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SOME REMARKS ON THE APPLICATION OF THE ENERGY VARIATIONAL PRINCIPLES IN THE FINITE ELEMENT METHOD

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1. Introduction

The energy variational principles are frequently used in the solution of complex problems, for approximate solutions and development of numerical methods, such as the finite element method. Although the finite element method became a common tool for solution of many engineering problems, stil there are some doubts and unclear things. For instance it is not clear enough why the "refined", high order elements do not give improved results. Why the elements with iregular shapes give bad results?

This paper is an attempt to give some answers to such questions. It will be shown that the most of the problems in the finite element method are due to some approximations in the application of the variational principles.

The application of the energy variational principles involves some approximations, which can result in not allways reliable solutions. There are so called "parasitic" stresses which in the equations introduce some terms with different physical meaning. For instance in the equations of equilibrium of the bending moments there could be shear forces and twisting moments. The stresses in the equations and the final stresses are computed in different ways, and consequently the final stresses at the interelement boundaries are not in equilibrium, although the system equations represent equilibrium equations for the interelement nodes.

The disadvantages present in the application of the energy variational principles are not present in the direct method of development of finite elements [3]. That method in short will be described in chapter 3.

2. Aproximations in the application of the variational principles

Some of the shortcomings in the application of the energy variational principles, which are applied in the finite element method, will be ilustrated on the plate bending problem and plane stress problem. The plate bending problem is governed by the following differential equation,

$$\Delta \Delta w = p/D \tag{1}$$

where w is normal displacement, Δ — the well known differential operator, p — distributed load, D — stiffness of the plate.

The variation of the energy in an element of the plate is given by the following expression [1],

$$\delta U = \iint_F (D \Delta \Delta w - p) \delta w \, dx \, du + \int_s M_n \delta \left(\frac{\partial w}{\partial n} \right) ds -$$

$$- \int_s Q_s \delta w \, ds + \int_s M_{ns} \delta \left(\frac{\partial w}{\partial s} \right) ds \tag{2}$$

where M_n is a boundary normal moment, function of the second derivatives, Q_s — boundary shear force, function of the third derivatives, and M_{ns} — boundary twisting moment, function of the mixed second derivatives. The integration is carried out on the area F and the boundaries s of the element.

If the displacement shape function (DShF) of the element is assumed so that the following condition is satisfied,

$$\Delta \Delta w = 0 \tag{3}$$

the Exp. 2 becomes,

$$\delta U = \int_s M_n \delta \left(\frac{\partial w}{\partial n} \right) ds - \int_s Q_s \delta w \, ds + \int_s M_{ns} \delta \left(\frac{\partial w}{\partial z} \right) ds - \iint_F p \, \delta \, w dx dy \tag{4}$$

Kantorovich [2] for the biharmonic equation (1) has developed the following functional,

$$\Pi_k = \frac{1}{2 \, D \, (1 - \nu^2)} \iint_F [M_x{}^2 + M_y{}^2 - 2 \nu M_x M_y + 2 (1 + \nu) M_{xy}{}^2] \, dx \, dy +$$

$$+ \int_s M_n \frac{\partial w}{\partial n} ds - \int_s V_s w \, ds - \iint_F pw \, dx \, dy \tag{5}$$

where V_s is an equivalent shear force which contains the shear force Q_s and derivatives of the twisting moment M_{ns}. If the condition (3) is satisfied, the line integrals are twofold the first area integral and the functional (5) becomes,

$$\Pi_k = \frac{1}{2} \int_s M_n \frac{\partial w}{\partial n} ds - \frac{1}{2} \int_s Q_s w ds + \frac{1}{2} \int_s M_{ns} \frac{\partial w}{\partial s} ds - \iint_F p \, w \, dx \, dy \tag{6}$$

The Eq. 4 actually represents varied Eq. 6. It means that those two expressions lead to development of same elements.

In the case of stiffness (deformation) element the DShF explicitely provides interelement continuity of the function w and its first derivatives. The equations which have to be writen have to provide continuity of the second (moments) and the third (shear forces) derivatives at the interelement nodes. Therefore the element matrix actually should represent such equivalent nodal forces. The Eqs. 4 and 6 give such forces in function of the boundary forces, i.e. they give a way of concentrating them to the nodes.

However, if $\Delta\Delta w \neq 0$, besides the nodal forces computed in that way, which are due to the boundary forces, there is a contribution of the first integral of Eq. 2. The nodal forces resulting from Eq. 6 or 4 give the fundamental portion of the element stiffness matrix K, and the first integral of Eq. 2 gives an additional portion ΔK_1. The summ of the coefficients in one row of ΔK_1 is,

$$\Sigma \Delta \mathbf{K}_{lij} = 0$$

Therefore, when the size of the finite elements on which the system is subdivided becomes smaller and smaller, the displacements and the rotations in one element tend to constant, and consequently the contribution of ΔK_1 to the system equations tends to zero. For instance the nodal rotations and diplacements within an element become constant, and their product with the corresponding coefficients of ΔK_1, because the summ of those coefficients in every row is equal to zero, tends to to zero. As a result of that, good and converging final results are derived. However, when the mesh size is rough, or there is a high graident of variation of the stresses, the results which gives the element with ΔK_1 are bad and could be unreliable.

If the DShF of an element like that on Fig. 1 is defined by the following polinomial,

$$w = a_1 + a_2 x + a_3 y + \ldots . a_{11} x^3 y + a_{12} xy^3 \tag{7}$$

where the first 10 terms represent the full third order polinomial, the condition (3) – $\Delta\Delta w = 0$ is satisfied. Therefore that function leads to $\Delta K = 0$, and the element developed on the base of such function gives quite good results.

By variation of the functional (6), let say on the rotation $w_{3,x} = \theta_{3x}$ (Fig. 1), equivalent nodal moment M_{3x} should be obtained, and finaly equation of equilibrium of the moments M_x at that node in the system. However, besides the nodal moment which is derived from the first integral of (6), there is certain contribution of the second and the third integrals, because they are function of the rotation θ_{3x} also. But the physical meaning of Q_s and M_{ns}, which are present in those integrals, is not bending moments. It means that in the equations of equilibrium of the bending moments there is contribution of the shear forces and the twisting moments. And consequently, although those equations will be completely satisfied, the final moments computed by derivation of (7) will be not in equilibrium. The bending moments at the nodes computed from two adjacent elements will be different because of that additional contribution of the shear forces and twisting moments.

Another reason for such results of the boundary moments is the neglection of the contribution of the distributed load. That is the contribution of the last, area integral of Eq. 6, which in the derivation of the equations (element matrix) is taken into account, but neglected in the computation of the final moments.

The main portion of the element stiffness K° in conection with the rotations is computed from the first integral of (6), as a contribution of the bending moments, and from the remaining two integrals additional stiffness ΔK_2 is computed. The characteristics of ΔK_2 are similar to those of ΔK_1 i.e. —$\Sigma \Delta K_{2ij} = 0$ in every row. Therefore the element with stiffness $K^\circ + \Delta K_2$ finaly gives converging results. But that is not allways the case and sometimes the results could be unreliable.

As an ilustrative example let us give a portion of ΔK_2 which is derived by variation of θ_{3x} (Fig. 1). A portion of ΔK_2 which is multiplied by the nodal displacements w_1+w_4 is,

$$\Delta K_2 = \frac{D}{b}\left[\frac{1-\nu}{10},\ -\frac{1+4\nu}{10},\ \frac{1+4\nu}{10},\ -\frac{1-\nu}{10}\right] \tag{8}$$

This portion of the stiffness is derived from the second and the third integrals of Eq. 6. The corresponding main portion of the stiffness derived from the first integral of (6) is,

$$K^0 = \frac{Db}{a^2}\left[-\frac{1}{2},\ \frac{1}{2},\ 1, -1\right] \tag{9}$$

As can be seen, $\Sigma\Delta K_{2ij}=0$. The same holds for the portion of ΔK_2 which corresponds to the rotations θ_{xi} and rotationa θ_{yi} separately.

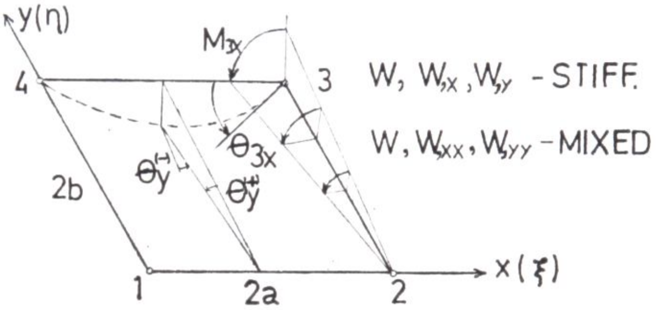

Fig. 1 Plate beuding element. stiffness oı mixed, with applied $\theta_{3x}=1$ or $M_{3x}=1$.

The additional stiffness ΔK_2 is a result of the so called "parasitic stresses". For instance in the case of the mixed element like that on Fig. 1 those stresses are the twisting moments $M_{xy}=f(M_{xi}, M_{yi})$, which are function of the nodal bending moments. The real twisting moments are function of the nodal displacements.

Let us now consider an element like that on Fig. 1, but with compatable shape function [6]. For instance the component of that function due to the rotation θ_{3x} is,

$$w=(1-\xi)(3-2\eta)\,\xi^2\eta^2\,a\,\theta_{3x} \tag{10}$$

$$\xi = x/2\,a,\quad \eta = y/2\,b$$

This function has second derivatives (moment) in x and y directions and mixed derivativ (twisting moment). In the application of the minimum potential energy variational principle all those moments have their own contribution, giving the following stiffness,

$$K_{33x} = \left[\frac{52}{35}\frac{b}{a} + \frac{4}{35}\frac{a}{b} + \frac{8}{25}\frac{a}{b}\right]D \tag{11}$$

The first portion of this stiffness is due to the moments M_x, the second due to M_y and the third due to M_{xy}. However, by variation of θ_{3x} one should get the equivalent nodal moment M_{3x}, in direction of θ_{3x}. Therefore that stiffness should be defined by the first portion of Exp. 11 only. The second and the third portions are due to the "parasitic stresses". It is interesting to note that in the computation of the final moments the moments M_y of (10) will give contribution to the moments M_y, not to the moments M_x like in the equations (element matrix), as was found out.

The same displacement component of the uncompatible element, with DShF (7) is as follows,

$$w=(1-\xi)\,\xi^2\,\eta\, a\,\theta_{3x} \tag{12}$$

The corresponding stiffness derived by application of the minimum potential energy variational principle is,

$$K_{33x}=\left[\frac{4}{3}\,\frac{b}{a}+\frac{4}{15}(1-\nu)\,\frac{a}{b}\right]=K^0+\Delta\,K_2 \tag{13}$$

The first portion of this stiffness is the contribution of the moments M_x and the second portion is due to the moments M_{xy}, which make the additional stiffness ΔK_2.

Because the DShF (10) does not satisfy the homogenious differential equation of the problem (3), the stiffness derived out of such function, which was given by Exp. 11, contains the additional stiffness ΔK_1 as well as the additional stiffness ΔK_2. It means that,

$$K_{33x}=K^0+\Delta\,K_1+\Delta\,K_2 \tag{14}$$

As one can see, the additional stiffness of the parasitic stresses in the case of compatable element (or high order element), —$\Delta K_1+\Delta K_2$ is bigger than the additional stiffness ΔK_2 of the uncompatable element (13). It means that the influence of the "parasitic stresses" in the case of "refined" elements, — elements with high order DShF, is more pronounced than in the case of elements with low order DShF. That is the main reason why the "refined" elements do not give satisfactory results

It is interesting to note what hapens when the shape of the element is irregular, for instance when $b<<a$. In that case the basic stiffness component K° of the both stiffnesses —(11) and (13) (the first portions) are decreased, while the additional stiffness portions are increased. For instance in the case of $a=2b$ the second portion of (11) (due to the parasitic moment M_y) becomes bigger than the first, main portion. Such influence of the parasitic stresses explains why the elements of iregular shapes give bad results.

Something similar hapens in the case of the plane stress problem [5]. Fort instance the element of Fig. 2 has the following DShF of the displacement com-s ponent u,

$$u=a_1+a_2\,x+a_3\,y+a_4\,x^2+a_5\,x\,y+a_6\,x^2\,y$$

This function was defined by the 4 corner strains (stresses) and the 2 displacemen components at the midsides (Fig. 2). The displacement u due to the normal stres N_{1x} is,

$$u=\frac{a}{4E}\,N_{1x}\left(x-\frac{x^2}{2\,a}\right)(1-y/b)$$

The displacements defined in that way have mixed derivatives (shear forces) N_{xy}, which are function of the normal forces N_x, N_y,

$$N_{xy}=G\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)=f(N_{xi}, N_{yi}) \tag{15}$$

In that way defined displacements and shear forces due to $N_{1x}=1$ are presented on Fig. 2. The corresponding equivalent nodal displacements should give the flexibility coefficients of the element matrix. For instance the flexibility in direction of the N_{1x} due to N_{1x} itself, is the corresponding area of the displacements along the side 1—4,

$$f_{11}=\frac{3\,a}{4\,E}\left(2\,b\,\frac{1}{3}\right)=\frac{a\,b}{2\,E}$$

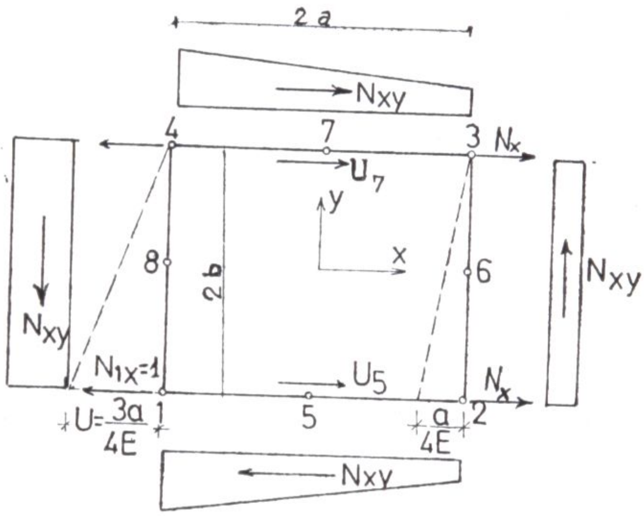

Fig. 2 Plane stress element, displacements and shear forces due to $N_{1x}=1$

However, if the energy approach is applied, the work of the shear forces on the applied displacements should be taken into account. That work gives an additional flexibility,

$$\Delta F_2=\int_s N_{xy}\,u\,ds \tag{16}$$

The physical meaning of this contribution is not clear. This contribution of the shear forces has meaning of additional nodal displacement. But apparently there is not any real additional displacement. Therefore those shear forces should be treated as "parasitic" and their contribution exluded. The real shear forces have to be expressed in terms of the nodal displacements only. In the development of mix ed element of Fig. 2 by application of the direct method of development those stresses were automatically eliminated [5]. The element developed in that way gives exelent results. Even one element gives the exact beam solution of the well known cantilever problem.

In the current practice the influence of the parasitic stresses is taken into account by selective, – reduced integration. The contribution of the twisting moments in (11), (13) and the shear forces of (15) is computed by one point Gauss integration at the middle of the element. At that point the twisting moments of (10) are equal to zero and in that way their influence is eliminated. However, the twisting moment of (12) at that point is not equal to zero consequently the influence of those parasitic moments is not completely eliminated.

Those are some of the disadvantages in the application of the energy variational principles, which make a lot of troubles in the FEM. All, those disadvantages can be overcomed by application of the direct method of development of finite elements [3]. For the completeness of the paper, that method will be described in the following chapter in short. More details on the direct method are given in [3].

3. The direct method

The start of the direct method is the same as in the standard FEM. The displacement shape function of the element, for instance as is (7), is defined by the nodal parameters, — nodal displacements w_i and nodal bending moments M_{xi}, M_{yi} (mixed element). From the DShF defined in that way, for instance for $x=0$ (Fig. 1) the rotations along the side 1—4 can be computed directly.

$$\theta_{14}=\frac{\partial w}{\partial x}_{(x=0)}=f(w_i,\ M_{xi},\ M_{yi}) \tag{17}$$

The equivalent nodal rotations can be computed by consideration of the side 1—4 as a simply supported beam (equivalent to mixed beam element) loaded with a distributed load, for which the reactions have to be found. For instance the equivalent nodal rotation of the node (1) computed in that way is,

$$\theta_{1x}=\int_0^{2b}\theta_{14}(1-y/b)\,dy \tag{18}$$

The function (1—y/b) in this expression can be understood as representing transverse distribution of the moments due to $M_{1x}=1$, along the side 1—4. This expression can be developed as a work of the boundary moments on the corresponding boundary rotations. For instance the boundary moments due to M_{1x} are,

$$M_{14x}=M_{1x}(1-y/b)$$

Their work on the boundary rotations θ_{14} is,

$$U=\int_{(1)}^{(4)}\theta_{14}\,M_{1x}(1-y/b)\,dy$$

The variation of this work δu on the nodal moment M_{1x} gives the equivalent nodal rotation θ_{1x} as defined by Exp. 18. That expression actually defines one row of the element flexibility matrix.

As was shown, directly from the DShF, without application of any variation, all nodal rotations and nodal forces can be computed and in that way the element matrix developed. However, the application of the variational approach in the

FEM has deep roots. Therefore it seems advisible to present the direct method in a form of a variational approach. As was mentioned, the equivalent nodal values can be obtained starting from the work of the boundary forces F_s on the corresponding boundary deformations w_s. That work and the work of the external forces represent total potential energy or functional,

$$\Pi_t = \int_S F_s\, w_s\, ds + \int\int_F p\, w\, dx\, dy$$

The boundary forces and displacements can be defined from the DShF and expressed in terms of the nodal parameters d_i as follows,

$$\boldsymbol{F}_s = \boldsymbol{F}'\, \boldsymbol{d}$$

$$\boldsymbol{w}_s = \boldsymbol{\Phi}_s\, \boldsymbol{d}$$

By substitution of these values into the previous equation the following functional is developed,

$$\Pi_p = d\,(\int_s \Phi_s^t F'\, ds)\, d + (\int\int_F p\, \Phi\, ds)\, d \tag{19}$$

The variation of this functional on the nodal parametars d_i leads to the necessery equations. The term in the perenthesis of the first integral gives the element matrix,

$$K_e = \int_S \Phi_s^t F'\, ds \tag{20}$$

The functional (19) is similar to the functional (6). Both of them contain line integrals only, from which the element matrix is developed. However, the meaning of the partiqular terms of the both functionals is different. Here the functionals of the plate bending problem will be given, from which those differences clearly can be seen. The variation of the functionals of the stiffness and mixed elements of Fig. 1 can be defined as follows,

$$\delta\Pi_s = \int_s \delta\theta_s(\theta_i)\, M_s\, ds + \int_s \delta\, w_s(w_i)\, V_s\, ds \tag{21}$$

$$\delta\Pi_m = \int_s \delta\, M_s(M_i)\frac{\partial w}{\partial n}\, ds + \int_s \delta\, w_s(w_i)\, V_s\, ds \tag{22}$$

where: $\delta\,\theta_s(\theta_i)$ — is variation of the boundary rotations on the nodal rotations θ_i, M_s — boundary bending moments, $\delta\, w_s(w_i)$ — variation of the boundary displacements on the nodal displacements w_i, V_s — boundary shear forces, which contain the contribution of the twisting moments also, $\delta\, M_s(M_i)$ — variation of the boundary moments on the nodal moments M_i, $\dfrac{\partial w}{\partial n}$ — boundary rotations.

The matrix of the mixed element of Fig. 1 can be represented in the following way,

$$F_k = \begin{bmatrix} F_m & F_w \\ F_w^t & K \end{bmatrix} \tag{23}$$

The first row submatrices represent flexibility portion of the element matrix and is defined by the first integral of (22). The submatrix F_m derived in that way contains a portion ΔF_{m1}, which takes into account the uncompatibility of the inter-element slopes that the DShF (7) has. Actually the element matrix gives the real slopes and the meaning of the system equations is continuity of the slopes, like in the three moment rool of the one dimensional problem. However, the coefficients of that submatrix are such that $f_{ij} \neq f_{ji}$, i.e. the Betti-Maxwell rule does not hold completely. The simple explanation of that very unusual finding should be searched in the fact that the DShF represents an approximate solution only, may be much different from the exact solution. By application of the energy variational principles the off diagonal terms of the element matrix are derived as an average of f_{ij} and f_{ji}, and consequently allways are simmetric. However, as a result of such averaging, when the elements in the system are of irregular shapes, the final results are bad.

As an ilustration of that what was said here will be given a portion of the additional submatrix ΔF_{m1} of the mixed element of Fig. 1, in direction of M_{3x}, due to the moments M_{yi},

$$\Delta F_{m1} = \frac{b^3}{360\, a\, D(1-\nu^2)} [-8, 8, 7, -7]$$

The additional contribution of the moment M_{3x} on the equivalent nodal rotations in directions of the nodal moments M_{yi}, which should be transposed ΔF_{m1}, is

$$\Delta F^t_{m1} = \frac{a^3}{360\, \mathrm{b}\, D(1-\nu^2)} [-8, -7, 7, 8]$$

These values are due to the rotations θ_y of Fig. 1. Those rotations make the element uncompatible, but by introduction of ΔF_{m1} the uncompatibility is taken into account and the element behaves like a compatible one. However, as can be seen, ΔF_{m_1} is not transposed ΔF^t_{m1}. Even the signs of some of the corresponding terms are different. In addition, one of them is proportional to b^3/a and the other one is proportional to a^3/b. So ,when a and b are much different f_{ij} and f_{ji} are very much different. The DShF in this case, due to M_{3x}, Fig. 1, gives streight line transverse distribution, However, the exact solution of a plate supported on 4 corners, which corresponds to this case, subjected to a moment M_{3x}, is quite different from the assumed DShF. It seems that could be the explanation of the unsimmetry $f_{ij} \neq f_{ji}$.

The submatrix F_m derived by application of the direct method does not contain the contribution of the parasitic stresses (equivalent to ΔK_1 and ΔK_2 of the stiffness element). The second row of the element matrix (23) represents the stiffness portion and is derived from the second integral of (22). As can be seen, that integral gives shear (normal) forces only, while the first integral gives rotations only. Similar is the case with the Exp. 21, for the stiffness element. The first integral, which defines the equations of equilibrium of the moments, contains moments only. In those equations there will be no contribution of the shear forces and twisting moments. Contrary to that, as was mentioned, the functional (6) in the equations of equlibrium of the moments introduces contribution of the shear forces and twisting moments.

By application of the direct method it has been shown that the independent assumption of the deformations and the stresses in the element, frequently used in the FEM, is quite correct. For instance the displacements and the moments of the element of Fig. 1 can be assumed as follows.

$$w = \boldsymbol{\Phi}\, \mathbf{w}_i$$

$$M = \boldsymbol{\Phi}\, \mathbf{M}_i$$

$$\boldsymbol{\Phi} = \left[1 - \frac{x}{2a} - \frac{y}{2b} + \frac{xy}{4ab}, \frac{x}{2a} - \frac{xy}{4ab}, \frac{xy}{4ab}, \ 1 - \frac{x}{2a} + \frac{y}{2b} - \frac{xy}{4ab} \right]$$

$$\mathbf{w}_i^t = [w_1 \cdots w_4]$$

$$\mathbf{M}_i^t = [M_1 \cdots M_4]$$

where M_i can be M_{xi} and M_{yi}. Such assumptions have been made in the development of the early mixed and hybried elements. The elements developed on the base of such assumption give quite good results. The parasitic twisting moments $M_{xy} = f(M_{xi}, M_{yi})$, which give the additional element matrix ΔF_{m2} (ΔK_2) are eliminated. That is the main reason for the good results of those elements. However, the interelement slope discontinuity, which results in the additional submatrix ΔF_{m1} in this case is not taken into account. As a result of that, the final results which those elements give, oscilate around the exact solution.

4. Conclusions

In the application of the energy variational principles in the finite element method there are some approximations, which can be summerized as follows:

When the DShF is a polinomial of high order there is an additional stiffnes (ΔK_1), i.e. there are additional nodal forces, which are not result of the boundary forces. Consequently the final forces at the interelement nodes are not in equilibrium.

The boundary forces into the equations of equilibrium are computed in one way (energetic) and the final boundary forces in another way (by differentiation). That is another reason for the nodal forces to be not in equilibrium. Besides the contribution of the corresponding boundary forces to the nodal forces, there is a contribution of components whose physical meaning is something else. For instance in the equations of equilibrium of the bending moments there is contribution of the shear forces and twisting moments. Those forces which are called "parasitic", give an additionals tiffness (ΔK_2), as a result of which the accuracy and the reliabillity of the final results can be questionable. That is partiqulary true in the case of elements of irregular shapes.

By selection of the DShF as a polinomial which satisfies the homogenious differential equation of the problem the parasitic stiffne ΔK_1 is eliminated. By application of the direct method the additional stiffness ΔK_2 is eliminated also, and the uncompatibility taken into account. In that way developed elements give improved and reliable results.

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НЕКОТОРИЕ ЗАМЕЧАНИЯ ПРИМЕНЕНИЮ ЭНЕРГЕТИЧЕСКИХ ВАРИЯЦИОННЫХ ПРИНЦИПОВ В МЕТОД КОНЕЧНЫХ ЭЛЕМЕНТОВ

Рассматриваются недостатки энергетических вариационных принципов в методе конечных элементов. В случае когда функция дистрибуции высшего порядка появляются дополнительные узловые усилия, которые не являются последствием пересекающих сил. (ΔK_1). При энергетическом подходе входят и величины с различным физическим значением (ΔK_2 – „паразитные напряжения“). Эти влияния являются главной причиной того, что элементы высшего порядка или неправильной формы дают неудовлетворительные результаты. Все эти недостатки успешно отклоняются применением прямого метода развития конечных элементов.

NEKE PRIMEDBE NA PRIMENU ENERGETSKIH VARIJACIONIH PRINCIPA U METODU KONAČNIH ELEMENATA

Razmatraju se neki nedostaci energetskih varijacionih principa kod primene u metodu konačnih elemenata. Kada je interpolaciona funkcija polinom višeg reda javljaju se dopunske čvorne sile koje nisu posledica presečnih sila (ΔK_1). Kod energetskog pristupa u matrici elemenata ulaze i veličine čije fizičko značenje je različito (ΔK_2 — "parazitska naprezanja"). Ti uticaji su glavni razlog za to što elementi višeg reda ili nepravilne forme daju nezadovoljavajuće rezultate. Svi ti nedostaci uspešno se savlađuju primenom direktnog metoda razvijanja konačnih elemenata, koji se ukratko iznosi.

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