UDK 531

# ON THE STABILITY OF A STATIONARY MOTION OF A RHEONOMIC SYSTEM

*Aleksandar Bakša*

(Received 13. 10. 1986., revised 5. 6. 1987.)

The stability of stationary motion and the state of equilibrium of a mechanical system belongs to a narrow field of problems in stability, for which general criteria can be found, according to which it is possible to discuss the stability. That is the reason why quite a number of papers have been written on this topic, starting from, now already classical, works of J. L. Lagrange, E. J. Routh, A. M. Lyaponov, N. G. Chetayev, and other well known scientists till the present times. Although in this field there exist important results, the problem has not been entirely solved. One of the questins to which we have not obtained an answer yet is which conditions of stability of the state of equilibrium and of a stationary motion of a rheonomic system are necessary and sufficient. In literature this problem has been considered, but mostly marginally, and not in a complex manner to the extent which this highly interesting topic deserves, [2] [3] [4].

An attempt has been made in the present paper to arrive at some general statements on stability and non-stability of the stationary motion of rheonomic system. The question of the stability of the state of equilibrium, as a special case of the problem mentioned above, has also been discussed to some extent. The presence of the parameter denoting time in equations of the disturbed motion leads to a number of peculiarities, because of which this problem is distinguished from an analogical problem for scleronomic systems. This, on the one hand, makes the problem more interesting, but on the other, leads to a series of new, not naive at all, problems. The questions considered here are not of theoretical importance only. There exist quite a number of practical problems whichg belong to this part of mechanics, e.g. stationary motion of gyroscopic systems on a moving platform.

## 1. Stationary Motion and State of Equilibrium of a Rheonomic System

We propose to consider a system of particles the motion of which is constrained by holonomic constraints

$$\rho_a\ (t, \vec{r}_1, \ldots, \vec{r}_N) = 0 \qquad (a = 1, \ldots, K) \tag{1.1}$$

where $\vec{r}_\nu$ is the radius vector of the particle $M_\nu$ with respect to the inertial system of reference. If $q^i$ $(i = 1, ..., n = 3N - K)$ are independent generalized coordinates, and if we introduce the notation $q = (q^1, ..., q^n) \in R^n$, then we can write

$$\vec{r}_\nu = \vec{r}_\nu (t, q) \tag{1.2}$$

The kinetic energy of the system is

$$T = \frac{1}{2} a_{ij} \dot{q}^i \dot{q}^j + a_i \dot{q}^i + a_0 \tag{1.3}$$

where $a_{ij}$, $a_i$ and $a_0$ are functions of both $t$ and $q$. Let the system be acted upon by forces, the potential of which is $\Pi = \Pi\ (t, q)$ as well as non-potential generalized forces $Q_i = Q_i\ (t, q, \dot{q})$.

The motion of this system will be described by equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = Q \quad (L = T - \Pi, \ \ Q = (Q_1, ..., Q_n)) \tag{1.4}$$

We shall assume that the system considered has $\ell$ $(\ell < n)$ cyclic coordinates, and that the numeration of coordinates is carried out in such a way that $q^k$ $(k = 1, ..., m = n - \ell)$ are positional (not cyclic) coordinates, while $q^\alpha$ $(\alpha = m + 1, ..., n)$ are cyclic coordinates. From (1.4) there follows that the first integrals of motion (cyclic integrals)

$$p_\alpha (t, q^1, ..., q^m, \dot{q}^1, ..., \dot{q}^n) = c_\alpha \quad (c_\alpha = \text{const.}) \tag{1.5}$$

coorrespond to the cyclic coordinates, where $p_\alpha$ are the generalized impulses which correspond to the cyclic coordinates

$$p_\alpha = a_{\alpha i} \dot{q}^i + a_\alpha \tag{1.6}$$

In our further deliberations we shall use the notations

$$q' = (q^1, ..., q^m),\ q'' = (q^{m+1}, ..., q^n),\ p = (p_{m+1}, ..., p_n) \tag{1.7}$$

It is well known that a system with cyclic coordinates can, under certain conditions, carry out a stationary motion, i.e. a motion in which position coordinates have constant values. For a stationary motion, thus, there follows

$$q' = q'_0,\ \dot{q}' = 0, \quad p = c \tag{1.8}$$

Provided the integrals (1.5) do not depend explicitly on time, generalized velocities, which correspond to cyclic coordinates, will also be constant in a stationary motion, for instance, in case of scleromonic sistems. Such a motion we shall call a stationary motion in a narrower sense of the word, or mero-static (μέρος – a part, portion). Especially if $\dot{q}'' = 0$ the equalities (1.8) become

$$q = q_0, \quad \dot{q} = 0 \tag{1.9}$$

and determine the postition of relative equilibrium of the representative point on a configurational multiplicity of a rheonomic system depends on time (time being taken as a parameter), i.e. it is formulated by the mapping

$$M^n \times R \to E^{3N} \qquad (q, t) \to (\vec{r}_1, ..., \vec{r}_n)$$

which means that the point $q_0 \in M^n$ ($q_0$ = const.) is mapped to a fixed point $(\vec{r}_1^{\,0}, ..., \vec{r}_N^{\,0}) \in E^{3N}$, and vice versa, constant generalized coordinates need not correspond to a fixed point in $E^{3N}$. Only in the case if from (1.2), for $\forall t \in R$ and $q = q_0$, we obtain

$$\vec{r}_\nu (t, q_0) = \vec{r}_\nu^{\,0} \qquad (\nu = 1, ..., N).$$

A state of equilibrium of the system considered with respect to the inertial system of reference will correspond to the point $(q_0, 0) \in TM^n / TM^n$ – a tangent bundle). From this it follows also that the state of equilibrium of the system, in generalized co-ordinates, can be determined by the function $q = q(t)$ for which there holds

$$\vec{r}_\nu (t, q(t)) = \vec{r}_\nu^{\,0}.$$

It is quite natural to raise the question of which conditions should be satisfied by the constraints (1.1) in order that the system could be in its position of equilibrium. That a position of the system $\{\vec{r}_\nu^{\,*}\}$, allowed by the constraints, could be a position of equilibrium, it is necessary that the class of possible velocities in that position should comprise a "zero element" $\{\vec{v}_\nu = 0\}_{\nu=1}^{N}$. For the constraints (1.1), the possible velocities are those that satisfy the conditions

$$\frac{\partial f a}{\partial \vec{r}_\nu} \cdot \vec{v}_\nu + \frac{\partial f a}{\partial t} = 0$$

That a system could have its position of equilibirum $\{\vec{r}_\nu^{\,*}\}$, it is necessary that position be allowed by the constraints, and that in that position the following equalities

$$\left. \frac{\partial f a}{\partial t} \right|_{\vec{r}_\nu = \vec{r}_\nu^{\,*}} = 0, \qquad t \in R \tag{1.10}$$

hold good. In a position of the system, in which the equilities (1.10) are valid, possible displacements are simultaneously virtual displacements.

NOTE. Systems for which possible displacements are simultaneously virtual displacements are called catastatic [1]/κατά – according to/. A characteristic of these system is that they can have a state of equilibrium. Since the systems in which there exists at least one point in which (1.10) is valid, can also have this property, we shall consider them too as catastatic. □

The motion of system with cyclic coordinates can be described conveniently by E. J. Routh's equations. If Routh's function is defined by

$$R\,(t, q', \dot{q}', p) = L - p_\alpha \dot{q}^\alpha = R_2 + R_1 - \Pi^*$$

then Rout's equations are

$$\frac{d}{dt}\frac{\partial R_2}{\partial \dot{q}^k} - \frac{\partial R_2}{\partial q^k} = g_{kj}\dot{g}^j - \frac{\partial \widetilde{a}_k}{\partial t} - \frac{\partial \Pi^*}{\partial q^k} + Q_k \tag{1.11}$$

$$\dot{q}^\alpha = \frac{\partial R_1}{\partial p_\alpha} - \frac{\partial \Pi^*}{\partial p_\alpha}, \quad \dot{p}_\alpha = 0 \quad (g_{kj} = -g_{jk})$$

where

$$2R_2 = (a_{jk} - a^{\alpha\beta} a_{\alpha j} a_{\beta k})\dot{q}^j \dot{q}^k = \widetilde{a}_{jk}\dot{q}^j \dot{q}^k$$

$$R_1 = (a_j^\alpha \, p_\alpha + a_j - a_{j\alpha} a^\alpha)\, \dot{q}^j = \widetilde{a}_j \dot{q}^j \tag{1.12}$$

$$\Pi^* = \Pi + \frac{1}{2} a^{\alpha\beta} p_\alpha p_\beta - a^\alpha p_\alpha + \frac{1}{2} a^{\alpha\beta} a_\alpha a_\beta - a_0$$

In order to make the stationary motion (1.8) a solution of the system (1.11), it is necessary and sufficient that for each

$$-\frac{\partial}{\partial t}\widetilde{a}_k\,(t, q_0', c) - \frac{\partial}{\partial q^k}\Pi^*\,(t, q_0', c) + Q_k\,(t, q_0', O, c) = 0 \tag{1.13}$$

holds good. The following geometrical interpretation can be ascribed to these equations. Let us denote

$$\sigma_k = -\frac{\partial \widetilde{a}_k}{\partial t} - \frac{\partial \Pi^*}{\partial q^k} + Q_k.$$

The system of equations

$$\sigma_k\,(t, q', c) = 0 \tag{1.14}$$

in which $c$ is considered as fixed, determines a sub–manifold $\gamma \subset M^n$ of a dimension not higher than $n - m$, which depends on time (considered as a parameter). Those solutions of (1.14) which do not depend on $t$, correspond to the stationary motion, i.e. the solutions $q' = q'(c)$. Thus, stationary motions of the system (1.11) belong to the set $\gamma_0$ of fixed points of the multiplicity $\gamma$.

In the case if the system is moving under the influence of potential forces only, Routh's potential on the stationary motion will have a stationary value provided only that

$$\frac{\partial}{\partial t}\,\widetilde{a}_k\,(t, q_0', c) = 0$$

which is evident from (1.13). This condition is trivially satisfied for gyroscopic non–constrained systems ($\widetilde{a}_k = 0$), as well as for system in which $\widetilde{a}_k$ do not depend explicitly on $t$ (for example, scleronomic systems).

If the integrals (1.5) do not depend on $t$, the equations (1.13) will, for $c = 0$, represent necessary and sufficient conditions for the representative point $q_0 \in M^n$ to be motionless with respect to a configurational multiplicity (a state of relative equilibrium). In a general case (if the system has no cyclic coordinates) the necessary and sufficient conditions for the relative equilibrium are, (which follows from (1.4)).

$$\frac{\partial a_i}{\partial t} + \frac{\partial a_0}{\partial q^i} - \frac{\partial \Pi}{\partial q^i} + Q_i = 0. \tag{1.15}$$

Finally, here we shall add yet another remark, which will be useful later on for our considerations. The function

$$\mathrm{H} = R_2 + \Pi^* \tag{1.16}$$

has a derivative in terms of time, this derivative being composed in the sense of the equations (1.11),

$$\frac{d\mathrm{H}}{dt} = Q_k \dot{q}^k - \frac{\partial R}{\partial t} \tag{1.17}$$

If $R$ does not depend explicitly on $t$, and if $Q_k \dot{q}^k = 0$, it follows the first integral $R_2 + \Pi^* = \text{const.}$ (Jacobi's integral).

## 2. Equations of the disturbed motion

Let us assume that there exists a value of the parameter $c$ for which the set $\gamma_0$ is not empty, i.e. that the sustem can perform stationary motion

$$q' = 0, \quad \dot{q}' = 0, \; p = c \tag{2.1}$$

(an appropriate selection of coordinates can always make sure that $q_0' = 0$). We propose to consider the stability of this motion by ignoric cyclic coordinates $q''$ (that will be a stability with respect to part of variables $q', \dot{q}'$ and $p$). If we denote the disturbed motion

$$q' = q'(t), \; p = c + \eta(t) \tag{2.2}$$

the equations of the disturbed motion are

$$\text{i)} \quad \frac{d}{dt}\frac{\partial R_2}{\partial \dot{q}^k} - \frac{\partial R_2}{\partial q^k} = g_{kj}\dot{q}^j - \frac{\partial \widetilde{a}_k}{\partial t} - \frac{\partial \Pi^*}{\partial q^k} + Q_k \qquad \text{ii)} \; \dot{\eta}_\alpha = 0 \tag{2.3}$$

where all terms of these equations should be considered as functions of the variables (2.2)

From (2.3 ii) it is evident that in the disturbed motion the integrals $\eta_\alpha = \gamma_\alpha(\gamma_\alpha =$ const., $p = c + \gamma)$ are valid. If these values are substituted in (2.3 i), we shall obtain a closed sub–system of differential equations in terms of $q'$. We shall note that $q' = 0$ is not a solution of that system because

$$\sigma_k (t, O, c + \gamma) \neq 0.$$

Thus we can say that (2.3 i) are the equations of the disturbed state of equilibrium $q' = 0$, $\dot{q}' = 0$, of the "reduced" system under constant action of disturbations. The functions $\sigma_k$ can be expressed in the form of

$$\sigma_k (t, q', \dot{q}', c + \gamma) = \sigma_k (t, q', \dot{q}', c) + \rho_k (t, q', \dot{q}', c + \gamma).$$

Since $\sigma_k (t, O, O, c) = 0$, there follows that the functions $\rho_k$ are generating constant disturbations. This means that in this case disturbations are represented by a known function (the so–called potential disturbations).

In an analogous way it is also possible to derive equations of the disturbed state of equilibrium. If the system of coordinates is selected in such a way that the state of equilibrium is at the point $q = 0$, and the notations of coordinates $q$ and $\dot{q}$ retained for the disturbations, then the equations of the disturbed state of equilibrium are of the form (1.6), provided we assume that the generalized forces are annulled in the state of equilibrium.

## 3. Conditions of stability

We propose to investigate the stability of the stationary motion considered by the application of Lyapunov's direct method. We shall begin our investigations by considering the system for which the following assumptions are valid:

1° Gneralized forces $Q_k$ on the non–disturbed motion are annulled;

2° $\dfrac{\partial \tilde{a}_k}{\partial t} = 0$

3° $R_2$ is positive definite in terms of $\dot{q}'$, and it has an infinitely small upper limit when $\dot{q}' \to 0$, and

4° In the vicinity of the point $(0, c)$, $\Pi^* (t, q', p)$ is uniformaly continuous in terms of $t$.

Under these assumptions we see that

$$V (t, q', \dot{q}') = R_2 (t, q', \dot{q}') + \Pi^* (t, q', p) - \Pi^*(t, O, c)$$

is a Lyapounoff's functiona. Its derivative in terms of time, composed in the sense of equations of the disturbed motion (2.3), is

$$\dot{V} = Q_k \dot{q}^k - \frac{\partial}{\partial t} [ R_2 (t, q', \dot{q}') - \Pi^* (t, q', p) + \Pi^* (t, O, c)]$$

On that basis, for systems which satisfy the assumptions 1° to 4°, it is possible to formulate

THEOREM 1. If Routh's potential $\Pi^*$ at the point $q' = 0, p = c$ has an isolated minimum for each $t \in R$, and if in the vicinity of the point $q' = 0, \dot{q}' = 0, p = c$, the inequality

$$\frac{\partial}{\partial t} [R_2(t, q', \dot{q}') - \Pi^*(t, q', p) + \Pi^*(t, O, c)] - Q_k \dot{q}^k \geqslant 0$$

is valid, then the stationary motion (2.1) is a stable one with respect to the variables $q'$ and $\dot{q}'$.

If, in addition, $\Pi^*$ has an infinitely small upper limit when $q' \to 0$, and the function $p(t, q', \dot{q}')$ is a uniformly continuous on in terms of $t$, the stationary motion is uniformly stable with respect to the variables $q', \dot{q}', p$.

The proof of this Theorem follows from the general theorems on stability of the direct Lyapunov method.

The Theorem 1 provides ample conditions of stability of the stationary motion for a considerably narrower (although not a rare one) class of mechanical systems. In a general case, the analogous conditions of stability are unknown to us. For the analysis of problems we can use the following method. Since the function $\Pi^*$ does not necessarily have a stationary value on the non–disturbed motion (2.1), an attempt should be made that it be substituted by the function $w = w(t, q', \eta) \in C^1$ which should be such that $R_2 + w$ is a positive definite function. In that case, sufficient conditions of stability are provided by the

THEOREM 2. If there exists a function $w(t, q', \eta) \in C^{(1,1,1)}$ and strictly increasing functions

$$\theta_s : R \to R, \quad \theta_s(0) = 0 \qquad (s = 1, 2, 3, 4)$$

such that in the vicinity of the point $q' = 0, \dot{q}' = 0, \eta = 0$ the following is valid

1° $\theta_1(\|\dot{q}'\|) \leqslant R_2(t, q', \dot{q}') \leqslant \theta_2(\|\dot{q}'\|)$

2° $\theta_3(\|q'\| + \|\eta\|) \leqslant w(t, q', \eta) \leqslant \theta_4(\|q'\| + \|\eta\|)$

3° $\left(\frac{\partial w}{\partial q^k} - \frac{\partial \Pi^*}{\partial q^k} + Q_k\right)\dot{q}^k - \frac{\partial}{\partial t}(R_2 + R_1 - w) \leqslant 0$

then the stationary motion (2.1) is a stable one with respect to $q'$ and $\dot{q}'$.

If in 3° the sign of equality is valid when and only when $q' = 0, \dot{q}' = 0, \eta = 0$, then the stability is asymptotic. □

The proof of the Theorem 2 follows from the general theorems on stability of the direct Lyapunov method.

The stability conditions in terms of $q', \dot{q}', \eta$, as well as the conditions of a uniform stability can be formulated in a way analogous to one used in the Theorem 1.

Since the state of equilibrium can be considered a special case of the stationary motion, we shall not divulge any more on the investigations of conditions of such a stability. An attempt, however, should be made to find an analogon to the famous Lagrange –Direchlet theorem.

We shall assume that the system (1.4) is moving in a field having a potential $\Pi(t, q)$ $(Q = 0)$, which has a strict minimum at the point $q = 0$. In order to enable this point to correspond to the state of equilibrium of the system it is necessary, according to (1.15), that

$$\frac{\partial a_0}{\partial q^i} - \frac{\partial a_i}{\partial t} = 0 \qquad (i = 1, \dots, n)$$

and that the constraints satisfy the condition (1.10). Sufficient conditions of stability of this state of equilibrium are provided by the

THEOREM 3. If the potential energy in the state of equilibrium has a strict minimum, and if in the vicinity of that position

$$\frac{\partial}{\partial t}(T - \Pi) - \frac{d T_0}{d t} \geqslant 0$$

then that position of equilibrium is stable. □

The proff of the Theorem 3 follows from the Lyapunov's first theorem, provided we take $V = T_2 + \Pi$ for Lyapunov's function, while bearing in mind that $T_2$ is a positive definite quadratic form in terms of $\dot{q}$.

## 4. Conditions of non–stability

The determinations of conditions of non–stability of a stationary motion is a complex task which even for scleronomic systems has not been solved completely. The problem is complicated by the presence of terms having the character of gyroscopic forces in equations of the disturbed motion, since they, as is well known, can be instrumental in providing stabilization of a non–stable motion (the so–called gyroscopic stabilization). This makes considerably more difficult the application of the direct Lyapunov method in the in vestigation of non–stability. We shall consider here the non–stability of a stationary motion in case if equations of the disturbed motion have first integral.

THEOREM 4. If the equations of the disturbed motion (2.3) have first integral

$$g(t, q', \dot{q}', \eta) \in C^{(1,1,1)}, \qquad g(t, 0, 0, 0) = 0$$

and if

1° for an arbitrary $\epsilon > 0$, there exist sets

$$G = \{(t, q', \dot{q}', \eta) | g(t, q', \dot{q}', \eta)\} > 0 \quad , B_\epsilon = \{(t, q', \dot{q}', \eta) | \, \|q'\| + \|\dot{q}'\| + \|\eta\| < \epsilon\}$$

such that $G \cap B_\epsilon \neq \emptyset$

$$2° \quad \emptyset \neq \psi = \left\{ (t, q', \dot{q}', \eta) \mid \frac{\partial R}{\partial \dot{q}^k} q^k > 0 \right\} \subset G$$

$$3^{\circ}\quad 2R_2 + R_1 + \left(\frac{\partial R}{\partial q^k} + Q_k\right) q^k > 0 \quad \text{for} \quad (q', \dot{q}', \eta) \in \psi$$

and thus, the stationary motion (2.1) is non–stable. □

**Proof.** The function

$$V = g\,\frac{\partial R}{\partial \dot{q}^k}\, q^k$$

satisfies the conditions of Chetayev's theorem in the sense of the equations (2.3), on the basis of which there follows the statement of the theorem.

**Consequence.** If there exists the first integral of equations of the disturbed motion of the form

$$g(t, q', \dot{q}', \eta) = \mathrm{H}(t, q', \dot{q}', \eta) + \gamma\,(t, q, \eta)$$

the condition 3° can be replaced by the non–equality

$$3.a^{\circ}\quad \Pi + \gamma + \left(\frac{\partial \Pi}{\partial q^k} - Q_k\right) q^k < 0.$$

Proof of the consequence. Since on an arbitrary disturbed motion, which starts from the region $\psi$ the following

$$g = R_2 + \Pi^* + \gamma = c > 0 \Rightarrow R_2 = c - \gamma - \Pi^*$$

is valid, the expression of the left–hand side of the non–equality 3° can be transformed in the following way

$$2R_2 + R_1 + \left(\frac{\partial R}{\partial q^k} + Q_k\right) q^k > -(\Pi + \gamma) - \left(\frac{\partial \Pi}{\partial q^k} - Q_k\right) q^k.$$

On the basis of this relationship and the condition 3°, there follows 3.a°, which was to be proved.

In the case of conservative system, the Theorem 4 is reduced to the well known result of M. Laloy, [5].

EXAMPLE 1. A particle is moving along a smooth ellipsoid

$$c^2(t)\,(x^2 + y^2) + a^2 z^2 = a^2 c^2(t), \quad a \in R,\ c(t) \in C^{(1)}$$

in a force field, the potential of which is $2\Pi = -mk^2\,(x^2 + y^2),\ k \in R$.

If the independent coordinates $\theta$ and $\varphi$ are introduced by a substitution

$$x = a \cos\theta \cos\varphi, \quad y = a \cos\theta \sin\varphi, \quad z = c\,(t) \sin\theta$$

the kinetic potential $L$ is

$$2L = m\,[(a^2\sin^2\theta + c^2\cos^2\theta)\dot\theta^2 + a^2\dot\varphi^2\cos^2\theta + c\dot c\,\dot\theta\sin 2\theta + \dot c^2\sin^2\theta] + mk^2a^2\cos^2\theta$$

A generalized impulse $p = ma^2\dot\varphi\cos^2\theta$ corresponds to the cyclic coordinate $\varphi$, while the corresponding Routh's function is

$$R = \frac{m}{2}\,[(a^2\sin^2\theta + c^2\cos^2\theta)\dot\theta^2 + c\dot c\,\dot\theta\sin 2\theta] - \frac{p^2}{2ma^2\cos^2\theta} +$$

$$+ \frac{m}{2}\,(k^2a^2\cos^2\theta + \dot c^2\sin^2\theta)$$

The equations of motion of the particle have the cyclic integral $p = \gamma$. For the existence of the stationary motion $\theta = \theta_0$ it is necessary and sufficient that

$$(c\ddot c + k^2a^2)\sin 2\theta + \frac{2p^2\sin\theta}{m^2a^2\cos^3\theta} = 0$$

whence there follows

1) $\theta_1 = 0$ – stationary motion along the trajector $C\colon z = 0, x^2 + y^2 - a^2 = 0$ (exists for an arbitrary function $c(t)$),

2) $\theta_2$ – solution of the equation

$$\cos^4\theta_2 = \gamma^2/m^2a^2\,(c\ddot c + k^2a^2)$$

– stationary motion that exists only provided the function $c(t)$ satisfies the condition $c\ddot c =$ = const.,

3) for $\gamma = 0$, states of equilibrium are obtainable:

a) $\theta_3 = 0$ – equilibrium multiplicity $x^2 + y^2 - a^2 = 0, z = 0$;

b) $\theta_{3,4} = \pm\pi/2\ (x = y = 0, z = \pm c\,(t))$.

We shall now investigate the stability of the stationary motion 1). Since

$$\Pi^* = \frac{p^2}{2ma^2\cos^2\theta} - \frac{1}{2}mk^2a^2\cos^2\theta - \frac{1}{2}m\dot c^2\sin^2\theta$$

we obtain

$$\left(\frac{\partial\Pi^*}{\partial\theta}\right)_{\theta=0} = 0, \qquad \left(\frac{\partial^2\Pi^*}{\partial\theta^2}\right)_{\theta=0} = \frac{\gamma^2}{ma^2} + m(a^2k^2 - \dot c^2).$$

This motion will – according to Theorem 1 – be stable provided

$$\gamma^2 + m^2a^2(a^2k^2 - \dot c^2) > 0 \quad (a^2\sin^2\theta + 2c\dot c\cos^2\theta)\dot\theta^2 - 2\dot c\ddot c\sin^2\theta \geqslant 0$$

It is raedily seen that there exist functions $c\,(t)$ for which these conditions are satisfied (e.g. $(c_0^2 + \alpha^2 t)^{1/2}$).

## 5. Systems with quasi–cyclic coordinates

Let us assume that in the system considered with $n$ degres of freedom of motion, the coordinates $q^k$ ($k = 1, ..., m$) are position coordinates, while $q^\alpha(\alpha = m+1, ..., n)$ are quasi–cyclic coordinates (if in the system there exist cyclic coordinates as well, we shall assume that they too belong to this set of coordinates). In Routh's form, the equations of motion of this system are

$$\frac{d}{dt}\frac{\partial R_2}{\partial \dot{q}^k}-\frac{\partial R_2}{\partial q^k}=g_{kj}\dot{q}^j-\frac{\partial \tilde{a}_k}{\partial t}-\frac{\partial \Pi^*}{\partial q^k}+Q_k^*$$

$$\dot{q}^\alpha=-\frac{\partial R}{\partial p_\alpha},\quad \dot{p}_\alpha=Q_\alpha,\qquad Q_k^*=Q_k-\frac{\partial \tilde{a}_k}{\partial p_\alpha}Q_\alpha \tag{5.1}$$

The existence of quasi–cyclic coordinates, as compared with cyclic ones, does not require the existence of first integrals of motion. Only in the case if there exist functions $f_\alpha\,(t, q')$ such that the generalized forces which correspond to cyclic coordinates can be expressed in the form of $Q_\alpha = \dot{f}\,(t, q')$ will there exsist integrals of motion

$$p_\alpha(t,q',\dot{q}')-\delta_\alpha(t,q')=c_\alpha. \tag{5.2}$$

For example, if the forces generally potential, the potential being $V(t, q', \dot{q}') = \theta_i \dot{q}^i$, $\theta_i = \theta_i\,(t, q')$, we shall have

$$Q_\alpha=\frac{d}{dt}\frac{\partial V}{\partial \dot{q}^\alpha}-\frac{\partial V}{\partial q^\alpha}=\frac{d\theta_\alpha}{dt}$$

so that the corresponding first integral is $p_\alpha - \theta_\alpha = c_\alpha$

The system with quasi–cyclic coordinates is able to perform a stationary motion

$$q^k=q_0^k,\ p_\alpha=c_\alpha\quad (q_0, c=\text{const.}) \tag{5.3}$$

provided the following, on that motion, is valid

$$-\frac{\partial \tilde{a}_k}{\partial t}-\frac{\partial \Pi^*}{\partial q^k}+Q_k^*=0,\qquad Q_\alpha=0. \tag{5.4}$$

It is readily seen that the equalities (5.4) represent necessary and sufficient conditions of the stationary motion of the system (5.1). There are as many these equalities as there are parameters (5.3) so that, under certain conditions, they can have isolated solutions in terms of $q^k$ and $c_\alpha$, which, provided they do not depend on $t$, represent stationary motions of the mechanical system.

In a special case, if $Q_\alpha = 0$, form (5.1) we obtain

$$\dot{q}^{\alpha} = -\frac{\partial R}{\partial p_{\alpha}} = -\frac{\partial}{\partial p_{\alpha}} [\widetilde{a}_k \dot{q}^k - \Pi^*(t, q', p)]$$

so that on the stationary motion

$$\dot{q}^{\alpha} = \frac{\partial}{\partial p_{\alpha}} \Pi^*(t, c, q_0) = \varphi^{\alpha}(t) \Rightarrow q^{\alpha} = q^{\alpha}(t).$$

Let us assume that the system considered can carry out a stationary motion

$$q^k = 0, \quad p_{\alpha} = c_{\alpha}, \quad q^{\alpha} = q^{\alpha}(t). \tag{5.5}$$

If that motion is considered as non–disturbed, let us designate a motion disturbed with respect to it by

$$q^k = q^k(t), \; p_{\alpha} = c_{\alpha} + \eta_{\alpha}(t), \; q^{\alpha} = q^{\alpha}(t) + \xi^{\alpha}(t) \tag{5.6}$$

The equations of the disturbed motion are

$$\text{i)} \quad \frac{d}{dt}\frac{\partial R_2}{\partial \dot{q}^k} - \frac{\partial R_2}{\partial q^k} = g_{kj}\dot{q}^j - \frac{\partial \widetilde{a}_k}{\partial t} - \frac{\partial \Pi^*}{\partial q^k} + Q_k^*$$

$$\text{ii)} \quad \dot{\xi}^{\alpha} = -\frac{\partial R}{\partial p_{\alpha}} - \frac{\partial \Pi_0^*}{\partial p_{\alpha}}; \qquad \dot{\eta}_{\alpha} = Q_{\alpha} \tag{5.7}$$

$$\Pi_0^* = \Pi^*(t, O, c), \; \widetilde{a}_k = \widetilde{a}_k(t, q', c+\eta), \; R, Q = \mathit{fonct}\,(t, q', \dot{q}', c+\eta).$$

Since the disturbances appear only in equations 5.7 ii), we can, in order to simplifi the problem, in this case as well, ignore them, and proceed to investigate the stability of motion (5.5) with respect to a part of variables $q'$ and $p$.

THEOREM 5. If on a stationary motion Routh's potential $\Pi^*$ has an isolated minimum, and the function $R_2$ is positive definite in terms of $\dot{q}'$, and if in the vicinity of that motion

$$Q_k \dot{q}^k - Q_{\alpha}\frac{\partial R}{\partial p_{\alpha}} - \frac{\partial R}{\partial t} \leqslant 0$$

the motion is stable with respect to the variables $q'$, $\dot{q}'$, $\eta$. □

The proof of this theorem follows from the general Lyaponov theorem on stability provided we assume the Lyapunov function to be $R_2 + \Pi^*$.

Since the system considered can perform a stationary motion on which the potential $\Pi^*$ does not have an isolated minimum, it appears that the application of this theorem is rather limited. In a general case an attempt should de made to find a positive definite function

$$w(t, q', \eta) \in C^{(1,1,1)}$$

such that

$$\frac{d}{dt}(R_2 + w) = (Q_k - \frac{\partial \Pi^*}{\partial q^k} + \frac{\partial w}{\partial q^k})\dot{q}^k + (\frac{\partial w}{\partial \eta_\alpha} - \frac{\partial R_1}{\partial \eta_\alpha})Q_\alpha - \frac{\partial}{\partial t}(R_2 + R_1 - w) \leqslant 0$$

If, in addition, $R_2$ is positive definite in terms of $\dot{q}'$ too, the stationary motion will be stable.

In a special case if there exist integrals (5.2), the investigation of the stability can be simplified to some extent. After working out the generalized impulses, from (5.2), and after their substitution in the first group of equations (5.1), the problem of investigation of stability is reduced to the one we considered in Section 2.

EXAMPLE 2. Heavy particles $M_1$ (of mass $m_1$) and $M_2$ (of mass $m_2$) are connected by a non–stretchable, perfectly flexible thread passing through an arbitrarily small opening $O$ in a horizontal smooth plane $\pi$, which moves translatorily according to the law of $\zeta = \zeta(t)$. The particle $M_1$ moves in the plane $\pi$, while the particle $M_2$ moves along the perpendicular through $O$. The particle $M_1$ is acted upon by a force, the centre of which is at $O$, its intensity being $F(t)$.

If $r$ and $\varphi$ are chosen for independent variables ($r, \varphi, z$ ($z \perp \pi$) – are cylindrical co-ordinates), the kinetic energy $T$ and the potential energy $\Pi$ are

$$2T = m\dot{r}^2 + 2m_2\dot{\zeta}\dot{r} + m_1 r^2\dot{\varphi}^2 + m\dot{\zeta}^2, \qquad \Pi = m_2 g(r + \zeta),\ m = m_1 + m_2$$

The cyclic coordinate is $\varphi$, and the corresponding generalized impulse is

$$p = m_1 r^2 \dot{\varphi} \quad \Rightarrow \quad \dot{\varphi} = p/m_1 r^2$$

For Routh's function we obtain

$$R = \frac{1}{2} m\dot{r}^2 + m_2\dot{\zeta}\dot{r} - \frac{p^2}{2m_1 r^2} - m_2 g r$$

while the corresponding equations of motion are

$$m\ddot{r} + m_2\ddot{\zeta} - \frac{p^2}{m_1 r^3} + m_2 g = Q_r, \quad \dot{p} = 0.$$

In order that the stationary motion $r = r_0$, $p = c$, should exist, it is necessary and sufficient that

$$Q_r = m_2 g - \frac{c^2}{m_1 r_0^3} + m_2\ddot{\zeta}$$

(which is feasible, for instance, whit $Q_r = m_2\ddot{\zeta}, c^2 = m_1 m_2 g r_0^3$).

The equations of the disturbed motion are

$$m\ddot{\xi} - \frac{(c+\eta)^2}{m_1(r_0+\xi)^3} + \frac{c^2}{m_1 r_0^3} = 0, \qquad \dot{\eta} = 0 \ (\Rightarrow \eta = \eta_0)$$

and they do not comprise $t$ (the autonomous system thus obtained can be analyzed further on by classical methods).

The system could perform a stationary motion under the action of the force $Q_\varphi = -k^2 \dot{r}$, too (in that case, the coordinate $\varphi$ would be a quasi–cyclic one). Then we would have

$$\dot{p} = -k^2 \dot{r} \Rightarrow p = -k^2 r + \gamma$$

and the equations of the disturbed motion would be

$$m\ddot{\xi} - \frac{(c-k^2\xi)^2}{m_1(r_0+\xi)^3} + \frac{c^2}{m_1 r_0^3} = 0 \quad (c = -k^2 r_0 + \gamma)$$

$$\dot{\eta} = -k^2 \dot{\xi} \Rightarrow \eta = -k^2 \xi + \eta_0 .$$

REFERENCES

[1] L. A. Pars, A TREATISE ON ANALYTICAL DYNAMICS, Heinemann, London, 1964.

[2] N. Rouche, P. Habets, M. Laloy, STABILITY THEORY BY LIAPOUNOFF'S DIRECT METHOD, Springer–Verlag, 1977.

[3] N. Rouche et K. Peiffer, LE THEOREME DE LAGRANGE–DIRICHLET ET LA DEUXIEME METHODE DE LIAPOUNOFF, Annales de la Societe Scientifiques de Bruxelles, T. 81. I pp 19 – 33, 1967.

[4] V. A. Vujičić, THE ANALYTIC CRITERION FOR STABILITY OF MOTION OF RHEONOMIC SYSTEMS, Glas CCCI de l'Academie Serbe des Sciences et des Arts, Class des Sci, mathém. at nat., N° 41, 1977.

[5] M. Laloy, ON EQULIBIUM INSTABILITY FOR CONSERVATIVE AND PARTIALLY DISSIPATIVE MECHANICAL SYSTEMS, Louiain, 1975. Sem. Math. Appl. mech, 82.

## SUR LA STABILITE DE MOUVEMENT STATIONNAIRE DE SYSTEME RHEONOME

On considère un système dynymice holonome rheonome qui se ment dans un champ des forces avec potentiel sous l'action des forces généraliséis sans potentiel. On determine les conditions du mouvement stationnaire et on etudie la stabilité de ce mouvement.

## O STABILNOSTI STACIONARNOG KRETNJA REONOMNOG SISTEMA

Razmatra se reonomni holonomni sistem koji se kreće u polju sa potencijalom u prisustvu nepotencijalnih generalisanih sila. Pretpostavlja se da sistem ima kvaziciklične, odnosno, ciklične koordinate i da može da vrši stacionarno kretanje. Ispitani su uslovi pod kojima je moguće stacionarno kretanje (kao specijalan slučaj postavljeni su uslovi ravnoteže). Teoremama 1 i 2 dati su dovoljni uslovi stabilnosti stacionarnog kretanja, primenom direktnog Ljapunovljevog metoda. Za specijalnu klasu sistema (koji zadovoljavaju uslove 1°–4°) ispitana je stabilnost stacionarnog kretanja na kome Rautov potencijal ima minimum (teorema 1). Za slučaj kad jednačine poremećenog kretanja imaju prvi integral, dati su dovoljni uslovi nestabilnosti stacionarnog kretanja (teorema 4). Za sisteme sa kvazicikličnim koordinatama dovoljne uslove stabilnosti stacionarnog kretanja pri kome Rautov potencijal ima izolovani minimum daje teorema 5. Rezultati su ilustrovani primerima 1 i 2.

Aleksandar Bakša
Prirodno-matematički fakultet
Institut za mehaniku
Studentski trg 16, BEOGRAD