UDK 539

THERMODINAMICS OF INTERSTITIAL WORKING OF MICROPOLAR CONTINUA

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1. Introduction

In order to model fluid capillarity effects Korteweg [1] formulated a constitutive equation for the Cauchy stress that included density gradients. Specifically, he proposed for study a compressible fluid model in which the "elastic" or "equilibrium" portion of the Cauchy stress tensor **T** is given by

$$\mathbf{T} = \mathbf{T}\,(\rho, \theta, grad\ \rho, grad^2\ \rho)$$

$$= (-p + \alpha\Delta\rho + \beta/grad\ \rho/^2)\,\mathbf{I} + \delta\ grad\ \rho \otimes grad\ \rho + \gamma\ grad^2\ \rho \tag{1.1}$$

where $\rho = \rho\,(\mathbf{x}, t)$ is the density of the fluid at the place **x** at time t, where p, α, β, δ and γ are material functions of ρ and temperature θ, where $grad\ \rho$ and $grad^2\ \rho$ are, respectively, the first and the second (special) gradients of ρ with respect to **x**, with $\Delta\rho = tr\,(grad^2\ \rho)$.

To model viscous effects in the dynamic response of his fluids Korteweg added to the right-hand side of (1.1) the classical form of Cauchy and Poisson, i.e., $\lambda\,(tr\ \mathbf{D})\,\mathbf{I} + 2\,\mu\,\mathbf{D}$, where **D** is the usual stretching tensor of hydrodynamics, and where λ and μ, the usual viscosity coefficients, may depend on ρ and θ.

In the terminology of continuum mechanics, Korteweg form (1.1) is a special example of an elastic material of grade N, in which the constitutive equations are permitted to depend not only on the first gradient of deformation but also on all gradients of deformation less then or equal to the integer N. A troubling aspect of all these higher-grade models, however, is that they are in general incompatible with the usual continuum theory of thermodynamics. Indeed Kotreweg's model (1.1) is incompatible with conventional thermodynamics unless all the nonclassical coefficients, α, β, δ and γ vanish identically.

What is required then is a new, broader thermodynamics structure that admits nontrivial Korteweg type materials and, more generally, materials of arbitrary grade. To this end Dunn and Serrin [2] modified the energy balance following the idea which was suggested by Toupin [3] and in same way by Ericksen [4]. Specifically, for each process π they postulated the existence of a rate of supply of mechanical energy, the interstitial working $u = u\,(X, t, \mathbf{n})$ defined for all $(X, t) \in B \times R$ and unit vectors **n**, such that the balance of energy for each subdomain P with its boundary

$$\frac{d}{dt}\int_{P_t} \rho\,(\epsilon + \frac{1}{2}\,\mathbf{v}\,\mathbf{v})\,dv = \int_{\partial P_t} (\mathbf{v}\,\mathbf{T}\,\mathbf{n} - \mathbf{q}\,\mathbf{n})\,da + \int_{P_t} \rho\,(\mathbf{v}\,\mathbf{f} + h)\,dv \tag{1.2}$$

is replaced by

$$\frac{d}{dt}\int_{P_t} \rho\,(\epsilon + \frac{1}{2}\,\mathbf{v}\,\mathbf{v})\,dv = \int_{\partial P_t} (\mathbf{T}\,\mathbf{n}\,\mathbf{v} + u - \mathbf{q}\,\mathbf{n})\,da + \int_{P_t} \rho\,(\mathbf{f}\,\mathbf{v} + h)\,dv \tag{1.3}$$

where during the process $\pi = \{\mathbf{x}, \theta, \epsilon, \eta, \mathbf{T}, \mathbf{q}, \mathbf{f}, h\}$ at the particle X and time t,

(i) $\mathbf{x} = \mathbf{x}\,(X, t)$ is the motion,
(ii) $\theta = \theta\,(X, t)\;(> 0)$ is the absolute temperature,
(iii) $\epsilon = \epsilon\,(X, t)$ is the specific internal energy per unit mass,
(iv) $\eta = \eta\,(X, t)$ is specific entropy per unit mass,
(v) $\mathbf{T} = \mathbf{T}\,(X, t)$ is the Cauchy stress tensor,
(vi) $\mathbf{q} = \mathbf{q}\,(X, t)$ is the heat flux vector,
(vii) $\mathbf{f} = \mathbf{f}\,(X, t)$ is the specific body force per unit mass,
(viii) $h = h\,(X, t)$ is the radiant heating per unit mas

and where

$\mathbf{n} = \mathbf{n}\,(\mathbf{x}, t)$ is the outer unit normal to ∂P_t,

$\mathbf{v} = \dfrac{\partial}{\partial t}\,\mathbf{x}\,(X, t)$ is the velocity,

$\rho = \rho\,(X, t)$ is the mass density.

B is fixed reference configuration

($B \subseteq E$, where E is a three-dimensional Euclidian space).

In this paper we try to apply the idea of Dunn and Serrin to more general model of continuous media, namely to the model of micropolar continua.

2. Basic formulae and equations

Troughout this paper we use a fixed rectangular Cartesian system of axes. The deformation of the body B is characterized by two fields: the deformation gradient $\mathbf{F} = [x_{k,K}]$ and a rotation tensor $\chi = [\chi_{kK}]$, where X_k and x_k are material and spatial co-ordinates. Suffices range over the values 1, 2, 3 and the usual summation convention is applied, to all indices unless an statement to the contraty is made.

The equation of balance for a micropolar body (C. B. Kafadar and A. C. Eringen [5]) asserts.

Balance of momentum

$$\frac{d}{dt}\int_{P_t} \rho\,\mathbf{v}\,dv = \int_{\partial P_t} \mathbf{t}_{(n)}\,da + \int_{P_t} \rho\,\mathbf{f}\,dv \tag{2.1}$$

where $\mathbf{t}_{(n)} = \mathbf{n}\mathbf{T}$ is stress vector,

Balance of moment of momentum

$$\frac{d}{dt} \int_{P_t} (\mathbf{x} \times \rho \mathbf{v} + \rho \sigma)\, dv = \int_{\partial P_t} (\mathbf{x} \times \mathbf{t}_{(n)} + M_{(n)})\, da + \int_{P_t} (\mathbf{x} \times \rho \mathbf{f} + \rho \mathbf{l})\, dv \tag{2.2}$$

where

$M_{(n)} = m_{kl} n_k \mathbf{g}_e$, $M = \{m_{kl}\}$ is the couple stress density,

$\mathbf{g}_k$ are the spatial base vectors,

$\rho\,\sigma$ is the spin density,

$\rho\, l$ is the external body couple density,

Balance of energy

$$\frac{d}{dt} \int_{P_t} \rho \left(\epsilon + \frac{1}{2} \mathbf{v}\mathbf{v} + \sigma \mathbf{v}\right) dv = \int_{\partial P_t} (\mathbf{t}_{(n)}\, \nu + M_{(n)}\, \nu - \mathbf{q}\,\mathbf{n} + u)\, da + \int_{P_t} \rho\, (\mathbf{f}\,\mathbf{v} + l\, \nu + \mathrm{h})\, dv \tag{2.3}$$

where the angular velocity vector ν (microgyration) is given by a second order skew–summetric tensor ν_{kl} by the formula

$$\nu_k = -\frac{1}{2} \epsilon_{klm} \nu_{lm}, \quad \nu_{kl} = -\epsilon_{klm} \nu_m, \tag{2.4}$$

where ϵ_{klm} is Ricci tensor of alternation.

The gyration tensor ν_{kl} is associated with χ, so that

$$\nu_{kl} = -\nu_{lk} = \dot{\chi}_{kK} \chi_{lK} \tag{2.5}$$

The energy balance law (2.4) is modified by the term u, which represents a rate of supply of mechanical energy across every material surface in B, on the base of the spatial interactions of longer range. In fact $u = u\ (X, t, \mathbf{n})$ cannot really depend on an arbitrary fashion: an analogue of a standard theorem due to Caushy tells us that the balance law (2.3) can hold for all subdomains $P \subset B$ if and only if $u\ (X, t, \mathbf{n})$ is linear in $\mathbf{n}$, i.e. there must exist a vector field $\mathbf{u} = \mathbf{u}\ (X, t)$ such that

$$u\ (X, t, \mathbf{n}) = \mathbf{u}\ (X, t)\, \mathbf{n} \tag{2.6}$$

for every unit vector **n**. If is in terms of this interstitial work flux **u**, rather then the scalar density u, that we shall consider our theory. Further, we shall require u to be objective under a frame change, i.e.

$$\overset{x}{u}(X, \overset{x}{t}, \overset{x}{\mathbf{n}}) = u(X, t, \mathbf{n}), \tag{2.7}$$

which imply that u is objective, i.e.

$$\overset{x}{\mathbf{u}}(X, \overset{x}{t}) = \mathbf{Q}(t)\,\mathbf{u}(X, t). \tag{2.8}$$

Entropy inequality

$$\frac{d}{dt}\int_{P_t} \rho\,\eta\,dv + \int_{\partial P_t} \frac{\mathbf{q}\,d\mathbf{a}}{\theta} - \int_{P_t} \frac{\rho\,h}{\theta} \geqslant 0 \tag{2.9}$$

These integral laws lead to the following field equations (Momentum)

$$t_{kl,k} + \rho f_l = \rho \dot{v}_l \tag{2.10}$$

(Moment of momentum)

$$m_{kl,k} + \epsilon_{lmn}\, t_{mn} + \rho\, l_l = \rho\, \dot{\sigma}_l \tag{2.11}$$

(Energy)

$$\rho\,\dot{\epsilon} = t_{kl}\,(v_{l,k} + v_{kl}) + m_{kl}\, v_{l,k} - q_{k,k} + u_{k,k} + g\,h \tag{2.12}$$

(Entropy)

$$\rho\,\dot{\eta} + \left(\frac{q_k}{\theta}\right)_{,k} - \frac{\rho\,h}{\theta} \geqslant 0. \tag{2.13}$$

By eliminating ρh from (2.12) and substituting this into (2.13) one finds

$$\rho\,(\dot{\psi} + \eta\,\dot{\theta}) - t_{kl}\,(v_{l,k} + v_{kl}) - m_{kl}\, v_{l,k} - u_{k,k} + \frac{q_k\,\theta_{,k}}{\theta} \leqslant 0 \tag{2.14}$$

where $\psi = \epsilon - \theta\,\eta$ is the free energy.

If we take into account that

$$t_{kl}\, v_{l,k} = X_{K,l}\, t_{l,k}\, \dot{x}_{k,K} = \mathbf{T}^T\, \mathbf{F}^{T-1}\, \dot{\mathbf{F}}$$

$$t_{kl}\, v_{kl} = t_{kl}\, \dot{x}_{kK}\, \chi_{lk} = \mathbf{T}\, \chi\, \dot{\chi}$$

$$m_{kl}\, v_{l,k} = -\frac{1}{2} m_{kl}\, \epsilon_{lmn}\, (\dot{\chi}_{ml}\, \chi_{nL})_{,k}\, \chi_{K,k} =$$

$$= - m_{kmn}\, (\dot{\chi}_{mL}\, \chi_{nL})_{,k}\, \chi_{K,k} =$$

$$= - \left\{ \mathbf{M} \times \nabla (\dot{\chi}\, \chi^{T}) \right\} \mathbf{F}^{T-1} =$$

$$= - \left\{ \mathbf{M} \times \nabla \dot{\chi}\, \chi^{T} + \mathbf{M} \times \dot{\chi} \nabla \chi^{T} \right\} \mathbf{F}^{T-1} \qquad (*)$$

than (2.14) can be written in more compact form, i.e.

$$\rho\, (\dot{\psi} + \eta\, \dot{\theta}) - \mathbf{T}^{T} F^{T-1}\, \mathbf{F} - \mathbf{T}\, \chi\, \dot{\chi} +$$
$$+ \left\{ \mathbf{M} \times \nabla \dot{\chi}\, \chi^{T} + \mathbf{M} \times \dot{\chi} \nabla\, \chi^{T} \right\} \mathbf{F}^{T-1} - div\, \mathbf{u} + \frac{\mathbf{q}\, \mathbf{g}}{\theta} \leqslant 0, \qquad (2.15)$$

where, by definition,

$$\mathbf{M} = \left\{ m_{kmn} = \frac{1}{2} m_{kl}\, \epsilon_{lmn} \right\},$$

and $\mathbf{g} = grad\, \theta$ is the spatial temperature gradient.

We will refer to (2.14) as the dissipation inequality, and we will study the thermodynamic consequences of (2.10)–(2.14) for the constitutive structure arising from the assumption that $\epsilon, \eta, \mathbf{T}, \mathbf{M}, \mathbf{q}$ and $\mathbf{u}$ are given by smooth functions of the set variables

$$\Lambda = (\mathbf{F}, \nabla \mathbf{F}, \nabla^{2} \mathbf{F}, \chi, \nabla \chi, \nabla^{2} \chi, \nabla^{3} \chi, \theta, \mathbf{g}, \dot{\mathbf{F}}, \dot{\chi}), ^{(**)}$$

i. e.

$$\begin{aligned} \epsilon &= \hat{\epsilon}\,(\Lambda) & \mathbf{M} &= \hat{\mathbf{M}}\,(\Lambda) \\ \eta &= \hat{\eta}\,(\Lambda) & \mathbf{q} &= \hat{\mathbf{q}}\,(\Lambda) \\ \mathbf{T} &= \hat{\mathbf{T}}\,(\Lambda) & \mathbf{u} &= \hat{\mathbf{u}}\,(\Lambda). \end{aligned} \qquad (2.17)$$

of cause, once $\hat{\epsilon}\,(\Lambda)$ and $\hat{\eta}(\Lambda)$ are given, the relation $\psi = \epsilon - \theta\, \eta$ determines a function $\hat{\psi}\,(\Lambda)$ sucht that

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(*) For any two tensors, $\boldsymbol{\Gamma}$ and $\boldsymbol{\phi}$, we define $\boldsymbol{\Gamma} \times \boldsymbol{\phi}$ to be the second order tensor such that (in Cartesian components) $(\boldsymbol{\Gamma} \times \boldsymbol{\phi})_{ij} = \Gamma_{ipq \ldots}\, \phi_{pq \ldots j}$.

(**) Throughout this paper, as in Dunn and Serrin's [2], "∇" will denote differentiation with respect to the material coordinates X_k while "grad" will denote differentiation with respect to the spatial coordinates.

$$\psi = \hat{\psi}(\Lambda). \tag{2.18}$$

3. Thermodynamic compability

Now, we assert that the responce functions (2.17) must be such that, by the chain rule

$$\begin{aligned}
&\rho(\psi_\theta + \eta)\dot{\theta} + (\rho\,\psi_{\mathbf{F}} + \mathbf{T}^T\,\mathbf{F}^{T-1})\dot{\mathbf{F}} + \\
&+ [(\rho\,\psi_\chi - \mathbf{T}\chi)\dot{\chi} + (\mathbf{M}\times\dot{\chi}\,\nabla\,\chi^T)\,\mathbf{F}^{T-1}] + \\
&+ [\rho\,\psi_{\chi\nabla}\,\dot{\chi} + (\mathbf{M}\times\nabla\,\dot{\chi}\chi^T - \mathbf{u}_\chi\times\nabla\,\dot{\chi})\,\mathbf{F}^{T-1}] + \\
&+ [\rho\,\psi_{\mathbf{F}}\,\nabla\,\dot{\mathbf{F}} - (u_{\mathbf{F}}\times\nabla\,\dot{\mathbf{F}})\,\mathbf{F}^{T-1}] + \\
&+\rho\,\psi_{\nabla^2\mathbf{F}}\,\nabla^2\dot{\mathbf{F}} + \rho\,\psi_{\nabla^2\chi}\,\nabla^2\dot{\chi} + \rho\psi_{\nabla^3\chi}\,\nabla^3\dot{\chi} + \rho\psi_{\mathbf{g}}\,\dot{\mathbf{g}} + \rho\psi_{\dot{\chi}}\,\ddot{\chi} + \rho\,\psi_{\dot{\mathbf{F}}}\,\ddot{\mathbf{F}} - \\
&-(\mathbf{u}_{\mathbf{F}}\times\nabla\,\mathbf{F} + \mathbf{u}_{\nabla\mathbf{F}}\times\nabla^2\mathbf{F} + \mathbf{u}_{\nabla^2\mathbf{F}}\times\nabla^3\mathbf{F} + \mathbf{u}_\chi\times\nabla\,\chi + \mathbf{u}_{\nabla\chi}\times\nabla^2\chi + \mathbf{u}_{\nabla^2\chi}\times\nabla^3\chi + \mathbf{u}_{\nabla^3\chi}\times \\
&\times\nabla^4\chi - \mathbf{u}_\theta\,\mathbf{g} - \mathbf{u}_{\mathbf{g}}\,\mathbf{G} + \frac{\mathbf{q}\,\mathbf{g}}{\theta} \leqslant 0,
\end{aligned} \tag{3.1}$$

where $\mathbf{G} = grad^2\ \theta$, must hold for every motion $\mathbf{x}$, every microrotation χ and every temperature field θ. In writing (3.1) we have used the fact that "∇" and "$\cdot$" commute, e.g. $\nabla\dot{\mathbf{F}} = \dot{\overline{\nabla\mathbf{F}}}$, etc.

It follows that (3.1) is essentially linear in twelve quantities

$$\dot{\theta}, \nabla\,\dot{\mathbf{F}}, \nabla\,\dot{\chi}, \nabla^2\dot{\mathbf{F}}, \nabla^2\dot{\chi}, \nabla^3\mathbf{F}, \dot{\mathbf{g}}, \ddot{\chi}, \ddot{\mathbf{F}}, \mathbf{G}, \nabla^3\mathbf{F}, \nabla^4\chi.$$

Therefore, (3.1) imply that

$$\psi_\theta + \eta = 0. \tag{3.3}$$

$$\rho\,\psi_{\nabla\mathbf{F}}\,\nabla\,\dot{\mathbf{F}} = (\mathbf{u}_{\mathbf{F}}\times\nabla\,\dot{\mathbf{F}})\,\mathbf{F}^{T-1}, \tag{3.4}$$

$$\rho\,\psi_{\nabla\chi}\,\nabla\,\dot{\chi} + (\mathbf{M}\times\nabla\,\dot{\chi}\,\chi^T - \mathbf{u}_\chi\times\nabla\,\dot{\chi})\,\mathbf{F}^{T-1} = 0, \tag{3.5}$$

$$\begin{aligned}
\psi_{\nabla^2\mathbf{F}} &= 0 & \psi_{\mathbf{g}} &= 0\\
\psi_{\nabla^2\chi} &= 0 & \psi_{\dot{\chi}} &= 0\\
\psi_{\nabla^3\mathbf{F}} &= 0 & \psi_{\dot{\mathbf{F}}} &= 0,
\end{aligned} \tag{3.6}$$

$$\mathbf{u}_{\mathbf{g}}\ \mathbf{G} = 0 \tag{3.7}$$

$$(\mathbf{u}_{\nabla^2 \mathbf{F}} \times \nabla^3 \mathbf{F})\ \mathbf{F}^{T-1} = 0 \tag{3.8}$$

$$(\mathbf{u}_{\nabla^3 \chi} \times \nabla^4 \chi)\ \mathbf{F}^{T-1} = 0. \tag{3.9}$$

What remains now of the dissipation inequality is the restriction that

$$(\rho\, \psi_{\mathbf{F}} - \mathbf{T}^T\ \mathbf{F}^{T-1})\, \dot{\mathbf{F}} + [(\rho\psi_\chi - \mathbf{T}\chi)\, \dot{\chi} + (\mathbf{M} \times \dot{\chi}_{\nabla} \chi^T)\ \mathbf{F}^{T-1}] -$$

$$- (\mathbf{u}_{\mathbf{F}} \times \nabla \mathbf{F} + \mathbf{u}_{\nabla \mathbf{F}} \times \nabla^2 \mathbf{F} + \mathbf{u}_\chi \times \nabla \chi + \mathbf{u}_{\nabla\chi} \times \nabla^2 \chi + \mathbf{u}_{\nabla^2 \chi} \times \nabla^3 \chi)\, \mathbf{F} - \mathbf{u}_\theta\, \mathbf{g} + \tag{3.10}$$

$$\frac{\mathbf{q}\ \mathbf{g}}{\theta} \leqslant 0$$

which is called the reduced dissipation inequality.

The conditions (3.6) tell us that $\hat{\psi}$ is independent of

$$\nabla^2 \mathbf{F},\ \nabla^2 \chi,\ \nabla^3 \mathbf{F},\ \mathbf{g},\ \dot{\chi},\ \dot{\mathbf{F}},$$

i. e.

$$\psi = \hat{\psi}\, (\mathbf{F},\ \nabla \mathbf{F},\ \chi,\ \nabla \chi,\ \theta). \tag{3.11}$$

In addition (3.3) shows that $\hat{\eta}$ is also independent of the quantities in (3.1) and that the entropy relation

$$\eta = \hat{\eta}\, (\mathbf{F}, \nabla \mathbf{F}, \chi, \nabla \chi, \theta) = - \hat{\psi}_\theta\, (\mathbf{F}, \nabla\ \mathbf{F}, \chi,\ \nabla \chi, \theta) \tag{3.12}$$

holds. Then, from the relation

$$\epsilon = \hat{\epsilon}\, (\mathbf{F}, \nabla \mathbf{F}, \chi, \nabla \chi, \theta) = \hat{\psi}\, (\mathbf{F}, \nabla \mathbf{F}, \chi, \nabla \chi, \theta) - \theta\ \psi_\theta\, (\mathbf{F}, \nabla \mathbf{F}, \chi, \nabla\ \chi, \theta) \tag{3.13}$$

and (3.12) follows that $\hat{\eta}$ and $\hat{\epsilon}$ are completely determined by the response function $\hat{\psi}$ for the free energy.

The conditions (3.4), (3.5), (3.7), (3.8) and (3.9) are harder to analyze. In terms of Cartesian coordinates they are, respectively

$$\rho\, \frac{\partial\, \hat{\psi}}{\partial\, x_{k,KL}}\ x_{l,k}\ x_{j,L} - \frac{\partial\, \hat{u}_l}{\partial\, \dot{x}_{k,K}}\ x_{j,K_{(l,j)}} = 0 \tag{3.14}$$

$$\rho\, \frac{\partial\, \hat{\psi}}{\partial\, \chi_{kK,L}}\ \dot{\chi}_{kK,L} + (m_{lkm}\ \chi_{mk} - \frac{\partial\, \hat{u}_l}{\partial\, \dot{\chi}_{kK}})\, X_{L,l}\ \dot{\chi}_{k\,K,L} = 0 \tag{3.15}$$

$$\frac{\partial \hat{u}_l}{\partial \theta_{,k}} \theta_{,kl} = 0 \tag{3.16}$$

$$\frac{\partial \hat{u}_l}{\partial x_{k,KLM}} X_{P,l} x_{k,KLMP} = 0 \tag{3.17}$$

$$\frac{\partial \hat{u}_l}{\partial x_{kK,LMN}} X_{P,l} x_{kK,LMNP} = 0 \tag{3.18}$$

The conditions (3.14) and (3.15), according to (3.11), may be written in equivalent forms

$$\frac{\partial \hat{u}_l}{\partial \dot{x}_{k,K}} X_{L,l} + \frac{\partial \hat{u}_l}{\partial \dot{x}_{k,L}} X_{K,l} = 2\rho \frac{\partial \hat{\psi}}{\partial x_{k,KL}} \tag{3.19}$$

$$\frac{\partial \hat{u}_l}{\partial \dot{\chi}_{kK}} \chi_{jK} - m_{lkj} = \rho \frac{\partial \hat{\psi}}{\partial \chi_{kK,L}} x_{l,L} \chi_{jK} \tag{3.20}$$

Next, we observe that the objectivity of $\mathbf{u}$ means that $\hat{\mathbf{u}}$ satisfies the equation

$$\begin{gathered} \mathbf{Q}\,\hat{\mathbf{u}}(\Lambda) = \hat{\mathbf{u}}(\mathbf{QF}, \mathbf{Q}\nabla\mathbf{F}, \mathbf{Q}\nabla^2\mathbf{F}, \mathbf{Q}\boldsymbol{\chi}, \mathbf{Q}\nabla\boldsymbol{\chi}, \mathbf{Q}\nabla^2\boldsymbol{\chi}, \\ \mathbf{Q}\nabla^3\boldsymbol{\chi}, \theta, \mathbf{Qg}, \mathbf{Q}\dot{\mathbf{F}} + \dot{\mathbf{Q}}\mathbf{F}, \mathbf{Q}\dot{\boldsymbol{\chi}} + \dot{\mathbf{Q}}\boldsymbol{\chi}) \end{gathered} \tag{3.21}$$

for all orthogonal tensor–valued functions $\mathbf{Q}(\tau)$, where $\dot{\mathbf{Q}} = \frac{d}{d\tau}\mathbf{Q}(\tau)$. If we take $\mathbf{Q}(\tau) = e^{\tau\mathbf{W}}$, where $\mathbf{W}$ is skew, and evaluate (3.21) at $\tau = 0$, we have

$$\mathbf{u}(\Lambda) = \hat{\mathbf{u}}(\mathbf{F}, \nabla\mathbf{F}, \nabla^2\mathbf{F}, \boldsymbol{\chi}, \nabla\boldsymbol{\chi}, \nabla^2\boldsymbol{\chi}, \nabla^3\boldsymbol{\chi}, \mathbf{g}, \dot{\mathbf{F}} + \mathbf{WF}, \dot{\boldsymbol{\chi}} + \mathbf{W}\boldsymbol{\chi}, \mathbf{Q}) \tag{3.22}$$

for all skew tensor $\mathbf{w}$. From (3.22) it follows that

$$\left(\frac{\partial \hat{u}_l}{\partial \dot{x}_{k,K}} x_{j,K} + \frac{\partial \hat{u}_l}{\partial \dot{\chi}_{kK}} \chi_{jK}\right)_{[kj]} = 0. \tag{3.23}$$

Recalling the expression for m_{lkj}, we see that $m_{lkj} = -m_{ljk}$. Then from (3.20) it follows that

$$m_{lkj} = \left(\frac{\partial \hat{u}_l}{\partial \chi_{kK}} - \rho \frac{\partial \hat{\psi}}{\partial x_{kK,L}} x_{l,L}\right) \chi_{jK_{[kj]}} \tag{3.24}$$

and

$$\left(\frac{\partial \hat{u}_l}{\partial \dot{\chi}_{kK}} - \rho \frac{\partial \hat{\psi}}{\partial \chi_{kK,L}} x_{l,L}\right) \chi_{jK_{(kj)}} = 0. \tag{3.25}$$

The expression (3.24) represents the constitutive equation for couple stress tensor m_{lkj}.

Further, we have to investigate the consequences implied by the conditions (3.19), (3.23), (3.25), (3.16), (3.17), (3.18) and the reduced dissipation inequality (3.10). Generally, this is very tedicus calculation. In any case, the presence of the strain rate $\dot{\mathbf{F}}$ and $\dot{\chi}$ in $\mathbf{u}$ is of crucial importance for our entire theory.

4. Interstitial working for elastic materials

From now on we apply the reduced dissipation inequality (3.10) to those of our materials, for which $\hat{\mathbf{T}}$, $\hat{\mathbf{M}}$ and $\hat{\mathbf{q}}$ are independent of $\dot{\chi}$.

For convinience in the next calculations the equations (3.20) can be written as

$$\frac{\partial \hat{u}_l}{\partial \dot{\chi}_{kM}} = m_{lkj} \chi_{jM} + \rho \frac{\partial \hat{\psi}}{\partial \chi_{kM,L}} x_{l,L} \equiv A_{lkM}, \tag{4.1}$$

where $A_{lkM} = A_{lkM}(\mathbf{F}, \nabla \mathbf{F}, \nabla^2 \mathbf{F}, \chi, \nabla \chi, \nabla^2 \chi, \nabla^3 \chi, \theta, \dot{\mathbf{F}}, \mathbf{g})$ i.e. $\frac{\partial \hat{u}_l}{\partial \dot{\chi}_{kM}}$ is independent of $\dot{\chi}$. Thus (4.1) is easily integrated, showing that $\hat{\mathbf{u}}$ depends on $\dot{\chi}$ at most affinely, i.e.

$$u_l = A_{lmM} \dot{\chi}_{mM} + w_l(\ldots, \dot{\mathbf{F}}, \mathbf{g}) \tag{4.2}$$

Then (3.14) may be written as

$$\left(\frac{\partial A_{lmM}}{\partial \dot{x}_{k,K}} \dot{\chi}_{mM} x_{j,K} + \frac{\partial W^E{}_l}{\partial x_{k,K}} x_{j,K}\right)_{(l,j)} = \rho \frac{\partial \hat{\psi}}{\partial x_{k,KL}} x_{l,K} x_{j,L},$$

wherefrom we obtain that

$$\frac{\partial A_{lmM}}{\partial \dot{x}_{k,K}} x_{j,K_{(l,j)}} = 0, \tag{4.3}$$

$$\frac{\partial W^E_l}{\partial \dot{x}_{k,K}} x_{j,K_{(l,j)}} = \rho \frac{\partial \hat{\psi}}{\partial x_{k,KL}} x_{l,K} x_{j,L}. \tag{4.4}$$

Further, from (3.23) and (4.2) it follows that

$$\frac{\partial A_{lmM}}{\partial x_{k,K}} x_{j,K_{[kj]}} = 0, \tag{4.5}$$

$$\left(\frac{\partial \hat{W}_l^E}{\partial \dot{x}_{k,K}} x_{j,K} + A_{lkK} \chi_{jK}\right)_{[kj]} = 0. \tag{4.6}$$

After some lengthy manipulations, it may be shown, from (4.3) and (4.5), that A_{lmM} does not depend on $\dot{\mathbf{F}}$,

$$A_{lmM} = A_{lmM}(\mathbf{F}, \nabla\mathbf{F}, \nabla^2\mathbf{F}, \chi, \nabla\chi, \nabla^2\chi, \nabla^3\chi, \theta) \tag{4.7}$$

and

$$W_l = b_{lmM}(\mathbf{F}, \nabla\mathbf{F}, \nabla^2\mathbf{F}, \chi, \nabla\chi, \nabla^2\chi, \nabla^3\chi, \theta)\, \dot{x}_{m,M} + s_l(\mathbf{g}) \tag{4.8}$$

what follows from (4.4) and (4.6).

Using (4.7) and (4.8) in (4.2), we obtain

$$\begin{aligned} u_l = {} & A_{lmM}(\mathbf{F}, \nabla\mathbf{F}, \nabla^2\mathbf{F}, \chi, \nabla\chi, \nabla^2\chi, \nabla^3\chi, \theta)\, \dot{\chi}_{mM} + \\ & + b_{lmM}(\quad)\dot{x}_{m,M} + s_l(\quad, \mathbf{g}) \end{aligned} \tag{4.9}$$

or, equivalently,

$$\mathbf{u} = \mathbf{A}(H)\dot{\chi} + \mathbf{B}(H)\dot{\mathbf{F}} + \mathbf{s}(\Gamma) \tag{4.10}$$

where

$$\begin{aligned} H &= \{\mathbf{F}, \nabla\mathbf{F}, \nabla^2\mathbf{F}, \chi, \nabla\chi, \nabla^2\chi, \nabla^3\chi, \theta\} \\ \Gamma &= H \cup \{\mathbf{g}\} \end{aligned} \tag{4.11}$$

If we now enter (4.11) into the restrictions (3.7) we find that $\mathbf{s}$ must meet the condition

$$\mathbf{s}_{\mathbf{g}}\, \mathbf{G} = 0. \tag{4.12}$$

The condition (4.12) means that the dependence of $\mathbf{s}$ on $\mathbf{g}$ can be at most affine, with skew linear part. i.e.

$$\mathbf{s} = \mathbf{\Omega}(H)\,\mathbf{g} + \mathbf{s}^E(H) \tag{4.13}$$

and we have shown that the interstitial work flux is oif the form

$$\mathbf{u} = \mathbf{A}\dot{\chi} + \mathbf{B}\dot{\mathbf{F}} + \mathbf{\Omega}\,\mathbf{g} + \mathbf{s}^E \tag{4.14}$$

In the next part of the paper we shall investigate the consequences of the conditions (3.8) and (3.9) and dissipative inequality (3.10) on the form of constitutive equations for **T**, **M** and **q**.

REFERENCES

[1] J. D. Korteweg: Arch. Neerl. Sci. Ex. Nat., (2) 6, 1901.
[2] J. E. Dunn and J. Serrin: Arch. Rat. Mech. Anal., 1985. (88) 95
[3] R. A. Toupin: Arch. Rat. Mech. Anal., 11, 1962.
[4] J. L. Ericksen: Trans. Soc. Rheology, 5, 1961.
[5] C. B. Kafadar and C. A. Eringen: Int. J. Engn. Sci., 3, 1971.

ТЕРМОДИНАМИКА ИНТЕРСТИЦИАЛЬНОГО ДЕИСТВИЯ МИКРОПОЛАРНОЙ СПЛОШНОЙ СРЕДЫ

Эта работа относиться к термодинамике интерстициального деиствия микрополарной среды. Согласно Дану и Серину [2] мы изменяем форму закона сохранения энергии для микрополарной сплошной среды и выводим услови которые должни быть выполнени флюксом интерстициального деиствия. Для специального класса микрополарной сплошной среды мы выводим явную форму вектора флюкса интерстициального деиствия.

TERMODINAMIKA INTERSTICIJALNOG DEJSTVA MIKROPOLARNOG KONTINUUMA

Rad se odnosi na problem nelokalnog dejstva u mikropolarnom materijalnom kontinuumu. U cilju obuhvatanja ovog dejstva Dun i Serin [2] modificiraju zakon balansa energije uvodeći novu konstitutivnu f-ju – koja karakteriše fluks intersticijalnog dejstva. Koristeći ovaj prilaz u radu je izveden ekplicitan oblik ovog fluksa za mikropolarni kontinuum.

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