UDK 531

ONE EXAMPLE OF EINSTEIN SPACES

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In this paper we propose a metric, besed on Gödel's cosmological model, which could be one example of Einstein spaces. The cosmological model of K. Gödel was published in 1949. [1]. That model is stationary, axisymmetric, and with a cosmological constant. Cosmological constant was introduced by Einstein in 1917. [2], to explain some global properties of physical space treated as a Rimannian space. Later investigation revealed that cosmological contant does not give an adequate explanation of some local properties of space.

In Einstein spaces the Ricci tensor is proprotional to the metric tensor [3].

We propose a metric of the following form

$$ds^2 = a^2 \left\{ \nu (dx^1)^2 + \frac{f^2}{2} (dx^2)^2 + \nu (dx^3)^2 - (dx^4 + f dx^2)^2 \right\} ,$$

$$a = \text{const}, \quad \nu = \nu(x^1), \quad f = f(x^4)$$

The metric tensor is represented by two functions: ν, depending on x^1 only, and f depending on time, x^4, only. First we shall calculate Christoffel symbols of the second kind and write down only those which are different from zero. The prime denotes the derivative of ν with respect to x^1 and a dot denotes the derivative of f with respect to x^4.

$$\left\{ {1 \atop 11} \right\} = \frac{1}{2} \frac{\nu'}{\nu}$$

$$\left\{ {2 \atop 22} \right\} = -\dot{f} \qquad \left\{ {3 \atop 13} \right\} = \frac{1}{2} \frac{\nu'}{\nu} \qquad \left\{ {4 \atop 22} \right\} = \frac{1}{2} f \dot{f}$$

$$\left\{ {2 \atop 24} \right\} = -\frac{\dot{f}}{f} \qquad \left\{ {4 \atop 24} \right\} = \dot{f}$$

$$\left\{ {1 \atop 33} \right\} = -\frac{1}{2} \frac{\nu'}{\nu}$$

$$\left\{ {2 \atop 44} \right\} = -2 \frac{\dot{f}}{f^2} \qquad \left\{ {4 \atop 44} \right\} = 2 \frac{\dot{f}}{f}$$

Next, we calculate a Ricci tensor. The result can be presented as the following matrix

$$R_{\alpha\beta} = \begin{Bmatrix} \frac{1}{2}\frac{\nu''}{\nu} - \frac{1}{2}\frac{\nu'^2}{\nu^2} & 0 & 0 & 0 \\ 0 & -\frac{1}{2} f \ddot{f} & 0 & -\ddot{f} \\ 0 & 0 & \frac{1}{2}\frac{\nu''}{\nu} - \frac{1}{2}\frac{\nu'^2}{\nu^2} & 0 \\ 0 & -\ddot{f} & & -\frac{\ddot{f}}{f} \end{Bmatrix}$$

The form of the Ricci tensor allows us to put the condition

$$R_{\alpha\beta} = \mathcal{H}\, g_{\alpha\beta}, \qquad \mathcal{H} = \text{const}$$

If there are solutions ν and f, satisfying this condition, the scalar curvature will be constant, as in [1].

The equations obtained from the conditions

$$R_{11} = \mathcal{H}\, g_{11}; \qquad R_{33} = \mathcal{H}\, g_{33}$$

are identical, and the equations obtained from the conditions

$$R_{22} = \mathcal{H}\, g_{22}; \qquad R_{24} = \mathcal{H}\, g_{24}\,; \qquad R_{44} = \mathcal{H}\, g_{44}$$

are equivalent. In this way we get the following system of two equations for determining ν and f, each of them containing only one unknown function

$$\frac{1}{2}\frac{\nu''}{\nu} - \frac{1}{2}\frac{\nu'^2}{\nu^2} = a^2 \mathcal{H}\, \nu$$

$$\frac{\ddot{f}}{f} = a^2 \mathcal{H}$$

The solutions of function f depend on the sign of $\mathcal{H}$:

$$\mathcal{H} > 0 \qquad\qquad \mathcal{H} < 0$$

$$f = c_1\, e^{a\sqrt{\mathcal{H}}(x^4 + D)} + c_2\, e^{-a\sqrt{\mathcal{H}}(x^4 + D)}; \quad f = M_1 \cos\omega\,(x^4 + D) + M_2 \sin\omega\,(x^4 + D)$$

The corresponding solutions of ν are of the form

$$\nu = \frac{D}{a^2 \mathcal{H}} \frac{e^{\sqrt{D}(x^1 - E)}}{[1 - e^{\sqrt{D}(x^1 - E)}]^2} \quad , \quad D > 0$$

$$\nu = \frac{1}{a^2 \mathcal{H}} \frac{1}{(x^1 - n^2)^2} \quad , \quad D = 0 \qquad \nu = \frac{D}{a^2 k^2} \frac{e^{\sqrt{D(x^1 + E)}}}{[1 + e^{\sqrt{D}(x^1 + E)}]^2} , \quad \begin{matrix} D > 0 \\ \mathcal{H} = -k^2 \end{matrix}$$

$$\nu = \frac{\omega^2}{a^2 \mathcal{H}} \frac{1}{\cos^2 \omega (x^1 - E)} , \quad D = -4\omega^2 < 0$$

We consider only positive solutions of ν, to preserve the correct signature.

This possibility enables us to interpret proposed metric, with ν and f we got, as one example of Einstein spaces.

Though we have no physical interpretation of this metric we shall present free–fall motion toward a singularity. We shall follow an example presented by Dirac in [4].

We take ν and f in this example to be

$$\nu = \frac{1}{a^2 \mathcal{H}} \frac{1}{(x^1 - n^2)^2} , \quad f = e^{a\sqrt{\mathcal{H}}(x^4 + D)}$$

First we solve the equations of geodesics:

$$\frac{d^2 x^1}{ds^2} + \frac{1}{2} \frac{\nu'}{\nu} \left(\frac{dx^1}{ds}\right)^2 - \frac{1}{2} \frac{\nu'}{\nu} \left(\frac{dx^3}{ds}\right)^2 = 0$$

$$\frac{d^2 x^2}{ds^2} - \dot{f} \left(\frac{dx^2}{ds}\right)^2 - 2 \frac{\dot{f}}{f} \frac{dx^2}{ds} \frac{dx^4}{ds} - 2 \frac{\dot{f}}{f^2} \left(\frac{dx^4}{ds}\right)^2 = 0$$

$$\frac{d^2 x^3}{ds^2} + \frac{\nu'}{\nu} \frac{dx^1}{ds} \frac{dx^3}{ds} = 0$$

$$\frac{d^2 x^4}{ds^2} + \frac{1}{2} f \dot{f} \left(\frac{dx^2}{ds}\right)^2 + 2 \dot{f} \frac{dx^2}{ds} \frac{dx^4}{ds} + 2 \frac{\dot{f}}{f} \left(\frac{dx^2}{ds}\right)^2 = 0$$

From the first and third equation we find

$$\frac{dx^1}{ds} = A (x^1 - n^2) \quad , \quad A = \text{const}$$

$$s = \frac{1}{A} \int_H^{n^2 + \epsilon} \frac{dx^1}{x^1 - n^2} = +\infty , \qquad \begin{matrix} H = \text{const} \\ H > n^2 \end{matrix}$$

From the second and fourth equation we find

$$\frac{dx^4}{ds} = \frac{1}{\sqrt{a\sqrt{\mathcal{H}}}} \sqrt{M - e^{-2a\sqrt{\mathcal{H}}(x^4+D)}}\ , \quad M = \text{const}$$

Following the example presented by Dirac

$$\frac{\dfrac{dx^4}{ds}}{\dfrac{dx^1}{ds}} = \frac{dx^4}{dx^1} = \frac{\dfrac{1}{\sqrt{a\sqrt{\mathcal{H}}}}\sqrt{M - e^{-2a\sqrt{\mathcal{H}}(x^4+D)}}}{A\,(x^1 - n^2)}$$

$$x^4 = -D - \frac{1}{2a\sqrt{\mathcal{H}}}\, ln \left.\frac{4\,BM(x^1 - n^2)}{[1 - B\,(x^1 - n^2)]^2}\right|_{n^2+\epsilon}^{H} = +\infty$$

In this example the proper time, s, and the coordinate time, x^4, are both infinite.

REFERENCES

[1] K. Godel: An example of a new type of cosmological solution of Einstein field equation of gravitation, Rev. Mod. Phys. 21, 447, (1949)

[2] A. Einstein: Cosmological considerations on the general theory of relativity, Dover, 1952.

[3] A. Z. Petrov: Einstein spaces, Pergamon Press, (1969)

[4] P. A. M. Dirac: General Theory of Relativity, John Wiley and Sons, 1975.

UNE EXAMPLE DES ESPACES D'EINSTEIN

Dans ce papier on propose une métrique fondée sur le modéle cosmologique de Gödel. Le tenseur métrique est representé par deux fonctions, ν et f, dependant de x^1 et x^4 respectivement. Deux fonctions de la métrique proposee, sont determinées de la condition que le tenseur de Ricci est proportionel a le tenseur métrique. En cette maniére on obtient une example des espaces d'Einstein.

JEDAN PRIMER AJNŠTAJNOVIH PROSTORA

U ovom radu prikazan je pokušaj da se konstruiše metrika koja spada u klasu Ajnštajnovih prostora, a bazirana je na nekim idejama vezanim za Gedelov kosmološki model. Metrički tenzor izražen je pomoću dve funkcije od kojih jedna zavisi samo od jedne prostorne koordinate a druga samo od vremena. Dve funkcije koje se pojavljuju u predloženoj metrici određene su tako da Ričijev tenzor krivine bude proporcionalan metričkom tenzoru. Na taj način dobijen je jedan primer Ajnštajnovih prostora.

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