equilibrium eq. (1), well known Navier-Lame equations develop

$$u_{j,\,kk} + \frac{1}{1-2\,\nu}\,u_{k,kj} + \overline{b}_j = 0 \tag{8}$$

where the generalised vector of body forces $\overline{b_j}$ is given by

$$\overline{b}_j = b_j - 2\,G\,\alpha\,\frac{1+\nu}{1-2\,\nu}\,\theta_{,j} \tag{9}$$

2.2. Elastoplasticity

Mechanical properties of materials may be divided into elastic, plastic and viscous. For the purpose of our elastoplastic formulation, viscous behaviour shall be neglected. Governing equations are given in the form of time derivatives. In this paper rates have to be multiplied by time increments ($dt \to \Delta t > 0$), transforming equations into time independent incremental forms, typical in the classical elastoplasticity.

For an elastoplastic increment of deformation, we have

$$\dot{\varepsilon}_{ij} = \frac{1}{2}(\dot{u}_{i,j} + \dot{u}_{j,i}) = \dot{\varepsilon}^{e}_{ij} + \dot{\varepsilon}^{th}_{ij} + \dot{\varepsilon}^{p}_{ij} \tag{10}$$

The equilibrium condition for the interior of a continuum applies also for increments

$$\dot{\sigma}_{ij,i} + \dot{b}_j = 0 \tag{11}$$

and on the boundary

$$\dot{p}_i = \dot{\sigma}_{ij} n_j \tag{12}$$

For the elastic part of total strain increment the constitutive law (7) with stress ncrements may be used, rendering

$$\dot{\sigma}_{ij} = 2G(\dot{\varepsilon}_{ij} - \dot{\varepsilon}^{th}_{ij} - \dot{\varepsilon}^{p}_{ij}) + \frac{2G\nu}{1-2\nu}(\dot{\varepsilon}_{kk} - \dot{\varepsilon}^{th}_{kk} - \dot{\varepsilon}^{p}_{kk})\delta_{ij} \tag{13}$$

In this equation plastic strain increments can be considered as "initial strains". Plastic stress contributions

$$\dot{\sigma}^{p}_{ij} = 2G\dot{\varepsilon}^{p}_{ij} + \frac{2G\nu}{1-2\nu}\dot{\varepsilon}^{p}_{kk}\delta_{ij} \tag{14}$$

may also be used, giving the "initial stress" formulation

$$\dot{\sigma}_{ij} = 2G(\dot{\varepsilon}_{ij} - \dot{\varepsilon}^{th}_{ij}) + \frac{2G\nu}{1-2\nu}(\dot{\varepsilon}_{kk} - \dot{\varepsilon}^{th}_{kk})\delta_{ij} - \dot{\sigma}^{p}_{ij} \tag{15}$$

For the description of the elastoplastic behaviour appropriate plasticity law is required. In addition to the Mises yield criterion

$$F(S_{ij}, k) = \frac{3}{2} S_{ij} S_{ij} - k^2 = 0, \quad S_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij} \tag{16}$$

the Levy-Mises flow rule is used

$$\dot{\varepsilon}^{p}_{ij} = \dot{\lambda} S_{ij} \tag{17}$$

where S_{ij} is deviatoric stress tensor and k unidimensional yield point, whereas the proportionality factor $\dot{\lambda} > 0$. The material hardening can be taken into account by tangential modulus values.

3. Governing integral formulation

The nonelastic description may be transcribed by the use of weighted residuals into the following integral formulation [4, 8]

$$\int b_k^* \dot{u}_k \, dV + \int p_k^* \dot{u}_k \, dA = \int u_k^* \dot{b}_k \, dV + \int u_k^* \dot{p}_k \, dA + \int \varepsilon_{jk}^* \left(\dot{\sigma}_{jk}^{th} + \dot{\sigma}_{jk}^{p} \right) dV \tag{18}$$

which corresponds to the 2nd Betti's reciprocal theorem, extended for thermal and plastic contributions. Taking into account fundamental solutions of Kelvin's infinite elastic field (*), the above record can be rewritten analogously to Somigliana's identity for increments

$$\dot{u}_i(\xi) = \int u_{ik}^* \dot{p}_k \, dA - \int p_{ik}^* \dot{u}_k \, dA + \int u_{ik}^* \dot{b}_k \, dV + \int \varepsilon_{ijk}^* \left(\dot{\sigma}_{jk}^{th} + \dot{\sigma}_{jk}^{p} \right) dV \tag{19}$$

Assuming that boundary values and distributions of body forces, thermal and plastic contributions, are known, the eq. (19) offers displacement increments in the interior of the continuum, expressed by integrals on the boundary. Applying a limit process makes possible to transfer the source point on the surface, rendering the governing integral equation of BEM for nonelastic computations ("initial strain" concept) [8]

$$C_{ik} \dot{u}_k(\xi) + \int p_{ik}^* \dot{u}_k \, dA = \int u_{ik}^* \dot{p}_k \, dA + \int u_{ik}^* \dot{b}_k \, dV + \int \varepsilon_{ijk}^* \left(\dot{\sigma}_{jk}^{th} + \dot{\sigma}_{jk}^{p} \right) dV \tag{20}$$

Volumetric plasticity integral may also be changed to

$$\int \varepsilon_{ijk}^* \dot{\sigma}_{jk}^{p} \, dV = \int \sigma_{ijk}^* \dot{\varepsilon}_{jk}^{p} \, dV \tag{21}$$

producing "initial stress" concept

$$C_{ik} \dot{u}_k(\xi) + \int p_{ik}^* \dot{u}_k \, dA = \int u_{ik}^* \dot{p}_k \, dA + \int u_{ik}^* \dot{b}_k \, dV + \int \varepsilon_{ijk}^* \dot{\sigma}_{jk}^{th} \, dV + \int \sigma_{ijk}^* \dot{\varepsilon}_{jk}^{p} \, dV \tag{22}$$

While the transformation of domain integrals to the contour form for thermal loading and body forces exists, plasticity integrals have to be evaluated numerically in the interior by dividing the domain into cells, but only where plastic zone is due to appear.

4. Evaluation of stress tensor for twodimensional plane cases

4.1. Streses on the contour

For the computation of stress tensor at the boundary nodes, a local coordinate system of the normal and tangent directions at a node can be used. Within this orthogonal system, normal and shear stress compoennts are given by [4, 8]

$$\dot{\bar{\sigma}}_{22} = \dot{\bar{p}}_2, \qquad \dot{\bar{\sigma}}_{12} = \dot{\bar{p}}_1 \tag{23}$$

The remainder of stress tensor components is obtained by numerical derivatives. For plane strain case and within the scope of an isochoric material behaviour ($\dot{\varepsilon}_{nn}^{p} = 0$) it gives

$$\dot{\bar{\sigma}}_{11} = \frac{2G}{1-\nu} \left(\dot{\bar{\varepsilon}}_{11} - \dot{\bar{\varepsilon}}_{11}^{th} \right) + \frac{\nu}{1-\nu} \dot{\bar{p}}_2 - 2G \left(\frac{\nu}{1-\nu} \dot{\bar{\varepsilon}}_{22}^{p} - \dot{\bar{\varepsilon}}_{11}^{p} \right) \tag{24}$$

and for the plane stress case the following form is applicable

$$\dot{\sigma}_{11}=\frac{2G}{1-\nu}\left(\dot{\varepsilon}_{11}-\dot{\varepsilon}_{11}^{th}\right)+\frac{\nu}{1-\nu}\dot{\bar{p}}_2-\frac{2G}{1-\nu}\dot{\varepsilon}_{11}^{p} \tag{25}$$

where $\nu=\nu/(1+\nu)$.

4.2. Stresses in the domain

In this case by differentiating eq. (19), and assuming constant values of plastic strain in the close vicinity of internal nodes the following formulation for the plane strain is obtained [8]

$$\dot{\sigma}_{ij}(\xi)-\dot{\bar{\sigma}}_{ij}(\xi)=\int\{\overset{*}{u}_{kij}(\dot{p}_k-\dot{\bar{p}}_k)-\overset{*}{p}_{kij}(\dot{u}_k-\dot{\bar{u}}_k)\}\,dA+$$

$$+\alpha(1+\nu)\{K_{0\theta}\int\bar{F}_{ij}\,dA+\int FN_{ij}\dot{\theta}\,dA+\int F_{ij}\dot{\theta}_{,n}\,dA\}-\delta_{ij}\frac{\alpha E}{1-2\nu}\dot{\theta}(\xi)+$$

$$+\int\overset{*}{\sigma}_{ijkm}\dot{\varepsilon}_{km}^{p}\,dV-\frac{1}{4(1-\nu)}\{2\dot{\varepsilon}_{ij}^{p}+\delta_{ij}(1-4\nu)\dot{\varepsilon}_{nn}^{p}\} \tag{26}$$

In the above formulation body force contributions are taken into account within particular solutions $\dot{\bar{\sigma}}_{ij}$, $\dot{\bar{u}}_i$, $\dot{\bar{p}}_i$, while thermal loads have been transfered to the contour.

Similarly the initial stress formulation can be obtained, where the two last terms of eq. (26) become

$$+\int\overset{*}{\varepsilon}_{ijkm}\dot{\sigma}_{km}^{p}\,dV-\frac{1}{8(1-\nu)}\{2\dot{\sigma}_{ij}^{p}+\delta_{ij}(1-4\nu)\dot{\sigma}_{nn}^{p}\} \tag{27}$$

5. Algebraization of the integral equation

Performing the discretization with boundary elements and internal integration cells for the computation of the domain integrals evaluating all source nodes on the contour, the eq. (22) yields the following system of linear equations

$$A\dot{x}=\dot{f}+S\,\dot{\epsilon}^{p} \tag{28}$$

where $\dot{x}$ represents unknown boundary values (increments of displacements and boundary stresses), while in $\dot{f}$ known contributions of boundary values, body forces and thermals loads are gathered.

The solution of the above system is

$$\dot{\underline{x}}=\mathbf{A}^{-1}\dot{\mathbf{f}}+(\mathbf{A}^{-1}\mathbf{S})\,\dot{\varepsilon}^{p} \tag{29}$$

or

$$\dot{\underline{x}}=\mathbf{m}+\mathbf{K}_1\,\dot{\varepsilon}^{p} \tag{30}$$

A similar procedure is used for the evaluation of stresses. By discretizing the system (26) for all internal points, and from eqs (23) to (25) for boundary nodes, it yields

$$\dot{\boldsymbol{\sigma}} = \overline{\mathbf{G}}\,\dot{\mathbf{p}} - \overline{\mathbf{H}}\,\dot{\mathbf{u}} + \dot{\overline{\mathbf{b}}} + (\overline{\mathbf{S}} + \mathbf{D})\,\dot{\varepsilon}^p \tag{31}$$

Within $\dot{\boldsymbol{\sigma}}$ there are vectorised tensor stress incremental components for all boundary and domain points. Accounting for known and unknown boundary values, the eq. (31) is further developed

$$\dot{\sigma} = \overline{\mathbf{A}}\,\dot{\mathbf{x}} + \dot{\overline{\mathbf{f}}} + (\overline{\mathbf{S}} + \mathbf{D})\,\dot{\varepsilon}^p \tag{32}$$

For unknown boundary values in $\dot{x}$ the solution of eq. (30) is used, producing

$$\dot{\boldsymbol{\sigma}} = \overline{\mathbf{A}}\,\dot{\mathbf{m}} + \dot{\overline{\mathbf{f}}} + (\overline{\mathbf{A}}\,\mathbf{K}_1 + \overline{\mathbf{S}} + \mathbf{D})\,\dot{\varepsilon}^p \tag{33}$$

or

$$\dot{\boldsymbol{\sigma}} = \dot{\mathbf{n}} + \mathbf{K}_2\,\dot{\varepsilon}^p \tag{34}$$

Matrices K_1 and K_2, containing geometrical information, have to be evaluated only once for a particular computation under the assumption of small deformations.

5. Incremental iterative procedure

In the preceeding formulations, increments of plastic strains have formally been considered as known parameters. It is now an urgent task to determine them iteratively within individual increment of looading. The algorithm developed is based on the procedure proposed by Mendelson [1] for initial strains.

With respect to the uniaxial state, effective plastic strain increment is defined by

$$d\,\varepsilon_v^p = \sqrt{2/3.\,(d\,\varepsilon_{ij}^p\,d\,\varepsilon_{ij}^p)} \tag{35}$$

The proportionality increment $d\lambda$ follows from eqs (17) and (35) to be

$$d\lambda = \frac{3}{2}\,d\,\varepsilon_v^p/k \tag{36}$$

In the very first computational step, pure elastic analysis of the problem is performed under full loading. At the mostly loaded point, a base loading factor is determined

$$L_0 = k_0/\sigma_v^{max} \tag{37}$$

by which the load is adjusted to the first yield. Incrementing $\Delta L = L_0\omega$ a new loading factor is determined

$$L_l = L_{l-1} + L_0\,\omega \tag{38}$$

Plastic strain contributions ε_{ij}^p are now split into

$$\varepsilon_{ij}^p(l) = \overline{\varepsilon}_{ij}^p(l-1) + \Delta\,\varepsilon_{ij}^p(l) \tag{39}$$

i.e. accumulated plastic strain from previous loading step and the newly calculated increment. The modified total strain can be written as

$$\varepsilon'_{ij}(l) = \varepsilon_{ij}(l) - \bar{\varepsilon}^{p}_{ij}(l-1) = \varepsilon^{e}_{ij}(l) + \varepsilon^{th}_{ij}(l) + \Delta\, \varepsilon^{p}_{ij}(l) \tag{40}$$

while the corresponding modified deviatoric strain reads as follows

$$e'_{ij} = \varepsilon'_{ij} - \frac{1}{3}\, \delta_{ij}\, \varepsilon'_{mm} = S_{ij}/2G + \Delta\, \varepsilon^{p}_{ij} \tag{41}$$

Using the deviatoric stress from eq. (17) in (35) the effective plastic strain increment is obtained

$$\Delta\, \varepsilon^{p}_{v} = \sqrt{2/3.\,(e'_{ij}\, e'_{ij})} - k/3G \tag{42}$$

Actual values of the yield limit k for the bilinear elastoplastic behaviour are obtained by

$$k(l) = k(l-1) + E_t/(1 - E_t/E).\, \Delta\, \varepsilon^{p}_{v}(l) \tag{43}$$

where E_t is the elastoplastic tangential modulus.

Increments of plastic strain tensor components are obtained from the uniaxial effective strain increment of step r

$$^{r}\{\Delta\, \varepsilon^{p}_{ij}\} = {}^{r-1}\left\{\Delta\, \varepsilon^{p}_{v}\, e'_{ij}/\sqrt{2/3 \cdot (e'_{ij}\, e'_{ij})}\right\} \tag{44}$$

The division of plastic strain into the accumulated values and presently computed increments by eq. (35) allows for the evaluation of total stresses in eq. (34) and boundary values with eq. (30).

Applying the last load increment, the unknowns on the boundary are

$$\underline{\boldsymbol{x}} = \boldsymbol{m} + \mathbf{K}_1 \{\boldsymbol{\epsilon}^{p}(l-1) + \Delta\, \varepsilon^{p}(l)\} = \boldsymbol{m} + \mathbf{K}_1\, \varepsilon^{p} \tag{45}$$

and stresses

$$\boldsymbol{\sigma} = \boldsymbol{n} + \mathbf{K}_2 \{\varepsilon^{p}(l-1) + \Delta\, \boldsymbol{\epsilon}^{p}(l)\} = \boldsymbol{n} + \mathbf{K}_2\, \varepsilon^{p} \tag{46}$$

6. Updated Lagrange routine

In the previously described procedure only small deformations have been dealt with (i.e. small displacements and small strains). The geometry has been considered invariable. However, with several engineering applications of small strains, large displacements can develop, where geometrical changes are considerable, what cannot be neglected in the computation. To sort this out, also for BEM the updated Lagrange routine (known with FEM [6]) has been developed, consisting of the following steps:

a) Elastoplastic computation until the displacement of a given node reaches a prescribed maximum value.

b) For a determined loading step, total displacements and stresses are evaluated on the boundary and in the domain.

c) Using displacements the new geometry is determined.

d) For all points relative rotation is determined in the global coordinate system.

e) Evaluated stresses and plastic strains are added to the previously accumulated values from preceeding Lagrange computation. Rotation directions are taken into account and values stored.

f) Correction of the basic yield criterion with respect to the reached level of plastification. For points in the elastic regime

$$k^{new} = k^{old} - \sigma_v \tag{47}$$

and for points within the plastification zone

$$k^{new} = 0 \tag{48}$$

g) Correction of boundary conditions if required (contacting problems).

7. Case study — metal plate bending

As an example of the descibed elastoplastic computation with the updated Lagrange routine, an analysis of a metal plate bending over a stiff cylinder has been performed (Fig. 1).

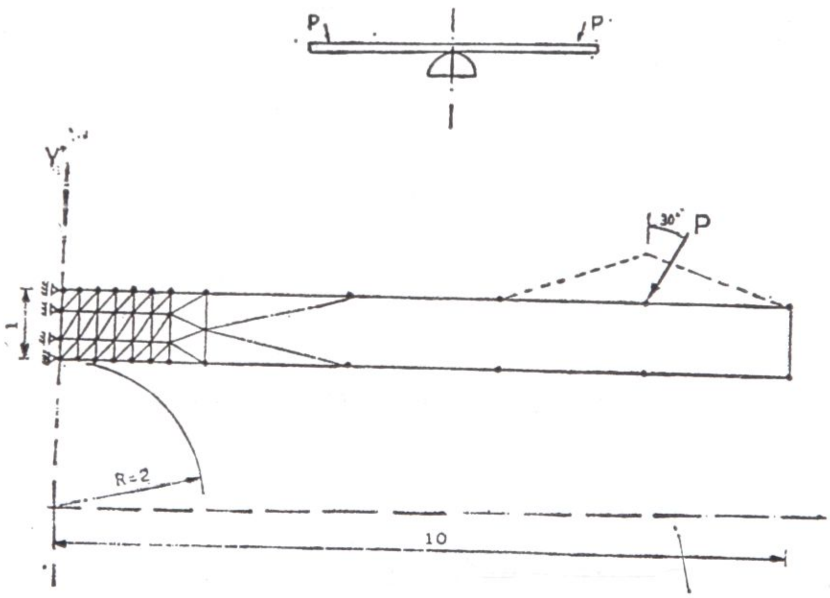

Fig. 1.

Elastoplastic computation has been initiated with a fixed node and continued until the next node reaches the support. At this point in time the computation has been broken and new geometry determined. Loading history was taken into account by correcting the yield point. Obtained stress values have been rotated with respect to the change of the geometry and separately freezed in the memory. Results for several steps of the updated Lagrange procedure have been computed, and in Fig. 2 plastic zone development after the fifth incremental step is illustrated. Material data used for the above computation have been: $E=20$ GPa, $\nu=0.3$, $k_0=200$ MPa and $E_t=2$ GPa [8, 9].

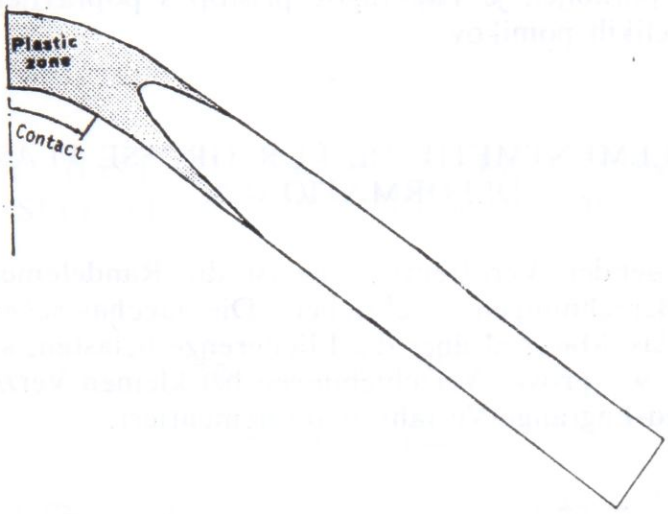

Fig. 2.

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ROBNI ELEMENTI ZA OBRAVNAVO VELIKIH POMIKOV PRI PLASTIFIKACIJI

V sestavku podajamo opis postopka robnih elementov za preračun plastičnosti. Upoštvene so mehanske in termične obremenitve, zaradi katerih prihaja do tečenja gradiva. Uporabljen je Lagrangov pristop s popravljanjem geometrijskih podatkov zaradi velikih pomikov.

RANDELEMENTMETHODE FÜR GROSSE PLASTISCHE DEFORMATIONEN

In der vorliegenden Veröffentlichung ist die Randelementmethode für die elastoplastischen Berechnungen beschrieben. Die mechanischen und thermischen Belastungen, die das Material über die Fließgrenze belasten, sind berücksichtigt. Für die Probleme wo grosse Verschiebungen bei kleinen Verzerrungen auftreten wurde ein Updated-Lagrange-Verfahren implementiert.

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